Discrete-log attacks and factorization Part I

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with some slides by
Daniel J. Bernstein

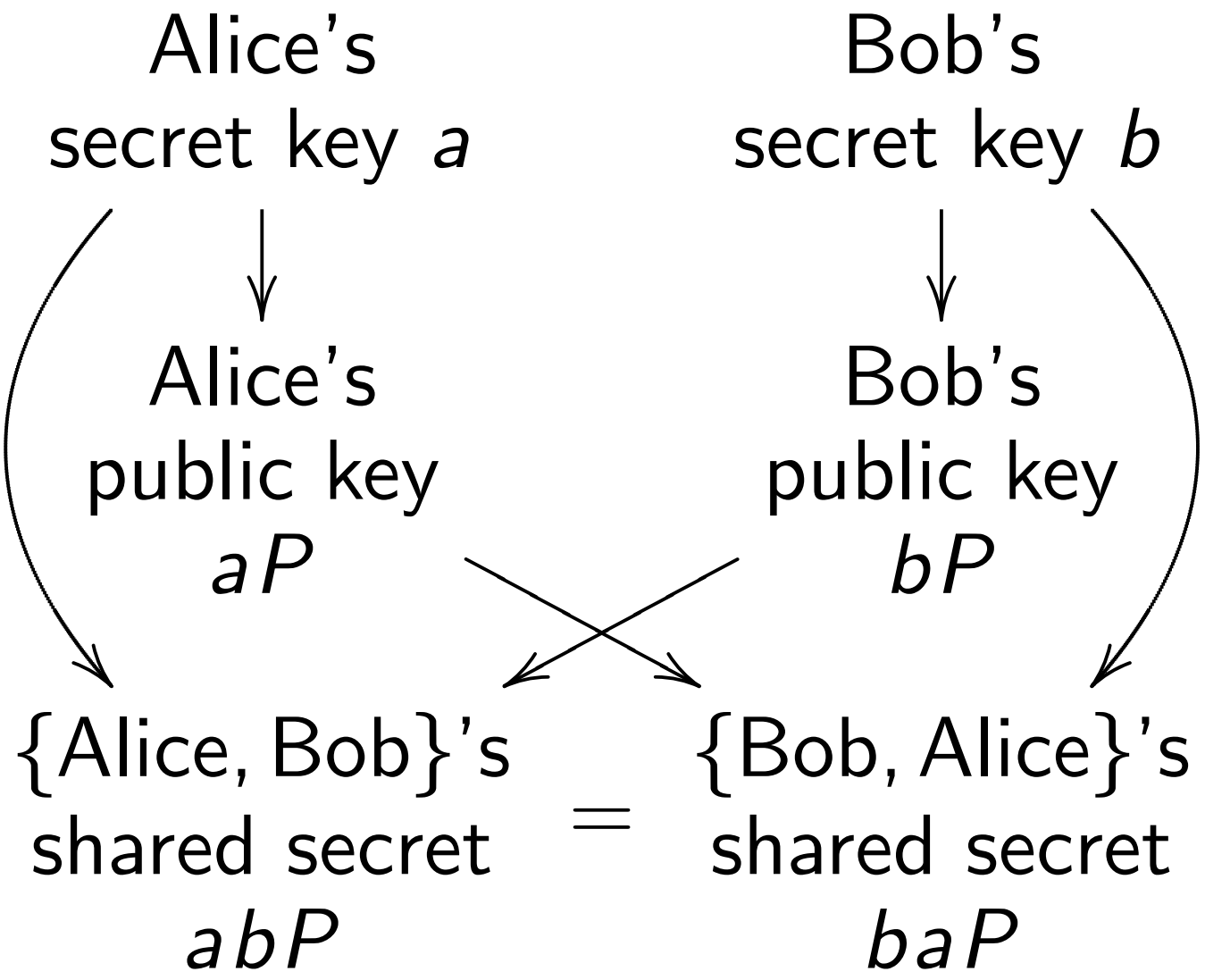
Main goal of this course:
We are the attackers.
We want to break ECC and RSA.

First need to understand ECC.

Main motivation for ECC:
Avoid index-calculus attacks
that plague finite-field DL.

Diffie-Hellman key exchange

Pick some *generator* P,
i.e. some group element
(using additive notation here).



