The state of factoring algorithms and other cryptanalytic threats to RSA

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Textbook RSA

Public Key

$N = pq$ modulus

e encryption exponent

Private Key

p, q primes

$d = e^{-1} \mod (p-1)(q-1)$
decryption exponent

Encrypt

$$c = m^e \pmod{N}$$

Decrypt

$$m = c^d \pmod{N}$$

Verify

$$m = s^e \pmod{N}$$

Sign

$$s = m^d \pmod{N}$$

Computational problems

Factoring

Problem: Given N, compute its prime factors.

- Computationally equivalent to computing private key d.
- Factoring is in NP and coNP $\rightarrow$ not NP-complete (unless P=NP or similar).

Computational problems

eth roots mod N

Problem: Given N, e, and c, compute x such that $x^e \equiv c \bmod N$.

- Equivalent to decrypting an RSA-encrypted ciphertext.
- Equivalent to selective forgery of RSA signatures.
- Conflicting results about whether it reduces to factoring:
  - "Breaking RSA may not be equivalent to factoring" [Boneh Venkatesan 1998]
    "an algebraic reduction from factoring to breaking low-exponent RSA can be converted into an efficient factoring algorithm"
  - "Breaking RSA generically is equivalent to factoring" [Aggarwal Maurer 2009]
    "a generic ring algorithm for breaking RSA in $\mathbb{Z}_N$ can be converted into an algorithm for factoring"
- "RSA assumption": This problem is hard.

Practical concern #1: Textbook RSA is insecure

RSA encryption is homomorphic under multiplication. This lets an attacker do all sorts of fun things:

Attack: Malleability

Given a ciphertext $c = m^e \bmod N$, $ca^e \bmod N$ is an encryption of ma for any a chosen by attacker.

Attack: Chosen ciphertext attack

Given a ciphertext c, attacker asks for decryption of $ca^e \bmod N$ and divides by a to obtain m.

Attack: Signature forgery

Attacker convinces a signer to sign $z = xy^e \bmod N$ and computes a valid signature of x as $z^d/y \bmod N$.

So in practice we always use padding on messages.

Practical concern #2: Efficiency

Choose e to be small and low hamming weight.

Use Chinese Remainder Theorem to speed up computations with d:

$$d_p = d \bmod p-1 \qquad d_q = d \bmod q-1$$

Compute

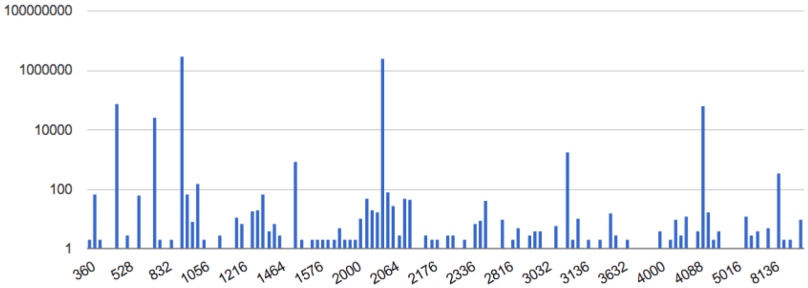
$$c_p = m^{d_p} \bmod p \qquad c_q = m^{d_q} \bmod q$$

$$c = \mathrm{crt}(c_p, c_q) \bmod N$$

Public-key cipher usage

	RSA	DSA	ECDSA	ElGamal	GOST
TLS	5,756,445	6,241	8		225
SSH	3,821,651	3,729,010	153,109		7
PGP	676,590	2,119,245		2,126,098	

