Factorization: state of the art

1. Batch NFS
2. Factoring into coprimes
3. ECM

D. J. Bernstein
University of Illinois at Chicago

Tanja Lange
Technische Universiteit Eindhoven

Finding small factors

Find smooth congruences
by finding small factors
of many congruences:

$$\begin{array}{c}\text{Never ending supply}\\ \text{of congruences}\\ \Big\downarrow \text{ select}\\ \text{Smallest congruences}\\ \Big\downarrow \text{ find small factors}\\ \text{Partial factorizations}\\ \text{using primes} \le y\\ \Big\downarrow \text{ abort non-smooth}\\ \text{Smooth congruences}\end{array}$$

How to find small factors?

Could use trial division:
For each congruence,
remove factors of 2,
remove factors of 3,
remove factors of 5,
etc.; use all primes $p \le y$.

$y^{3+o(1)}$ bit operations:
$y^{1+o(1)}$ per congruence.

Want something faster!

Early aborts

$$\begin{array}{c}
\text{Never ending supply} \\
\text{of congruences} \\
\downarrow \text{select} \\
\text{Smallest congruences} \\
\downarrow \\
\text{Partial factorizations} \\
\text{using primes} \le y^{1/2} \\
\downarrow \text{early abort} \\
\text{Smallest unfactored parts} \\
\downarrow \\
\text{Partial factorizations} \\
\text{using primes} \le y \\
\downarrow \text{final abort} \\
\text{Smooth congruences}
\end{array}$$

Find small primes by trial division.
Cost $y^{1/2+o(1)}$ for primes $\le y^{1/2}$.
Cost $y^{1+o(1)}$ for primes $\le y$.

Say we choose "smallest"
so that each congruence
has chance $y^{1/2+o(1)}/y^{1+o(1)}$
of surviving early abort.
Have reduced trial-division
cost by factor $y^{1/2+o(1)}$.

Fact: A y-smooth congruence
has chance $y^{-1/4+o(1)}$
of surviving early abort.
Have reduced identify-a-smooth
cost by factor $y^{1/4+o(1)}$.

Example from Andrew Shallue:
A uniform random integer in $[1, 2^{64} - 1]$ has chance about $2^{-8.1}$ of being 2^{15}-smooth, chance about $2^{-3.5}$ of having 2^7-unfactored part below 2^{44}, and chance about $2^{-9.8}$ of satisfying both conditions.

Given congruence, find primes $\leq 2^7$; abort if unfactored part is above 2^{44}; then find primes $\leq 2^{15}$. Compared to skipping the abort: about $2^{3.5}$ times faster, about $2^{1.7}$ times less productive; gain $2^{1.8}$.

More generally, can abort at $y^{1/k}$, $y^{2/k}$, etc. Balance stages to reduce cost per congruence from $y^{1+o(1)}$ to $y^{1/k+o(1)}$.

Fact: A y-smooth congruence has relatively good chance of surviving early abort.
Have reduced identify-a-smooth cost by factor $y^{(1-1/k)/2+o(1)}$.

Increase k slowly with y.
Find enough smooth congruences using $y^{2.5+o(1)}$ bit operations.

Want something faster!

Sieving

Textbook answer: Sieving finds enough smooth congruences using only $y^{2+o(1)}$ bit operations.

To sieve: Generate in order of p, then sort in order of i, all pairs (i, p) with i in range and $i(n+i) \in p\mathbf{Z}$.

Pairs for one p are (p, p), $(2p, p)$, $(3p, p)$, etc. and $(p - (n \bmod p), p)$ etc.

e.g. $y = 10$, $n = 611$, $i \in \{1, 2, \ldots, 100\}$:

For $p = 2$ generate pairs

$(2, 2), (4, 2), (6, 2), \ldots, (100, 2)$

and

$(1, 2), (3, 2), (5, 2), \ldots, (99, 2)$.

For $p = 3$ generate pairs

$(3, 3), (6, 3), \ldots, (99, 3)$ and

$(1, 3), (4, 3), \ldots, (100, 3)$.

For $p = 5$ generate pairs

$(5, 5), (10, 5), \ldots, (100, 5)$ and

$(4, 5), (9, 5), \ldots, (99, 5)$.

For $p = 7$ generate pairs

$(7, 7), (14, 7), \ldots, (98, 7)$ and

$(5, 7), (12, 7), \ldots, (96, 7)$.

Sort pairs by first coordinate:
$(1, 2)$, $(1, 3)$, $(2, 2)$, $(3, 2)$, $(3, 3)$,
$(4, 2)$, $(4, 3)$, $(4, 5)$, $\ldots$, $(98, 2)$,
$(98, 7)$, $(99, 2)$, $(99, 3)$, $(99, 5)$,
$(100, 2)$, $(100, 3)$, $(100, 5)$.

Sorted list shows that
the small primes in $i(n + i)$ are
$2, 3$ for $i = 1$;
2 for $i = 2$;
$\ldots$
$2, 7$ for $i = 98$;
$2, 3, 5$ for $i = 99$;
$2, 3, 5$ for $i = 100$.

In general, for $i \in \{1, \ldots, y^2\}$:

Prime p produces $\approx y^2/p$ pairs (p, p), $(2p, p)$, $(3p, p)$, etc. and produces $\approx y^2/p$ pairs $(p - (n \bmod p), p)$ etc.

Total number of pairs $\approx$ $\sum_{p \le y} 2y^2/p \approx 2y^2 \log\log y$.

Easily generate pairs, sort, and finish checking smoothness, in $y^2(\lg y)^{O(1)}$ bit operations. Only $(\lg y)^{O(1)}$ bit operations per congruence.

Hidden costs

Is that what we do
in record-setting factorizations?
No!

Sieving has two big problems.

First problem:
Sieving needs large i range.
For speed, must use batch of
$\geq y^{1+o(1)}$ consecutive i's.
Limits number of sublattices,
so limits smoothness chance.

Can eliminate this problem
using remainder trees.

Hidden costs, trees

Second problem with sieving,
not fixed by remainder trees:
Need $y^{1+o(1)}$ bits of storage.

Real machines don't have much
fast memory: it's expensive.

Effect is not visible for
small computations on
single serial CPUs,
but becomes critical in
huge parallel computations.

How to quickly find primes
above size of fast memory?

The rho method

Define $\rho_0 = 0$, $\rho_{k+1} = \rho_k^2 + 11$.

Every prime $\le 2^{20}$ divides $S = (\rho_1 - \rho_2)(\rho_2 - \rho_4)(\rho_3 - \rho_6) \cdots (\rho_{3575} - \rho_{7150})$.
Also many larger primes.

Can compute $\gcd\{c, S\}$ using $\approx 2^{14}$ multiplications mod c, very little memory.

Compare to $\approx 2^{16}$ divisions for trial division up to 2^{20}.

More generally: Choose z.
Compute $\gcd\{c, S\}$ where $S = (\rho_1 - \rho_2)(\rho_2 - \rho_4)\cdots(\rho_z - \rho_{2z})$.

How big does z have to be for all primes $\le y$ to divide S?

Plausible conjecture: $y^{1/2+o(1)}$; so $y^{1/2+o(1)}$ mults mod c.
Early-abort rho: $y^{1/4+o(1)}$ mults.

Reason: Consider first collision in $\rho_1 \bmod p, \rho_2 \bmod p, \ldots$.
If $\rho_i \bmod p = \rho_j \bmod p$ then $\rho_k \bmod p = \rho_{2k} \bmod p$ for $k \in (j-i)\mathbf{Z} \cap [i, \infty] \cap [j, \infty]$.

The $p-1$ method

$S_1 = 2^{232792560} - 1$ has prime divisors
3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 53, 61, 67, 71, 73, 79, 89, 97, 103, 109, 113, 127, 131, 137, 151, 157, 181, 191, 199 etc.