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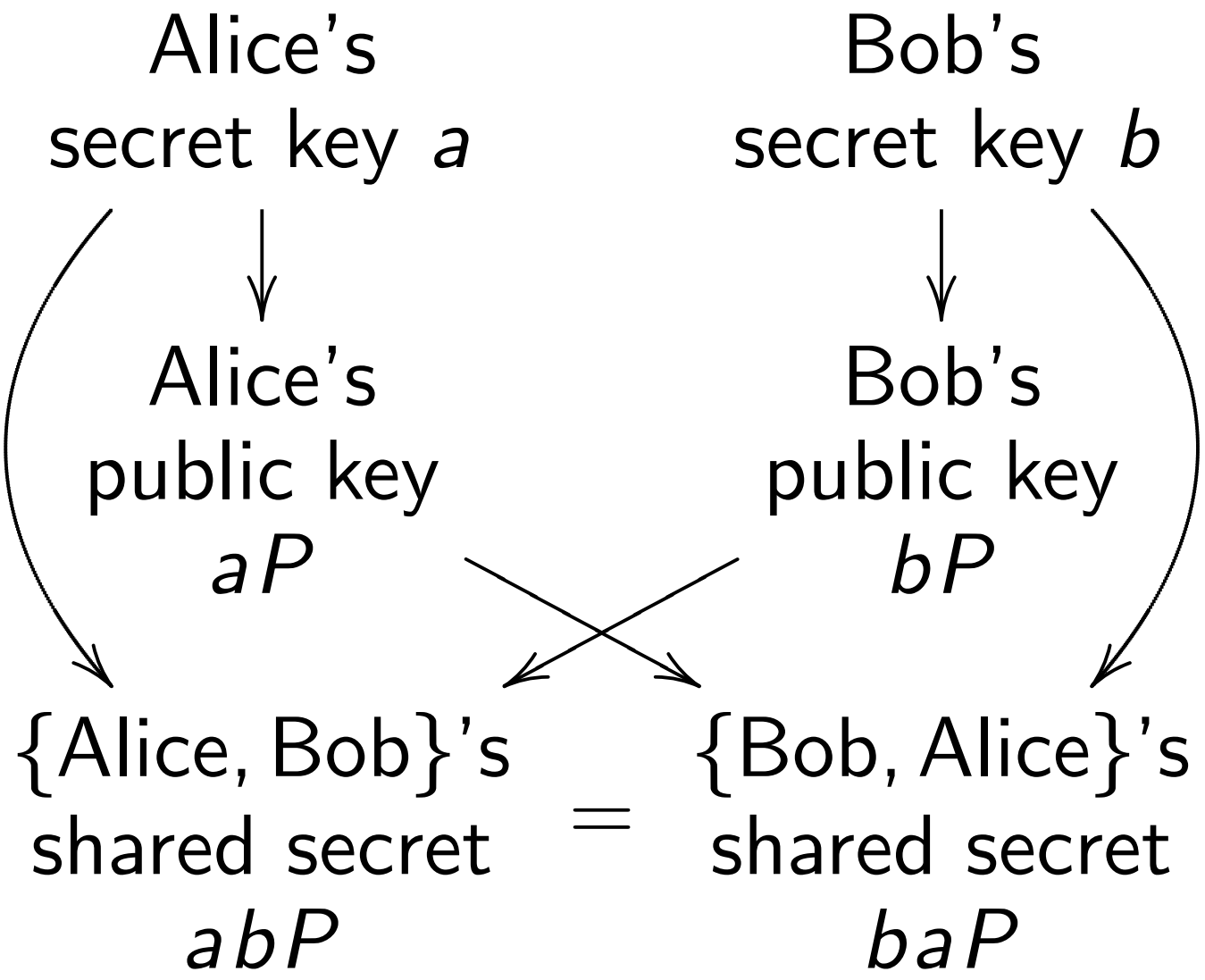
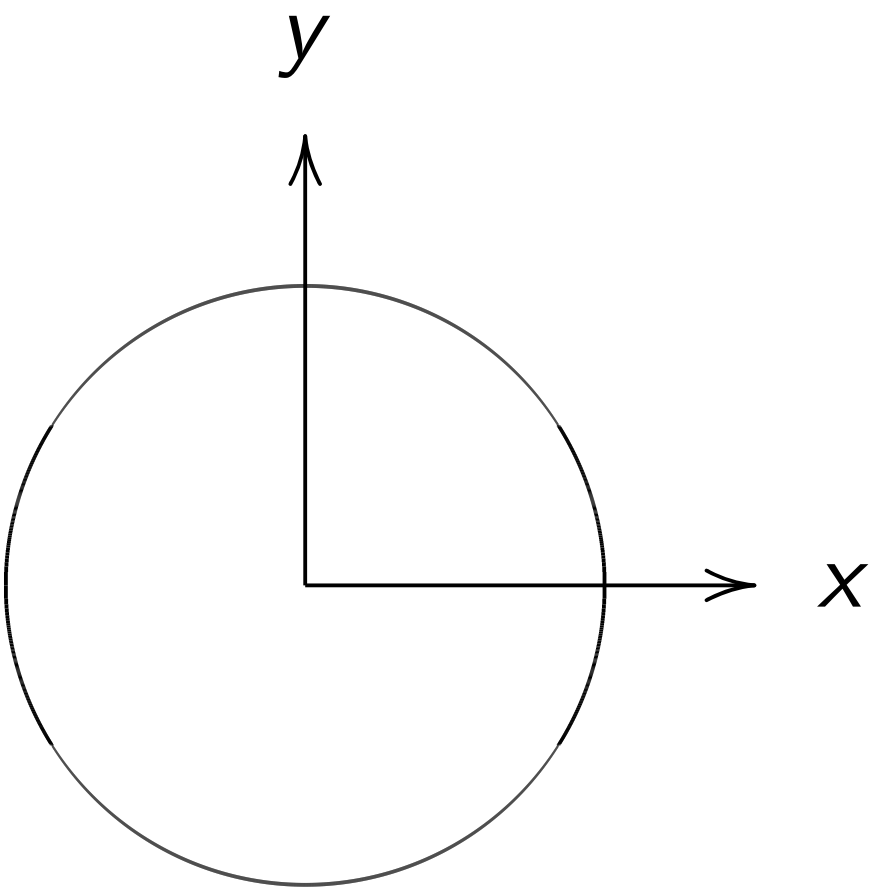