RSA key size distribution



TLS, November 2011

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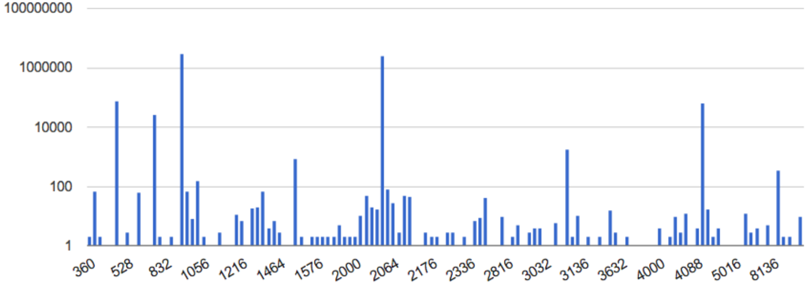
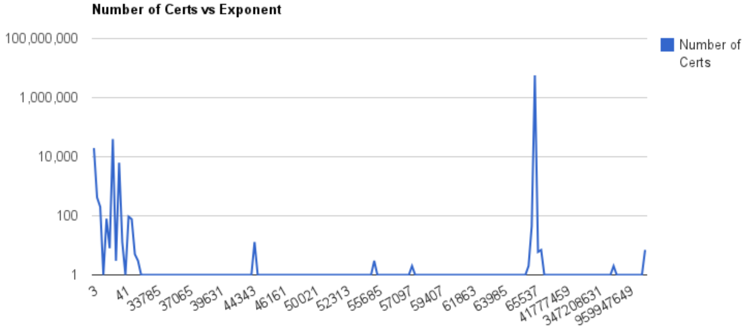


TLS, November 2011

RSA exponent distribution



TLS, November 2011

Implementation issues

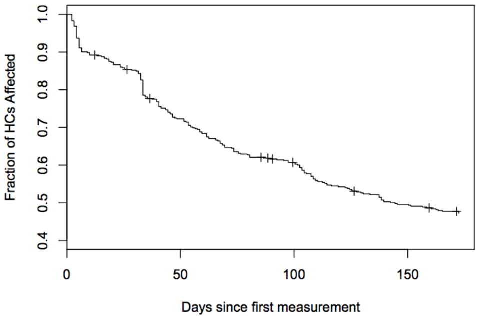
```
MD_Update(&m,buf,j);
```

The Debian OpenSSL entropy disaster

August, 2008: Discovered by Luciano Bello

Keys dependent only on pid and machine architecture:
294,912 keys per key size.

"When Private Keys are Public: Results from the 2008 Debian OpenSSL Vulnerability" [Yilek, Rescorla, Shacham, Enright, Savage 2009]



Searching for more entropy problems

Experiment

1. Acquire many public keys.
2. Look for obvious key-generation problems.

“Public keys” [Lenstra, Hughes, Augier, Bos, Kleinjung, Wachter Crypto 2012]

“Mining Your Ps and Qs: Detection of Widespread Weak Keys in Network Devices” [Heninger, Durumeric, Wustrow, Halderman Usenix Security 2012]

What could go wrong with RSA and entropy problems?

- Two hosts share N: $\rightarrow$ both know private key of the other.
- Two hosts share RSA moduli with a prime factor in common $\rightarrow$ outside observer can factor both keys by calculating the GCD of public moduli.

$$N_1 = pq_1 \qquad N_2 = pq_2$$

$$\gcd(N_1, N_2) = p$$

Time to factor 768-bit RSA modulus:
two years

Time to calculate GCD for 1024-bit RSA moduli:
15μs

Looking for problems: RSA common divisors

Speed-bump

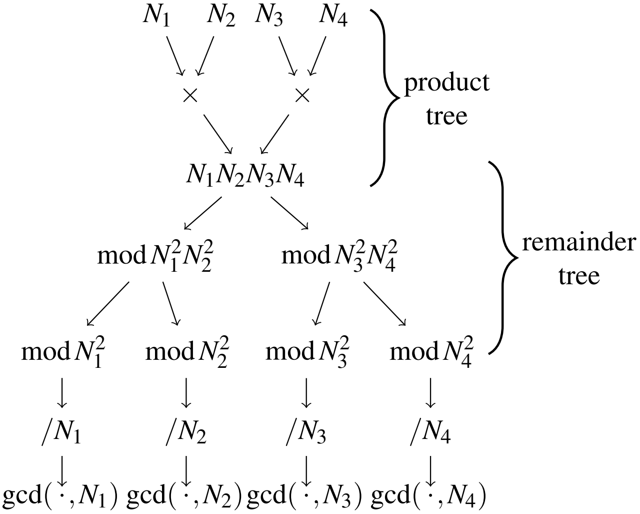
Computing pairwise $\gcd(N_i, N_j)$ for our dataset would take

$$15\mu s \times \binom{11 \times 10^6}{2} \text{pairs} \approx 30 \text{ years}$$

of computation time.