These divisors include
70 of the 168 primes $\leq 10^3$;
156 of the 1229 primes $\leq 10^4$;
296 of the 9592 primes $\leq 10^5$;
470 of the 78498 primes $\leq 10^6$;
etc.

An odd prime p
divides $2^{232792560} - 1$
iff order of 2 in the
multiplicative group $\mathbf{F}_p^*$
divides $s = 232792560$.

Many ways for this to happen:
232792560 has 960 divisors.

Why so many?
Answer: $s = 232792560$
$= \text{lcm}\{1, 2, 3, 4, 5, \ldots, 20\}$
$= 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$

Can compute $2^{232792560} - 1$ using 41 ring operations. (Side note: 41 is not minimal.)

Ring operation: 0, 1, $+$, $-$, $\cdot$.

This computation: 1; $2 = 1 + 1$; $2^2 = 2 \cdot 2$; $2^3 = 2^2 \cdot 2$; $2^6 = 2^3 \cdot 2^3$; $2^{12} = 2^6 \cdot 2^6$; $2^{13} = 2^{12} \cdot 2$; 2^{26}; 2^{27}; 2^{54}; 2^{55}; 2^{110}; 2^{111}; 2^{222}; 2^{444}; 2^{888}; 2^{1776}; 2^{3552}; 2^{7104}; 2^{14208}; 2^{28416}; 2^{28417}; 2^{56834}; 2^{113668}; 2^{227336}; 2^{454672}; 2^{909344}; 2^{909345}; $2^{1818690}$; $2^{1818691}$; $2^{3637382}$; $2^{3637383}$; $2^{7274766}$; $2^{7274767}$; $2^{14549534}$; $2^{14549535}$; $2^{29099070}$; $2^{58198140}$; $2^{116396280}$; $2^{232792560}$; $2^{232792560} - 1$.

Given positive integer n,
can compute $2^{232792560} - 1 \bmod n$
using 41 operations in $\mathbf{Z}/n$.
Notation: $a \bmod b = a - b\lfloor a/b \rfloor$.

e.g. $n = 8597231219$: ...

$$\begin{aligned}
2^{27} \bmod n &= 134217728;\\
2^{54} \bmod n &= 134217728^2 \bmod n\\
&= 935663516;\\
2^{55} \bmod n &= 1871327032;\\
2^{110} \bmod n &= 1871327032^2 \bmod n\\
&= 1458876811; \ldots;
\end{aligned}$$

$2^{232792560} - 1 \bmod n = 5626089344$.

Given positive integer n,
can compute $2^{232792560} - 1 \bmod n$
using 41 operations in $\mathbf{Z}/n$.
Notation: $a \bmod b = a - b\lfloor a/b \rfloor$.

e.g. $n = 8597231219$: ...

$$\begin{aligned}
2^{27} \bmod n &= 134217728;\\
2^{54} \bmod n &= 134217728^2 \bmod n\\
&= 935663516;\\
2^{55} \bmod n &= 1871327032;\\
2^{110} \bmod n &= 1871327032^2 \bmod n\\
&= 1458876811; \ldots;\\
2^{232792560}-1 \bmod n &= 5626089344.
\end{aligned}$$

Easy extra computation (Euclid):
$\gcd\{5626089344, n\} = 991$.

This $p-1$ method (1974 Pollard) quickly factored $n = 8597231219$. Main work: 27 squarings mod n.

Could instead have checked n's divisibility by $2, 3, 5, \ldots$. The 167th trial division would have found divisor 991.

Not clear which method is better. Dividing by small p is faster than squaring mod n. The $p-1$ method finds only 70 of the primes ≤ 1000; trial division finds all 168 primes.

Scale up to larger exponent
$s = \text{lcm}\{1, 2, 3, 4, 5, \ldots, 100\}$:
using 136 squarings mod n
find 2317 of the primes $\leq 10^5$.

Is a squaring mod n
faster than 17 trial divisions?

Or
$s = \text{lcm}\{1, 2, 3, 4, 5, \ldots, 1000\}$:
using 1438 squarings mod n
find 180121 of the primes $\leq 10^7$.

Is a squaring mod n
faster than 125 trial divisions?

Extra benefit:
no need to store the primes.

Plausible conjecture: if K is $\exp\sqrt{\left(\frac{1}{2}+o(1)\right)\log H \log\log H}$ then $p-1$ divides $\operatorname{lcm}\{1,2,\ldots,K\}$ for $H/K^{1+o(1)}$ primes $p \le H$.
Same if $p-1$ is replaced by order of 2 in $\mathbf{F}_p^*$.

So uniform random prime $p \le H$ divides $2^{\operatorname{lcm}\{1,2,\ldots,K\}}-1$ with probability $1/K^{1+o(1)}$.

$(1.4\ldots+o(1))K$ squarings mod n produce $2^{\operatorname{lcm}\{1,2,\ldots,K\}}-1 \bmod n$.

Similar time spent on trial division finds far fewer primes for large H.

Safe primes

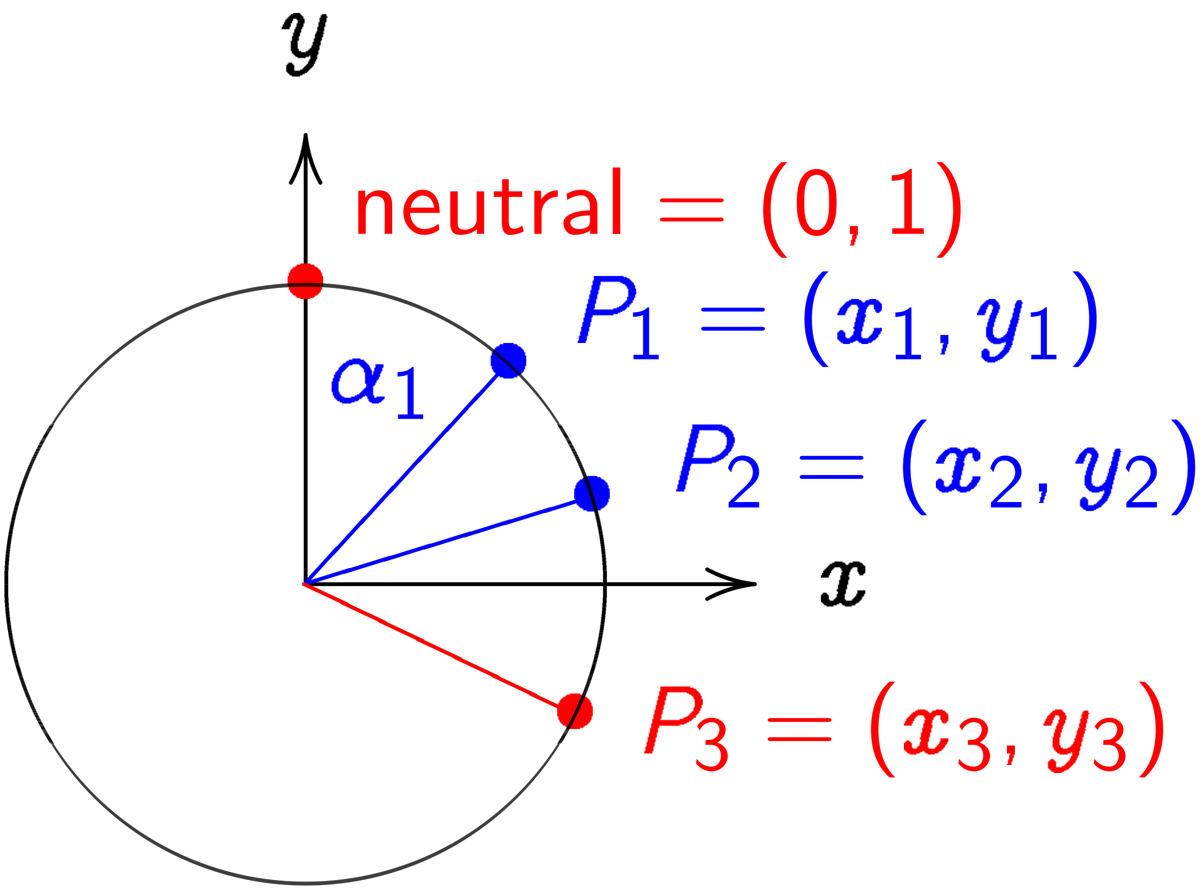
This means numbers are easy to factor if their factors p_i have smooth $p_i - 1$.