What does P look like &
how to compute $P + Q$?

The clock



This is the curve $x^2 + y^2 = 1$.

Warning:

This is *not* an elliptic curve.

"Elliptic curve" $\neq$ "ellipse."

Examples of points on this curve:

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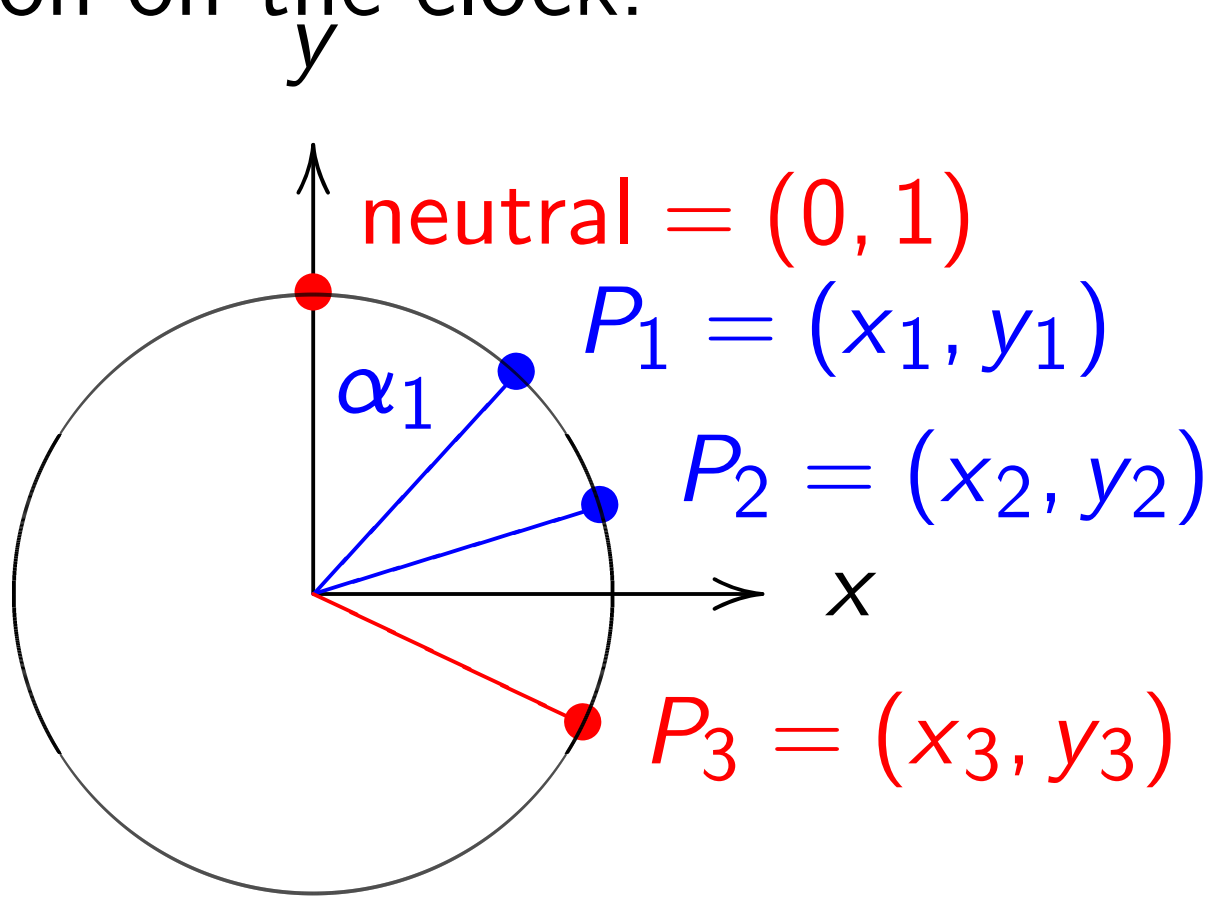