Efficient all-pairs GCDs

We implemented an efficient algorithm due to [Bernstein 2004].



Results

Repeated Keys

- $>$ 60% of TLS and SSH hosts have non-unique keys.
- $>$ 5% of TLS hosts and $>$ 10% of SSH hosts serve default or low-entropy keys
- 0.03% TLS hosts and 0.5% of SSH hosts serve Debian weak keys

Factored keys

- 0.5% of TLS hosts and 0.03% of SSH hosts keys factored

Flaw Found in an Online Encryption Method

By JOHN MARKOFF
Published: February 14, 2012

SAN FRANCISCO — A team of European and American mathematicians and cryptographers have discovered an unexpected weakness in the encryption system widely used worldwide for online shopping, banking, e-mail and other Internet services intended to remain private and secure.

The flaw — which involves a small but measurable number of cases — has to do with the way the system generates random numbers, which are used to make it practically impossible for an attacker to unscramble digital messages. While it can affect the transactions of individual Internet users, there is nothing an individual can do about it. The operators of large Web sites will need to make changes to ensure the security of their systems, the researchers said.

The potential danger of the flaw is that even though the number of users affected by the flaw may be small, confidence in the security of Web transactions is reduced, the authors said.

The system requires that a user first create and publish the product of two large prime numbers, in addition to another number, to generate a public "key." The original numbers are kept secret. To encrypt a message, a second person employs a formula that contains the public number. In

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Attributing vulnerabilities to implementations

Vast majority of compromised keys generated by headless or embedded network devices.

- Used information in certificate subjects, version strings, served over https or http, etc. to cluster hosts by implementation.
- Routers, firewalls, switches, server management cards, cable modems, VOIP devices, printers, projectors...

Vulnerabilities due mainly to generating keys on first boot with `/dev/urandom`, complicated interaction with application entropy pool behavior.

Disclosure and remediation

- Contacted 61 manufacturers of vulnerable products.
- After 9 months 13 of them have told us they fixed problem.
- 5 released security advisories.

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  Factored 103 Taiwan Citizen Digital Certificates
  (out of 2.26 million):
  smartcard certificates used for paying taxes etc.

  Names, email addresses, national IDs were public
  but **103 private keys** are now known.

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  Smartcard manufacturer:
  "Giesecke & Devrient: Creating Confidence."

Evaluating RSA's risk

Factoring keys is bad, but DSA (and ECDSA) are **worse** if you're worried about entropy problems.

Bad entropy from a single signature can compromise private key.

- e.g. A perfectly good DSA key used on a 2008 Debian system $\rightarrow$ compromised.
- e.g. 1% of DSA SSH host keys compromised from signatures with bad randomness after two scans.

Would be easy to fix in standard. (Make nonce deterministic: hash of message, secret salt.)

Side-channel attacks

Timing attacks

- **Hardware** [Kocher 96] "Timing attacks on implementations of Diffie-Hellman, RSA, DSS, and other systems."
- **Remote software** [Brumley Boneh 05] "Remote timing attacks are practical."

Cache timing

- **Inter-process software** [Percival 05] "Cache missing for fun and profit."
- **Cross-VM software** [Zhang Juels Reiter Ristenpart 12] "Cross-VM Side Channels and Their Use to Extract Private Keys"

Faults

- [Boneh, DeMillo, Lipton 96], [Lenstra 96]

Side-channel attacks

Side-channel structures relevant to RSA:

Exponentiation

- Square-and-multiply: different execution paths/instruction timing/power levels dependent on bits of private key.
- **Defense:** Exponent blinding, square and always multiply, never branch.

CRT coefficients

- Fault attacks can produce a value valid mod only one prime.
- **Defense:** Verify output.

Padding oracles

- Implementations differentiating between correct and incorrect decryption $\rightarrow$ chosen-ciphertext attacks.
- **Defense:** Don't distinguish failures.

Partial key recovery and related attacks

RSA particularly susceptible to partial key recovery attacks.

Theorem (Coppersmith/Howgrave-Graham)

We can find roots x of polynomials f of degree d mod divisors B of N, $B = N^{\beta}$, when $|x| \le N^{\beta^2/d}$.