To construct hard instances avoid such factors – that's it?

ANSI does recommend using "safe primes", i.e., primes of the form $2p' + 1$ when generating RSA moduli.

This does not help against the NFS nor against the following algorithms.

Interlude: Addition on a clock



$x^2 + y^2 = 1$, parametrized by

$x = \sin\alpha, \quad y = \cos\alpha$.

Sum of (x_1, y_1) and (x_2, y_2) is

$(x_1 y_2 + y_1 x_2, y_1 y_2 - x_1 x_2)$.

Examples of clock addition:

$$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right).$$

$$3\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{117}{125}, \frac{-44}{125}\right).$$

$$4\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{336}{625}, \frac{-527}{625}\right).$$

Many equivalent formulations.

e.g. Clock addition represents multiplication of norm-1 elements of $\mathbf{C} = \mathbf{R}[i]/(i^2 + 1)$.

$(x, y) \mapsto y + ix$;

$(4/5 + 3i/5)^3$

$\quad = -44/125 + 117i/125.$

The $p+1$ factorization method

(1982 Williams)

Define $(X, Y) \in \mathbf{Q} \times \mathbf{Q}$ as the 232792560th multiple of $(3/5, 4/5)$ in the group Clock$(\mathbf{Q})$.

The integer $S_2 = 5^{232792560} X$
is divisible by
82 of the primes $\leq 10^3$;
223 of the primes $\leq 10^4$;
455 of the primes $\leq 10^5$;
720 of the primes $\leq 10^6$;
etc.

Given an integer n,
compute $5^{232792560}X \bmod n$
and compute gcd with n,
hoping to factor n.

Many p's not found by $\mathbf{F}_p^*$
are found by Clock($\mathbf{F}_p$).

If -1 is not a square mod p
and $p+1$ divides 232792560
then $5^{232792560}X \bmod p = 0$.

Proof: $p \equiv 3 \pmod 4$, so
$(4/5 + 3i/5)^p = 4/5 - 3i/5$ and
so $(p+1)(3/5, 4/5) = (0, 1)$
in the group Clock($\mathbf{F}_p$)
so $232792560(3/5, 4/5) = (0, 1)$.

The elliptic-curve method

Fix $a \in \{6, 10, 14, 18, \ldots\}$.

Define $x_1 = 2$, $z_1 = 1$,
$x_{2i} = (x_i^2 - z_i^2)^2$,
$z_{2i} = 4x_i z_i (x_i^2 + a x_i z_i + z_i^2)$,
$x_{2i+1} = 4(x_i x_{i+1} - z_i z_{i+1})^2$,
$z_{2i+1} = 8(x_i z_{i+1} - z_i x_{i+1})^2$.

Define $S_a = z_{\mathrm{lcm}\{1,2,3,\ldots,B_1\}}$.

Have now supplemented S_1, S_2
with S_6, S_{10}, S_{14}, etc.
Variability of a is important.
. . . As many curves as you want!

Point of x_i, z_i formulas:

If $z_i(a^2 - 4)(4a + 10) \notin p\mathbf{Z}$
then ith multiple of $(2, 1)$
on the elliptic curve
$(4a + 10)y^2 = x^3 + ax^2 + x$
over $\mathbf{F}_p$ is $(x_i/z_i, \ldots)$.

If $(a^2 - 4)(4a + 10) \notin p\mathbf{Z}$
and lcm $\in$ (order of $(2, 1)$)$\mathbf{Z}$
then $S_a \in p\mathbf{Z}$.

Order of elliptic-curve group
depends on a but is always
in $[p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$.

e.g. $B_1 = 20$, $a = 10$,
$p = 105239$:

p divides S_{10}.

Have $232792560(2, 1) = \infty$
on the elliptic curve
$50y^2 = x^3 + 10x^2 + x$ over $\mathbf{F}_p$.

In fact, $(2, 1)$ has order
$13167 = 3^2 \cdot 7 \cdot 11 \cdot 19$
on this curve.

Number of $\mathbf{F}_p$-points of curve
is $105336 = 2^3 \cdot 3^2 \cdot 7 \cdot 11 \cdot 19$.

Good news (for the attacker):
All primes $\le H$
seem to be found after a
reasonable number of curves.

Plausible conjecture: if B_1 is
$\exp\sqrt{\left(\frac{1}{2}+o(1)\right)\log H \log\log H}$
then, for each prime $p \le H$,
a uniform random curve mod p
has chance $\ge 1/B_1^{1+o(1)}$ to find p.

If a curve fails, try another.
Find p using, on average,
$\le B_1^{1+o(1)}$ curves;
i.e., $\le B_1^{2+o(1)}$ squarings.
Time subexponential in H.

Overview of ECM

Stage 1: Point P on E over $\mathbf{Z}/n$, compute $R = sP$ for $s = \operatorname{lcm}\{2, 3, \ldots, B_1\}$.

Stage 2: Small primes $B_1 < q_1, \ldots, q_k \leq B_2$ compute $R_i = q_i R$.

If the order of P on the curve modulo p_i divides sq_i, R_i is the neutral element.

Let $\phi(\text{neutral}) = 0$, $\phi(P) \neq 0$. (Example uses Z-coordinate in Montgomery representation.)

Compute $\gcd\{n, \prod \phi(R_i)\}$.

Edwards curves

$x^2 + y^2 = 1 + dx^2y^2$

field k with $2 \neq 0$, $d \notin \{0, 1\}$.