Many more.

Addition on the clock:



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$x = \sin\alpha, \quad y = \cos\alpha.$

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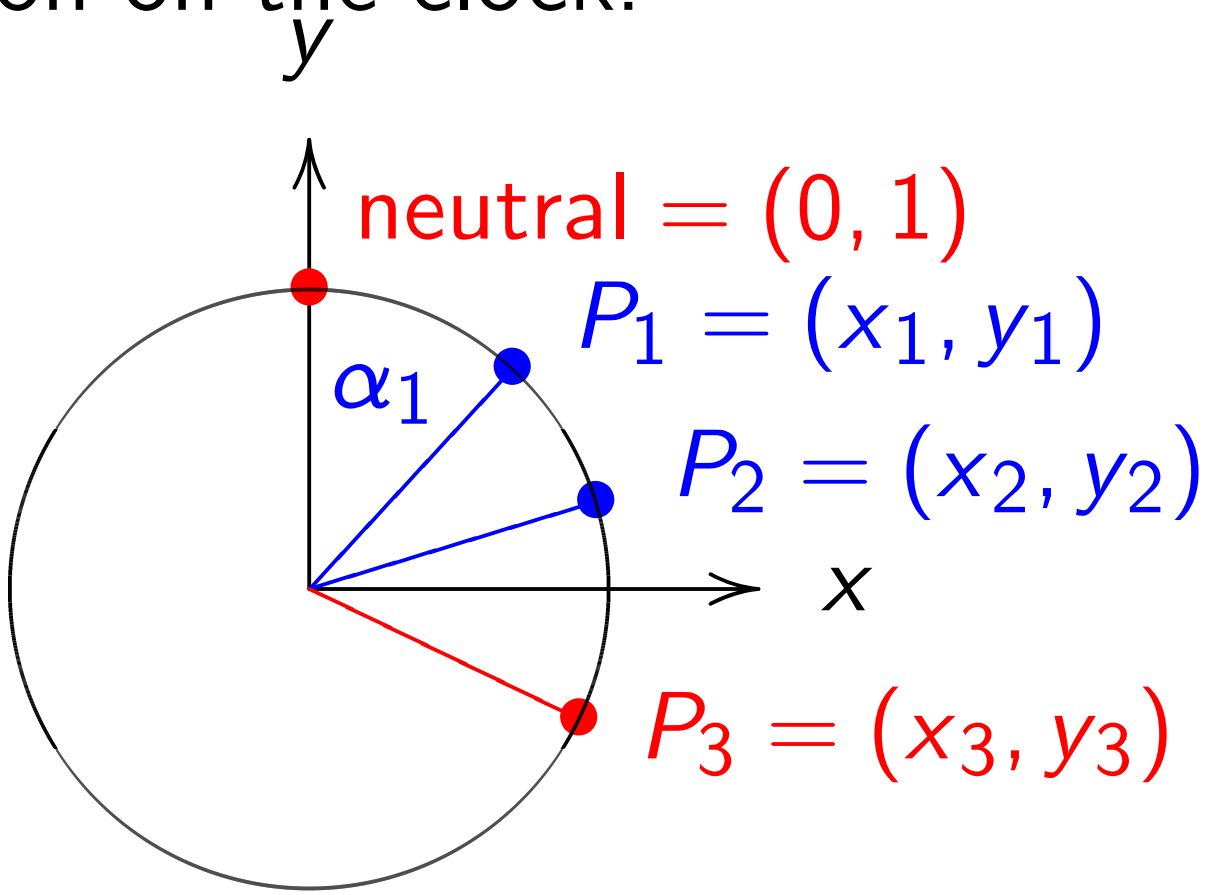