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- Can factor given $1/2$ bits of d_p. [Blömer May 03]

Also implies constraints on key choice:

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Message security: Least significant bit of message as secure as entire message. [Alexi Chor Goldreich Schnorr 88]

Protocol issues.

Padding schemes: Simple cryptanalyses

Fixed-pattern padding

Define a padding scheme $(P|m)$.
Coppersmith's theorem: With $e = 3$, if $|m| < N^{1/3}$ then can efficiently compute m as solution to

$$c - (P \cdot 2^t + x)^3 \bmod N$$

[Brier Clavier Coron Naccache 01] Existential forgery of signatures with $|m| > N^{1/3}$ by finding solutions to relation

$$(P + m_1)(P + m_2) = (P + m_3)(P + m_4) \bmod N$$

using continued fractions.

The agony and ecstasy of PKCS#1v1.5 and OAEP

PKCS#1: ($0x00\,0x02$|padding string|$0x00$|message)

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2012 **Bardou Focardi Kawamoto Simionato Steel Tsay:** Bleichenbacher attack works against RSA SecureID tokens, Estonian ID cards.

Shoup's "Simple RSA"

$$C_0 = r^e \bmod N \qquad r \text{ random}$$

$$k_0 || k_1 = H(r) \qquad H \text{ hash function}$$

$$C_1 = \mathrm{enc}_{k_0}(m) \qquad \mathrm{enc} \text{ a symmetric cipher}$$

$$T = \mathrm{mac}_{k_1}(C_1)$$

Output (C_0, C_1, T).
Very short and efficient security proof.

Factoring, aka. breaking RSA if nothing else went wrong.

Preliminaries: Using Sage

The following 2 parts use some code snippets to give examples using the free open source mathematics software Sage. http://www.sagemath.org/.

Sage looks like Python

```
sage: 2*3
6
```

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That's it, just `factor(N)`

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```
sage: factor(x^2-1)
(x - 1) * (x + 1)
```

Trial division

Factoring easy-to-factor numbers:

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```

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```

is obviously divisible by 5.

```
sage: N/5        # / is exact division
340282366920938463463374607431768211507
```

Searching for p by trial division takes time about $p/\log(p)$ (number of primes up to p) trial divisions.
Computers can test quickly for divisibility by a precomputed set of primes (using % or gcd with product). Can batch this computation for many moduli N using product and remainder trees.

Pollard rho

Do random walk modulo N, hope for collision modulo factor p.
E.g. using Floyd's cycle finding algorithm

```
N=698599699288686665490308069057420138223871
a=98357389475943875; c=10 # some random values
a1=(a^2+c) % N ; a2=(a1^2+c) % N
while gcd(N,a2-a1)==1:
    a1=(a1^2+c) %N
    a2=(((a2^2+c)%N)^2+c)%N
gcd(N,a2-a1)
```

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gcd(N,a2-a1) # output is 2053
```

Pollard's rho method runs till a prime p divides $a1 - a2$ and N.
By the birthday paradox expect collisions modulo p after $\sqrt{p}$ steps.

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Each step is more expensive than trial division, so don't use this to find 5 but to find 2053.