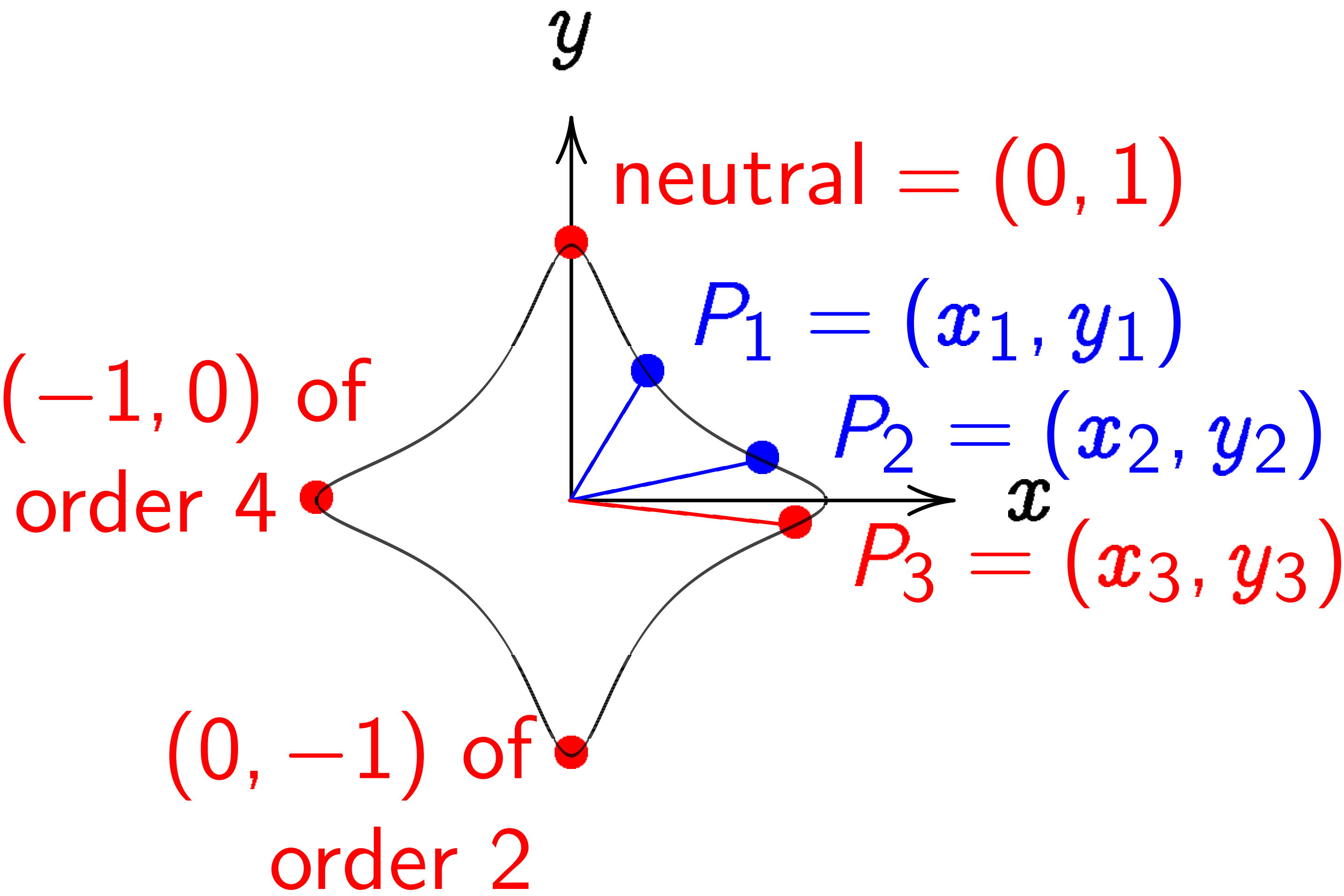
Edwards addition law:

$(x_1, y_1) + (x_2, y_2) =$
$\left(\frac{x_1y_2+y_1x_2}{1+dx_1x_2y_1y_2}, \frac{y_1y_2-x_1x_2}{1-dx_1x_2y_1y_2}\right)$.

Neutral element: $(0, 1)$.
Negation: $-(x_1, y_1) = (-x_1, y_1)$.

Projective point $(X_1 : Y_1 : Z_1)$
represents $(X_1/Z_1, Y_1/Z_1)$.
Addition costs $10\mathbf{M} + 1\mathbf{S} + 1\mathbf{M_d}$.
Doubling costs $3\mathbf{M} + 4\mathbf{S}$.

Example: $x^2 + y^2 = 1 - 30x^2y^2$



Compare to standard Jacobian $V^2 = U^3 - 3UW^4 + bW^6$:

Addition $11\mathbf{M} + 5\mathbf{S}$.

Edwards saves $4\mathbf{S} + 1\mathbf{M} - 1\mathbf{M_d}$.

Doubling $3\mathbf{M} + 5\mathbf{S}$.

Edwards saves $1\mathbf{S}$.

Example: $x^2 + y^2 = 1 - 30x^2y^2$

Compare to standard Jacobin $V^2 = U^3 - 3UW^4 + bW^6$:

Addition $11\mathbf{M} + 5\mathbf{S}$.

Edwards saves $4\mathbf{S} + 1\mathbf{M} - 1\mathbf{M_d}$.

Doubling $3\mathbf{M} + 5\mathbf{S}$.

Edwards saves $1\mathbf{S}$.

Twisted Edwards curves

$ax^2 + y^2 = 1 + dx^2y^2$ with $a \neq 0, d \neq 0, a \neq d$.

(2008 B.–Birkner–Joye–L.–Peters)

Addition law: $(x_1, y_1)+(x_2, y_2) = \left(\frac{x_1y_2+y_1x_2}{1+dx_1x_2y_1y_2}, \frac{y_1y_2-ax_1x_2}{1-dx_1x_2y_1y_2}\right)$.

Projective addition:

$10\mathbf{M} + 1\mathbf{S} + 1\mathbf{M_d} + 1\mathbf{M_a}$.

Projective doubling:

$3\mathbf{M} + 4\mathbf{S} + 1\mathbf{M_a}$.

Advantages of twisted Edwards

- More flexible:
  not necessarily a point of order 4.
- Covers all Montgomery curves.
- Covers even more curves
  by applying a 2-isogeny.
- Saves time when d is
  ratio of small integers.

2008–2010 B.–Birkner–L.–Peters
"ECM using Edwards curves"
(software: "EECM-MPFQ")
save time in ECM by using
(twisted) Edwards curves.

Fewer mulmods per curve

Measurements of EECM-MPFQ for $B_1 = 1000000$:

$b = 1442099$ bits in $s = \text{lcm}\{1, 2, 3, 4, \ldots, B_1\}$.

$P \mapsto sP$ is computed using 1442085 ($= 0.99999b$) DBL + 98341 ($0.06819b$) ADD.

These DBLs and ADDs use 5112988**M** ($3.54552b$**M**) + 5768340**S** ($3.99996b$**S**) + 9635920**add** ($6.68187b$**add**).

Compare to GMP-ECM 6.2.3:

$P \mapsto sP$ is computed using
2001915 ($1.38820b$) DADD $+$
194155 ($0.13463b$) DBL.

These DADDs and DBLs use
8590140**M** ($5.95669b$**M**) $+$
4392140**S** ($3.04566b$**S**) $+$
12788124**add** ($8.86772b$**add**).

Compare to GMP-ECM 6.2.3:

$P \mapsto sP$ is computed using
2001915 (1.388206) DADD +
194155 (0.134636) DBL.

These DADDs and DBLs use
8590140**M** (5.95669b**M**) +
4392140**S** (3.04566b**S**) +
12788124**add** (8.86772b**add**).

Could do better! 0.13463b**M**
are actually 0.13463$b\mathbf{M_d}$.
$\mathbf{M_d}$: mult by curve constant.
Small curve, small P, ladder
$\Rightarrow 4b\mathbf{M} + 4b\mathbf{S} + 2b\mathbf{M_d} + 8b\mathbf{add}$.
EECM still wins.

HECM handles 2 curves using $2b\mathbf{M} + 6b\mathbf{S} + 8b\mathbf{M_d} + \cdots$ (1986 Chudnovsky–Chudnovsky, et al.); again EECM is better.

HECM handles 2 curves using
$2b\mathbf{M} + 6b\mathbf{S} + 8b\mathbf{M_d} + \cdots$
(1986 Chudnovsky–Chudnovsky, et al.); again EECM is better.

What about NFS? $B_1 = 587$?
Measurements of EECM-MPFQ:

$b = 839$ bits in s.

$P \mapsto sP$ is computed using
833 ($0.992285b$) DBL +
131 ($0.156614b$) ADD.

These DBLs and ADDs use
3552$\mathbf{M}$ ($4.23361b\mathbf{M}$) +
3332$\mathbf{S}$ ($3.97139b\mathbf{S}$) +
6308**add** ($7.51847b$**add**).

Note: smaller window size
in addition chain,
so more ADDs per bit.

Compare to GMP-ECM 6.2.3:

Note: smaller window size
in addition chain,
so more ADDs per bit.

Compare to GMP-ECM 6.2.3:

$P \mapsto sP$ is computed using
4785**M** (5.70322b**M**) $+$
2495**S** (2.97378b**S**) $+$
7053**add** (8.40644b**add**).