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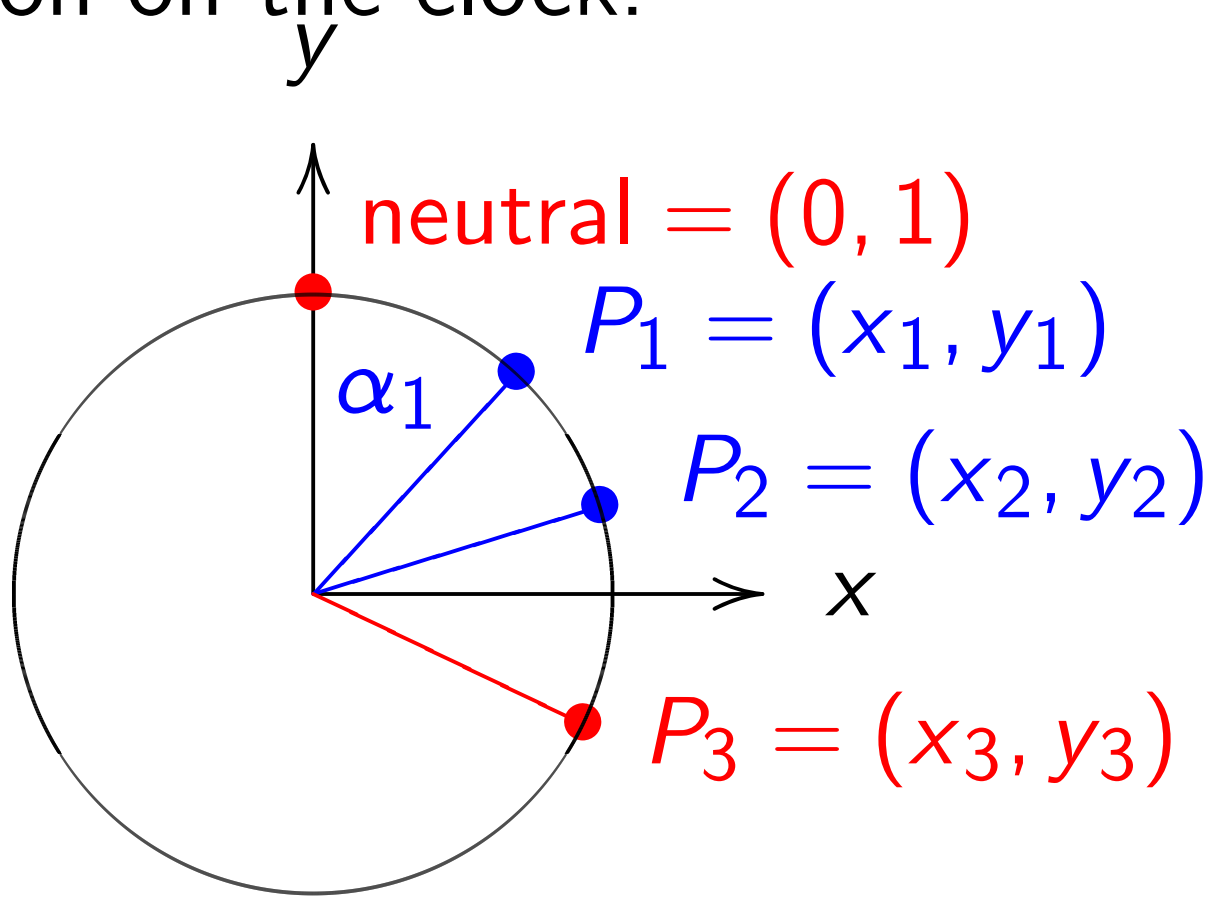