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N=44426601460658291157725536008128017297890787
4637194279031281180366057
r=lcm(range(1,2^22)) # this takes a while ...
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gcd(s-1,N) # output is 1267650600228229401496703217601
```

This method finds larger factors than the rho method (in the same time) but only works for special primes.

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This method finds larger factors than the rho method (in the same time) but only works for special primes. Here
$p-1 = 2^6 \cdot 3^2 \cdot 5^2 \cdot 17 \cdot 227 \cdot 491 \cdot 991 \cdot 36559 \cdot 308129 \cdot 4161791$
has only small factors (aka. $p-1$ is *smooth*).

Outdated recommendation: avoid such primes, use only "strong primes". ECM (next pages) finds all primes.

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Hasse's theorem: the order of an elliptic curve modulo p is in $[p+1-2\sqrt{p}, p+1+2\sqrt{p}]$. There are lots of smooth numbers in this interval.
Lenstra: Good distribution in the interval.

ECM – Math description

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ECM has the power to change the group; if E_1 does not work, go for $E_2, E_3, \ldots$ till a point has smooth order modulo a p.

EECM: Edwards ECM, Basic version

Use Elliptic curve in twisted Edwards form:
$E : ax^2 + y^2 = 1 + dx^2y^2$ with point $P = (x, y)$; $a, d \neq 0, a \neq d$.
Generate random curve by picking random nonzero a, x, y,
compute $d = (ax^2 + y^2 - 1)/x^2y^2$.

Multiplication in $p - 1$ method replaced by addition on E:

$$(x_1, y_1) + (x_2, y_2) = \left(\frac{x_1y_2 + x_2y_1}{1 + dx_1y_1x_2y_2}, \frac{y_1y_2 - ax_1x_2}{1 - dx_1y_1x_2y_1}\right).$$

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Compute $\gcd(\bar{x}, N)$; this finds primes p for which the order of P modulo p divides r.

ECM: production version

- Use special curves with
  - small coefficients for faster computation, e.g. $(1/23, 1/7)$ is a point on $25x^2 + y^2 = 1 - 24167x^2y^2$;
  - with better chance of smooth orders; this curve has a guaranteed factor of 12.
- Split computation into 2 stages:
  - stage 1 as described before with somewhat smaller t in `r=lcm(range(1,t))`;
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ECM is very efficient at factoring random numbers (once small factors are removed). Favorite method to kill RSA-360.

Factoring bad choices of N

Problem if one takes 'same size' too literally:

$$N = 1000000000000000000000000000000000000000000000000000000\
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0000000000000000000000000000000000000000000000000000000029\
9999999999999999999999999999999999999999999999999999999999\
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In real life would expect this with power of 2 instead of 10.

This problem happens not only for p and q too close to powers of 2 or 10. User starts search for p with some offset c as $p = \texttt{next\_prime}(2^{512} + c)$. Takes $q = \texttt{next\_prime}(p)$.

```
sage: N=11579208923731619542357098500872121122114628
262713908746538761285902758367353
sage: sqrt(N).numerical_approx(256).str(no_sci=2)
'340282366920938463463374607431817146356.999999999999
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340282366920938463463374607431817146421
```

Fermat factorization

We wrote $N = a^2 - b^2 = (a+b)(a-b)$ and factored it using $N/(a-b)$.

```
sage: N=11579208923731619544867939228200664041319989
0130332179010243714077028592474181
sage: sqrt(N).numerical_approx(256).str(no_sci=2)
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sage: sqrt(N).numerical_approx(256).str(no_sci=2)
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This always works eventually: $N = ((q+p)/2)^2 - ((q-p)/2)^2$ but searching for $(q+p)/2$ starting with $\lceil\sqrt{N}\rceil$ will usually run for about $\sqrt{N} \approx p$ steps.

An example of the quadratic sieve (QS)

Let's try Fermat to factor $N = 2759$. Recall idea:
if $a^2 - N$ is a square b^2 then $N = (a-b)(a+b)$.

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But the *product* $50 \cdot 490 \cdot 605$ is a square: $2^2 \cdot 5^4 \cdot 7^2 \cdot 11^2$.
QS computes $\gcd\{2759, 53 \cdot 57 \cdot 58 - \sqrt{50 \cdot 490 \cdot 605}\} = 31$.

Exercise: Square product has 50% chance of factoring pq.

QS more systematically

Try larger N. Easy to generate many differences $a^2 - N$:

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N = 314159265358979323
X = [a^2-N for a in range(sqrt(N)+1,sqrt(N)+500000)]
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Use linear algebra mod 2 to find a square:

```
M = matrix(GF(2),len(F),len(P),lambda i,j:P[j] in F[i][0])
for K in M.left_kernel().basis():
  x = product([sqrt(f[2]+N) for f,k in zip(F,K) if k==1])
  y = sqrt(product([f[2] for f,k in zip(F,K) if k==1]))
  print [gcd(N,x - y),gcd(N,x + y)]
```

Strategies to implement easyfactorizations

Trial-dividing $a^2 - N$ using primes in $[1, y]$ costs $y^{1+o(1)}$.
Four major directions of improvements:

- Early aborts: e.g., throw $a^2 - N$ away if unfactored part is uncomfortably large after primes in $[1, y^{0.5}]$.
  1982 Pomerance: optimized early aborts reduce cost of trial division to $y^{0+o(1)}$ while reducing effectiveness by factor $y^{0.5+o(1)}$.