Even for this small B_1,
EECM beats Montgomery ECM
in operation count.

Notes on current stage 2:

1. EECM-MPFQ jumps through the j's coprime to d_1.
GMP-ECM: coprime to 6.

2. EECM-MPFQ computes Dickson polynomial values using Bos–Coster addition chains.
GMP-ECM: ad-hoc, relying on arithmetic progression of j.

3. EECM-MPFQ doesn't bother converting to affine coordinates until the end of stage 2.

4. EECM-MPFQ uses NTL for poly arith in "big" stage 2.

Faster mulmods

ECM is bottlenecked by mulmods:

- practically all of stage 1;
- curve operations in stage 2 (pumped up by Dickson!);
- final product in stage 2, *except* fast poly arith.

GMP-ECM does mulmods with the GMP library.
. . . but GMP has slow API, so GMP-ECM has $\geq$ 20000 lines of new mulmod code.

```
$ wc -c<eecm-mpfq.tar.bz2
16031
```

Obviously EECM-MPFQ doesn't include new mulmod code!

```
$ wc -c<eecm-mpfq.tar.bz2
16031
```

Obviously EECM-MPFQ doesn't include new mulmod code!

MPFQ library (Gaudry–Thomé) does arithmetic in $\mathbf{Z}/n$ where number of n words is known at compile time.
Better API than GMP: most importantly, n in advance.

EECM-MPFQ uses MPFQ for essentially all mulmods.

GMP-ECM 6.2.3/GMP 4.3.2:

Tried 1000 curves, $B_1 = 2000$,
typical 240-bit n,
on 3.2GHz Phenom II x4.

Stage 1: $7.4 \cdot 10^6$ cycles/curve.

GMP-ECM 6.2.3/GMP 4.3.2:

Tried 1000 curves, $B_1 = 2000$,
typical 240-bit n,
on 3.2GHz Phenom II x4.

Stage 1: $7.4 \cdot 10^6$ cycles/curve.

EECM-MPFQ,
same 240-bit n, same CPU,
1000 curves, $B_1 = 2000$:
$5.2 \cdot 10^6$ cycles/curve.

Some speedup from Edwards;
some speedup from MPFQ.

What about stage 2?

GMP-ECM, 1000 curves,
$B_1 = 587$, $B_2 = 15366$,
Dickson polynomial degree 1:
$6.6 \cdot 10^6$ cycles/curve.
Degree 3: $9.5 \cdot 10^6$.

What about stage 2?

GMP-ECM, 1000 curves,
$B_1 = 587$, $B_2 = 15366$,
Dickson polynomial degree 1:
$6.6 \cdot 10^6$ cycles/curve.
Degree 3: $9.5 \cdot 10^6$.

EECM-MPFQ, 1000 curves,
$B_1 = 587$, $d_1 = 420$, range 20160
for primes $420i \pm j$:
$2.6 \cdot 10^6$ cycles/curve.
Degree 3: $3.1 \cdot 10^6$.

Summary: EECM-MPFQ uses fewer mulmods than GMP-ECM; takes less time than GMP-ECM; and finds more primes.

Summary: EECM-MPFQ uses fewer mulmods than GMP-ECM; takes less time than GMP-ECM; and finds more primes.

Are GMP-ECM and EECM-MPFQ fully exploiting the CPU? No!

Three recent efforts to speed up mulmods for ECM: Thorsten Kleinjung, for RSA-768; Alexander Kruppa, for CADO; and ours—see next slide.

Our latest mulmod speeds,
(working on update)
interleaving vector threads
with integer threads:

4×3GHz Phenom II 940:
$202 \cdot 10^6$ 192-bit mulmods/sec.

4×2.83GHz Core 2 Quad Q9550:
$114 \cdot 10^6$ 192-bit mulmods/sec.

6×3.2GHz Cell (Playstation 3):
$102 \cdot 10^6$ 195-bit mulmods/sec.

\$500 GTX 295
is one card with two GPUs;
60 cores; 480 32-bit ALUs.
Runs at 1.242GHz.

Our latest CUDA-EECM speed:
$481 \cdot 10^6$ 210-bit mulmods/sec.

For $\approx$ \$2000 can build PC
with one CPU and four GPUs:
$1300 \cdot 10^6$ 192-bit mulmods/sec.

Curve details – torsion points

Curve over $\mathbf{Q}$ has some torsion points: points of finite order.
All possible torsion groups (Mazur's theorem):
$\mathbf{Z}/m$ for $m \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12\}$,
$\mathbf{Z}/2 \times \mathbf{Z}/2m$ for $m \in \{1, 2, 3, 4\}$.

If a point has finite order on the curve over $\mathbf{Q}$ then the point has the same finite order over $\mathbf{Z}/n$ and over $\mathbf{F}_p$.

Don't choose P as a torsion point.

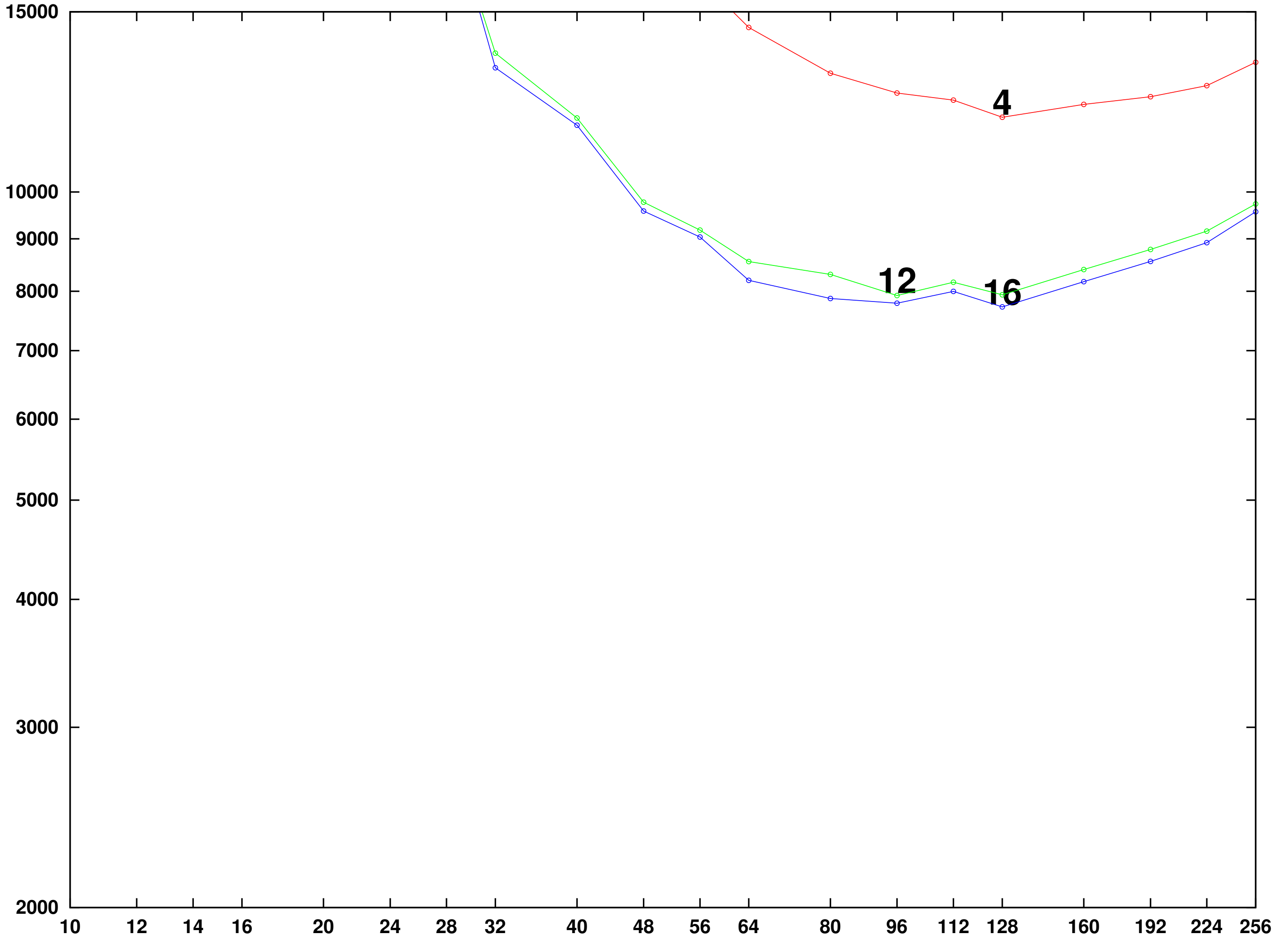
Minimize trouble by choosing curve with torsion $\mathbf{Z}/1$?
No: people try to use curves with many torsion points.