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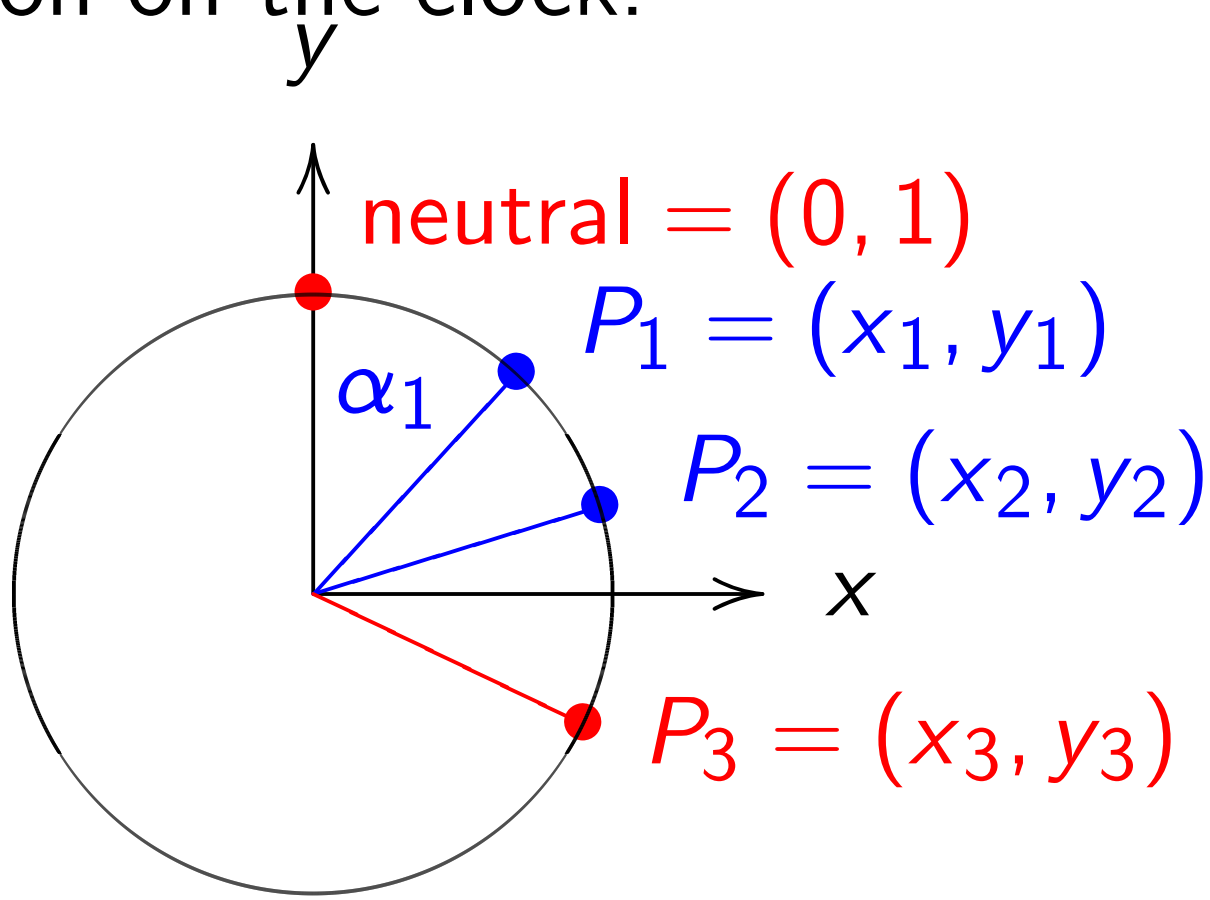

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$$(\sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2,$$
$$\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2).$$

Adding two points corresponds
to adding the angles α_1 and α_2.
Angles modulo 360° are a group,
so points on clock are a group.