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- rho, $p - 1$, $p + 1$, ECM. **Low memory, high parallelism.**

Sieving seemed very important 30 years ago. Today much less use: we care more about communication cost and lattice optimization.

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More careful analysis: $\approx 0.306y^2$.

How many integers in $[1, y^u]$ factor into primes in $[1, y]$?
Somewhat careful analysis: $\approx u^{-u}y^u$.
More careful analysis: e.g., $\approx 0.277 \cdot 10^{-10}y^u$ for $u = 10$.

QS scalability

QS is slow for small N ... but scales very well to larger N.

Choose $y = N^{1/u}$.
If differences $a^2 - N$ were random integers mod N
then they would factor into primes in $[1, y]$ with probability $\approx u^{-u}$.
(Actually $a^2 - N$ is closer to $\sqrt{N}$; even more likely to factor.)

Factorization exponent vectors produce linear dependencies
once there are $\approx u^u y / \log y$ differences.

Choose u on scale of $\sqrt{\log N / \log\log N}$
to balance u^u with $N^{1/u}$. Subexponential cost!

History of the world, part 1

1931 Lehmer–Powers, 1975 Morrison–Brillhart, "CFRAC": find small squares mod N using $\sqrt{N}$ continued fraction.

1977 Schroeppel "linear sieve": find square products of $ab(ab - N)$ by sieving $ab - N$; use a, b in small range around $\sqrt{N}$. This uses $\exp(O(\sqrt{\log N \log\log N}))$ operations.

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Applying ECM directly to N also uses $\exp((1 + o(1))\sqrt{\log N \log\log N})$ operations.

History of the world, part 2

1982 Schnorr, 1987 Seysen, 1988 A. Lenstra,
1992 H. Lenstra–Pomerance: another method
that *provably* uses $\exp((1 + o(1))\sqrt{\log N \log\log N})$ operations.

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"Over the last few years there has developed a remarkable six-way tie for the asymptotically fastest factoring algorithms. ... It might be tempting to conjecture that $L(N)$ is in fact the true complexity of factoring, but no one seems to have any idea how to obtain even heuristic lower bounds for factoring."

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1985 Odlyzko, commenting on the same conjecture:
"It is this author's guess that this is not the case, and that we are missing some insight that will let us break below the $L(p)$ barrier."

The number-field sieve (NFS)

1988 Pollard, independently 1989 Elkies,
generalized by 1990 Lenstra–Lenstra–Manasse–Pollard:
Use $(a + b\alpha)(a + bm)$ with $\alpha \equiv m \pmod{n}$.
$\exp((2.08\ldots + o(1))(\log N)^{1/3}(\log\log N)^{2/3})$.

1991 Adleman, 1993 Buhler–Lenstra–Pomerance:
$\exp((1.92\ldots + o(1))(\log N)^{1/3}(\log\log N)^{2/3})$.
Adleman estimated QS/NFS cutoff as $N \approx 2^{1100}$.

1993 Coppersmith:
$\exp((1.90\ldots + o(1))(\log N)^{1/3}(\log\log N)^{2/3})$.

1993 Coppersmith, batch NFS ("factorization factory"):
$\exp((1.63\ldots + o(1))(\log N)^{1/3}(\log\log N)^{2/3})$
after a precomputation independent of N.

So what does this mean for RSA-1024?

Complicated NFS analysis and optimization. Latest estimates:
Attacker breaks my 1024-bit key by scanning $\approx 2^{70}$ pairs (a, b).

Plan A: NSA is building a 2^{26}-watt computer center in Bluffdale.
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2^{26} watts of standard GPUs: 2^{84} floating-point mults/year.
Latest estimates: This is enough to break 1024-bit RSA.
... and special-purpose chips should be at least $10\times$ faster.
... and batch NFS should be even faster.

Quantum computers

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Major impact, part 2: 1994 Shor.
Factors N using *one exponentiation modulo N*.

Post-quantum RSA

Conventional wisdom:
Shor's algorithm supersedes all previous factorization methods.
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Concrete analysis suggests that RSA with 2^{31} 4096-bit primes provides $>2^{100}$ security vs. all known quantum attacks.
Key fits on a hard drive; encryption+decryption take only a week.