1987/1992 Montgomery,
1993 Atkin–Morain
had suggested using torsion
$\mathbf{Z}/12$ or $\mathbf{Z}/2 \times \mathbf{Z}/8$.

2008–2010 B.–Birkner–L.–Peters construct families of Edwards curves with torsion
$\mathbf{Z}/12$ or $\mathbf{Z}/2 \times \mathbf{Z}/8$.

Impact of large **Q**-torsion

20 bit primes, stage 1 only.
Multiplications per prime found
vs. B_1.



Why people want big torsion

Standard series of heuristic approximations for "random" elliptic curve E:

$\Pr[\text{prime } p \in [1, R] \text{ is found by } E]$

Why people want big torsion

Standard series of heuristic approximations for "random" elliptic curve E:

Pr[prime $p \in [1, R]$ is found by E]
$\overset{?}{\approx}$ Pr[prime $p \in [1, R]$ has smooth $\#\langle P \text{ in } E(\mathbf{F}_p)\rangle$]

Why people want big torsion

Standard series of heuristic approximations for "random" elliptic curve E:

Pr[prime $p \in [1, R]$ is found by E]
$\stackrel{?}{\approx}$ Pr[prime $p \in [1, R]$ has smooth $\#\langle P$ in $E(\mathbf{F}_p)\rangle$]
$\stackrel{?}{\approx}$ Pr[$p \in [1, R]$ has smooth $\#E(\mathbf{F}_p)$]

Why people want big torsion

Standard series of heuristic approximations for “random” elliptic curve E:

$$
\begin{aligned}
&\Pr[\text{prime } p \in [1, R] \text{ is found by } E] \\
&\overset{?}{\approx} \Pr[\text{prime } p \in [1, R] \text{ has smooth } \#\langle P \text{ in } E(\mathbf{F}_p)\rangle] \\
&\overset{?}{\approx} \Pr[p \in [1, R] \text{ has smooth } \#E(\mathbf{F}_p)] \\
&\overset{?}{\approx} \Pr[\text{integer} \in [1, R] \text{ is smooth}].
\end{aligned}
$$

Standard series of
heuristic approximations
when ECM uses a curve E
known to have t torsion points:

Pr[prime $p \in [1, R]$ is found by E]

Standard series of
heuristic approximations
when ECM uses a curve E
known to have t torsion points:

$$\Pr[\text{prime } p \in [1, R] \text{ is found by } E] \stackrel{?}{\approx} \Pr[\text{integer} \in t\mathbf{Z} \cap [1, R] \text{ is smooth}]$$

Standard series of heuristic approximations when ECM uses a curve E known to have t torsion points:

$$
\begin{aligned}
&\Pr[\text{prime } p \in [1, R] \text{ is found by } E] \\
&\overset{?}{\approx} \Pr[\text{integer} \in t\mathbf{Z} \cap [1, R] \\
&\quad \text{is smooth}] \\
&\overset{?}{\approx} \Pr[\text{integer} \in \mathbf{Z} \cap [1, R/t] \\
&\quad \text{is smooth}].
\end{aligned}
$$

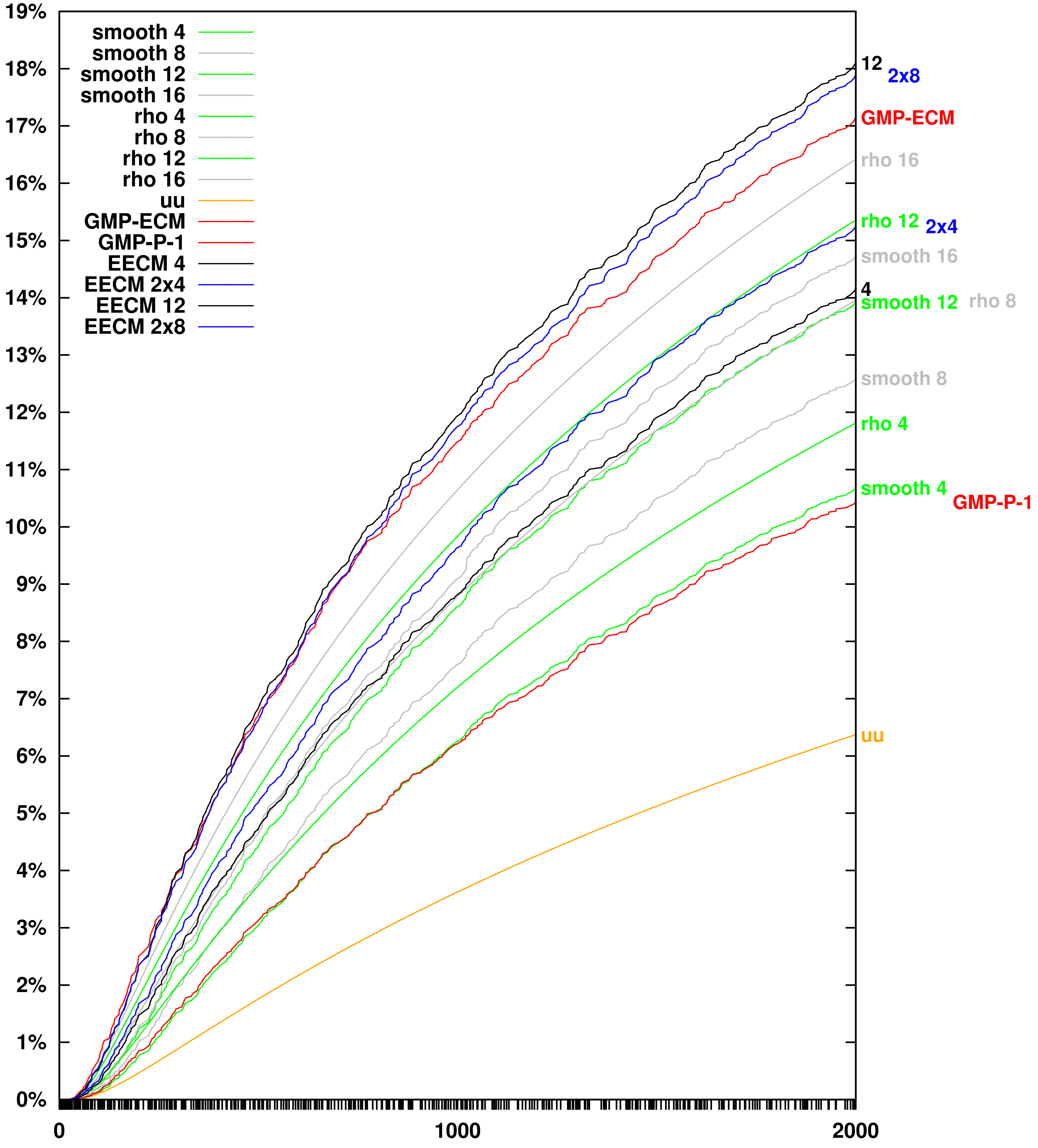
Standard series of heuristic approximations when ECM uses a curve E known to have t torsion points:

Pr[prime $p \in [1, R]$ is found by E]
$\stackrel{?}{\approx}$ Pr[integer $\in t\mathbf{Z} \cap [1, R]$ is smooth]
$\stackrel{?}{\approx}$ Pr[integer $\in \mathbf{Z} \cap [1, R/t]$ is smooth].

Larger $t \Rightarrow$ smaller R/t $\Rightarrow$ larger Pr.

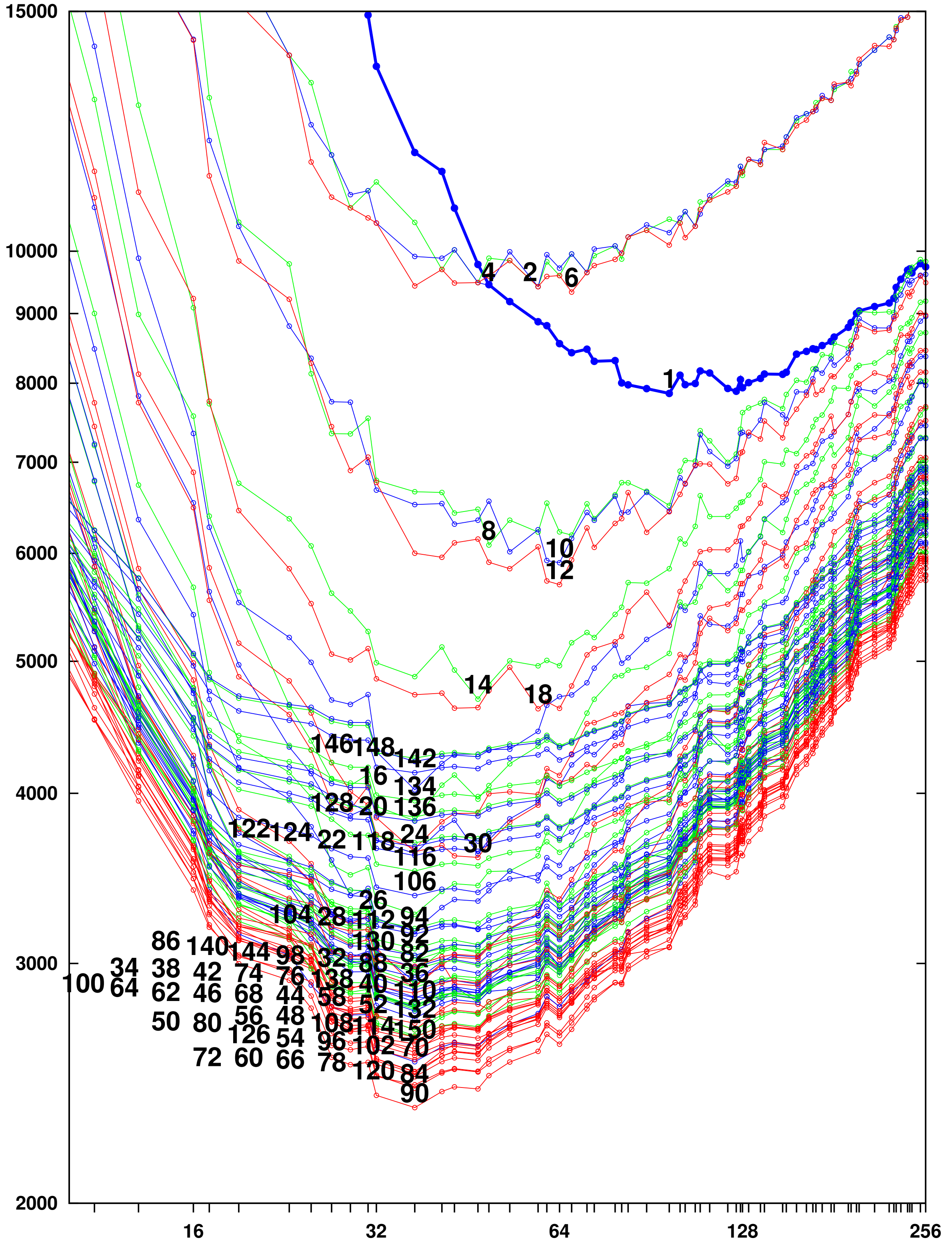
More primes per curve

Probability vs. B_1, 30-bit primes.



Influence of d_1

Multiplications per prime found vs. B_1; different d_1's, same E.



Faster twisted Edwards curves

Dual addition law by Hisil–Wong–Carter–Dawson.

$$\left(\frac{x_1y_1 + x_2y_2}{ax_1x_2 + y_1y_2}, \frac{x_1y_1 - x_2y_2}{x_1y_2 - y_1x_2}\right)$$

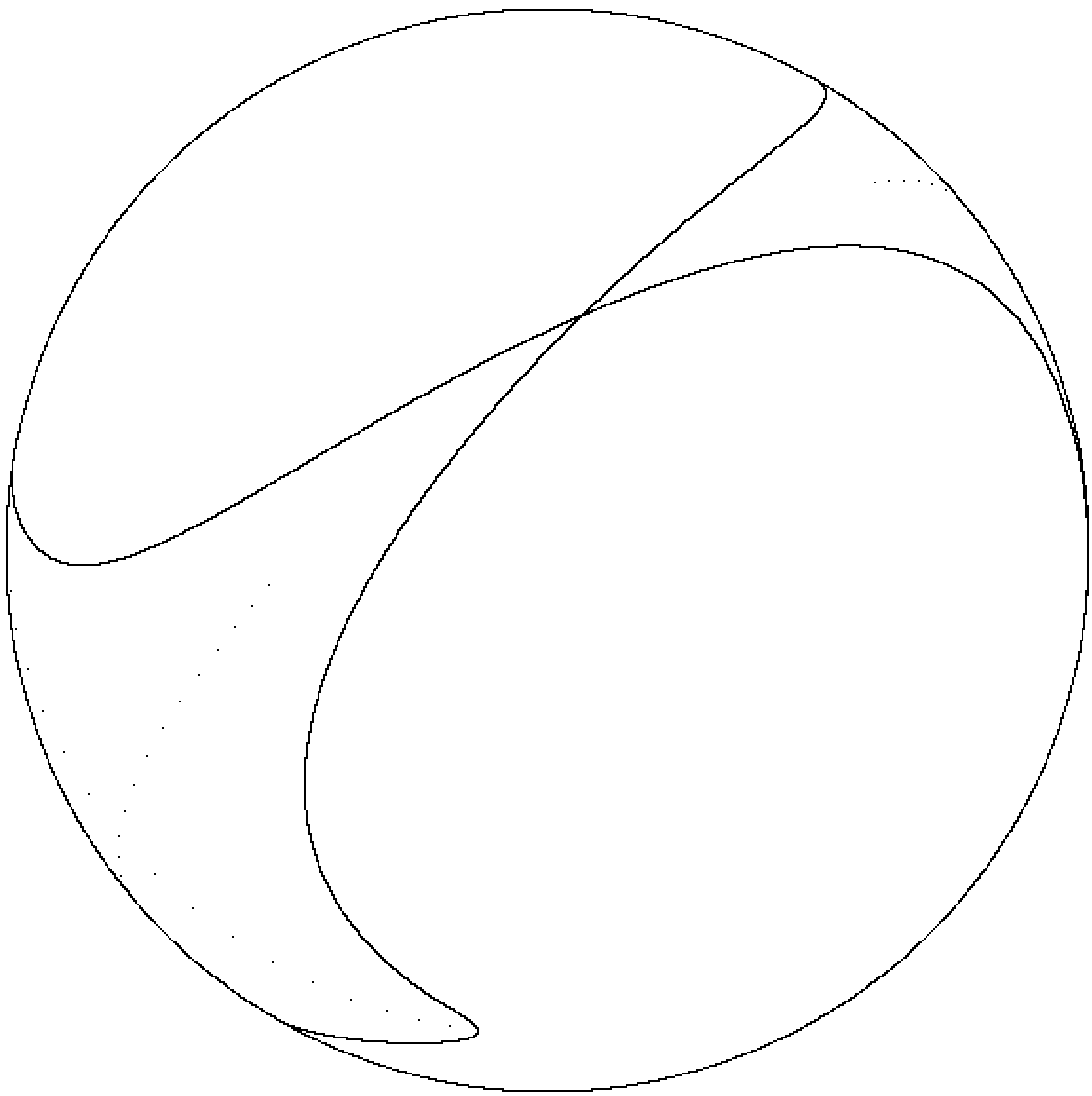
Use extended coordinates $(X : Y : Z : T)$ with $T = XY/Z$; bouncing between projective and extended coordinates.

Addition: $9\mathbf{M} + 1\mathbf{M_a}$.
Only $8\mathbf{M}$ for $a = -1$.
Doubling: $3\mathbf{M} + 4\mathbf{S} + 1\mathbf{M_a}$.

Note the addition speedup for $a = -1$.

Faster ECM?

Let's look closer at
$-x^2 + y^2 = 1 - 30x^2y^2$:

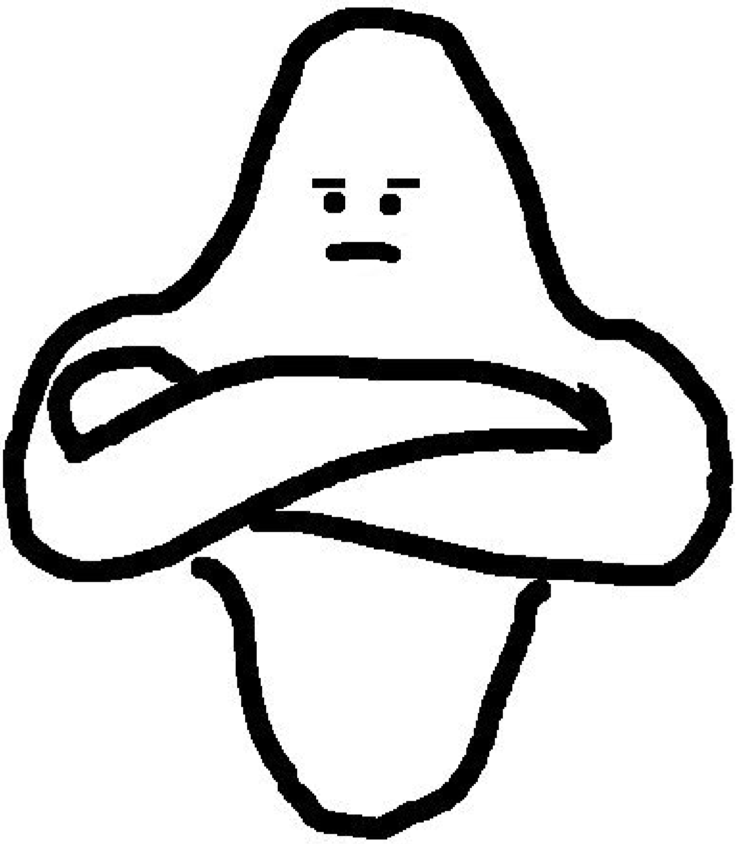


Singularity at infinity blows up to two points of order 2.

EECM paper proved:
arbitrary d with $a = -1$
cannot achieve highest torsion
such as $\mathbf{Z}/12$ and $\mathbf{Z}/2 \times \mathbf{Z}/8$.

Singularity at infinity blows up to two points of order 2.

EECM paper proved: arbitrary d with $a = -1$ cannot achieve highest torsion such as $\mathbf{Z}/12$ and $\mathbf{Z}/2 \times \mathbf{Z}/8$.



“Starfish on strike”

Is the sacrifice in torsion justified by the ADD speedup? Modified EECM-MPFQ to support new curves.

“Starfish on strike”

Is the sacrifice in torsion justified by the ADD speedup? Modified EECM-MPFQ to support new curves.

Happy observation:
Gain in # modular multiplications per curve outweighs loss in # primes found per curve.

“Starfish on strike”

Is the sacrifice in torsion justified by the ADD speedup? Modified EECM-MPFQ to support new curves.

Happy observation:
Gain in # modular multiplications per curve outweighs loss in # primes found per curve.

Surprising phenomenon: $\mathbf{Z}/6$ $-x^2+y^2=1+dx^2y^2$ family finds *more* primes than $\mathbf{Z}/12$.

“Starfish on strike”

Is the sacrifice in torsion justified by the ADD speedup? Modified EECM-MPFQ to support new curves.

Happy observation: Gain in # modular multiplications per curve outweighs loss in # primes found per curve.

Surprising phenomenon: $\mathbf{Z}/6$ $-x^2 + y^2 = 1 + dx^2y^2$ family finds *more* primes than $\mathbf{Z}/12$.

Even more benefit from precomputing best curves.

Number of b-bit primes found by 1000 different curves $-x^2 + \cdots$ with $\mathbf{Z}/2 \times \mathbf{Z}/4$ torsion:

b	20	21	22
curve	$\frac{12}{343}, \frac{1404}{1421}$	$\frac{12}{343}, \frac{1404}{1421}$	$\frac{12}{343}, \frac{1404}{1421}$
#1	15486	22681	46150
curve	$\frac{27}{11}, \frac{5}{13}$	$\frac{27}{11}, \frac{5}{13}$	$\frac{27}{11}, \frac{5}{13}$
#2	14845	21745	43916
curve	$\frac{63}{20}, \frac{1}{244}$	$\frac{3}{14}, \frac{1}{17}$	$\frac{3}{14}, \frac{1}{17}$
#3	14537	21428	43482
#500	13706	19979	40993
#1000	13379	19475	40410

Number of b-bit primes
found by 1000 different curves
$x^2 + \cdots$ with $\mathbf{Z}/12$ torsion:

b	20	21	22
curve #1	. . . 16276	. . . 23991	. . . 48076
curve #2	. . . 16275	. . . 23970	. . . 48028
curve #3	. . . 16273	. . . 23965	. . . 48020
#500	15977	23590	47521
#1000	15313	22714	45987

Number of b-bit primes found by 1000 different curves $-x^2 + \cdots$ with $\mathbf{Z}/6$ torsion:

b	20	21	22
curve	[932]	$\frac{825}{2752}, \frac{1521}{1504}$	$\frac{336}{527}, \frac{80}{67}$
#1	16328	24160	48424
curve	[94]	[982]	$\frac{825}{2752}, \frac{1521}{1504}$
#2	16289	24119	48378
curve	[785]	[265]	[306]
#3	16287	24113	48357
#500	16037	23735	47867
#1000	15399	22790	45828

1. “ECM using Edwards curves.”
Prototype software: GMP-EECM.
New rewrite: EECM-MPFQ.

2. “ECM on graphics cards.”
Prototype CUDA-EECM.

3. “The billion-mulmod-per-second PC.”
Current CUDA-EECM,
plus fast mulmods on
Core 2, Phenom II, and Cell.

4. “Starfish on strike.”
Integrated into EECM-MPFQ.

5. Not covered in this talk:
early-abort ECM optimization.