Neutral element: angle $\alpha = 0$;
point $(0, 1)$; "12:00".
The point with $\alpha = 180^\circ$
has order 2 and equals 6:00.
3:00 and 9:00 have order 4.
Inverse of point with α
is point with $-\alpha$
since $\alpha + (-\alpha) = 0$.
There are many more points
where angle α is not "nice."

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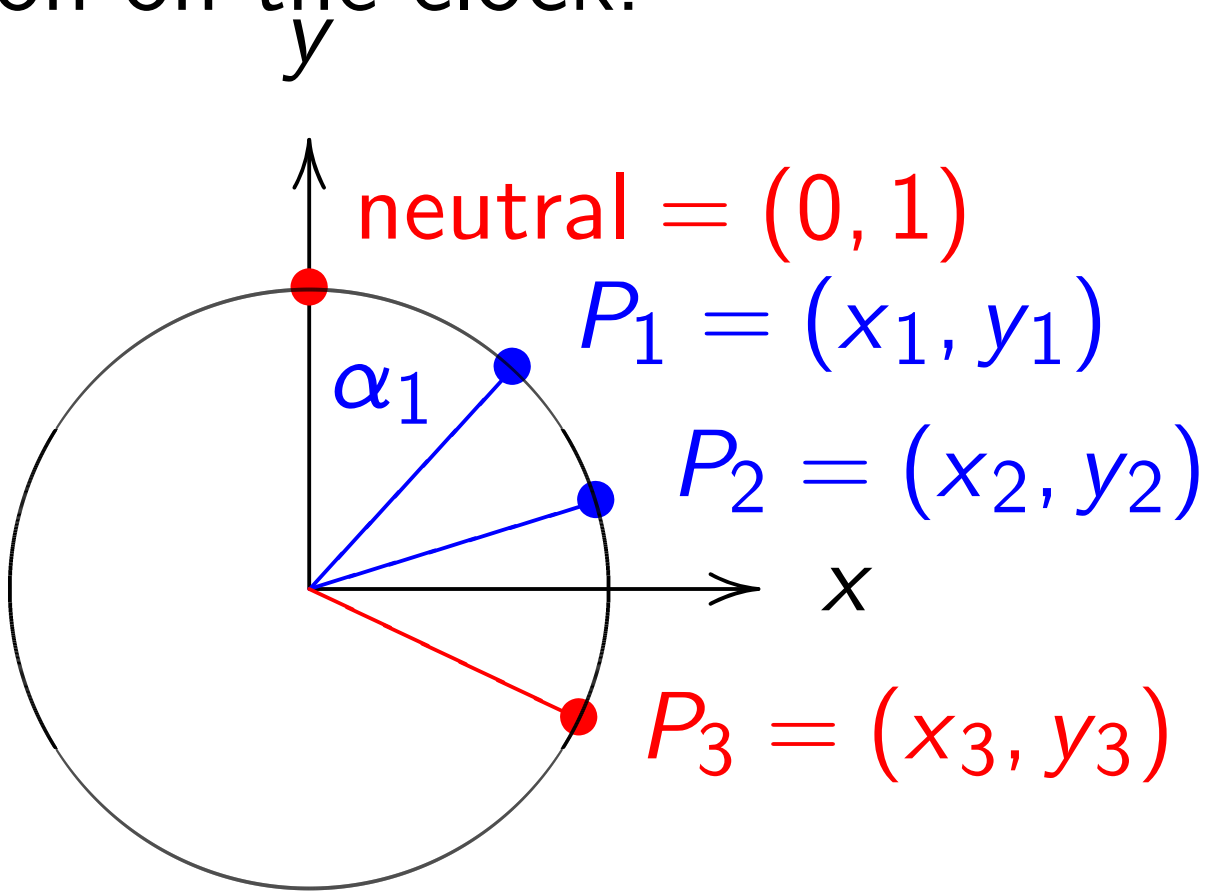

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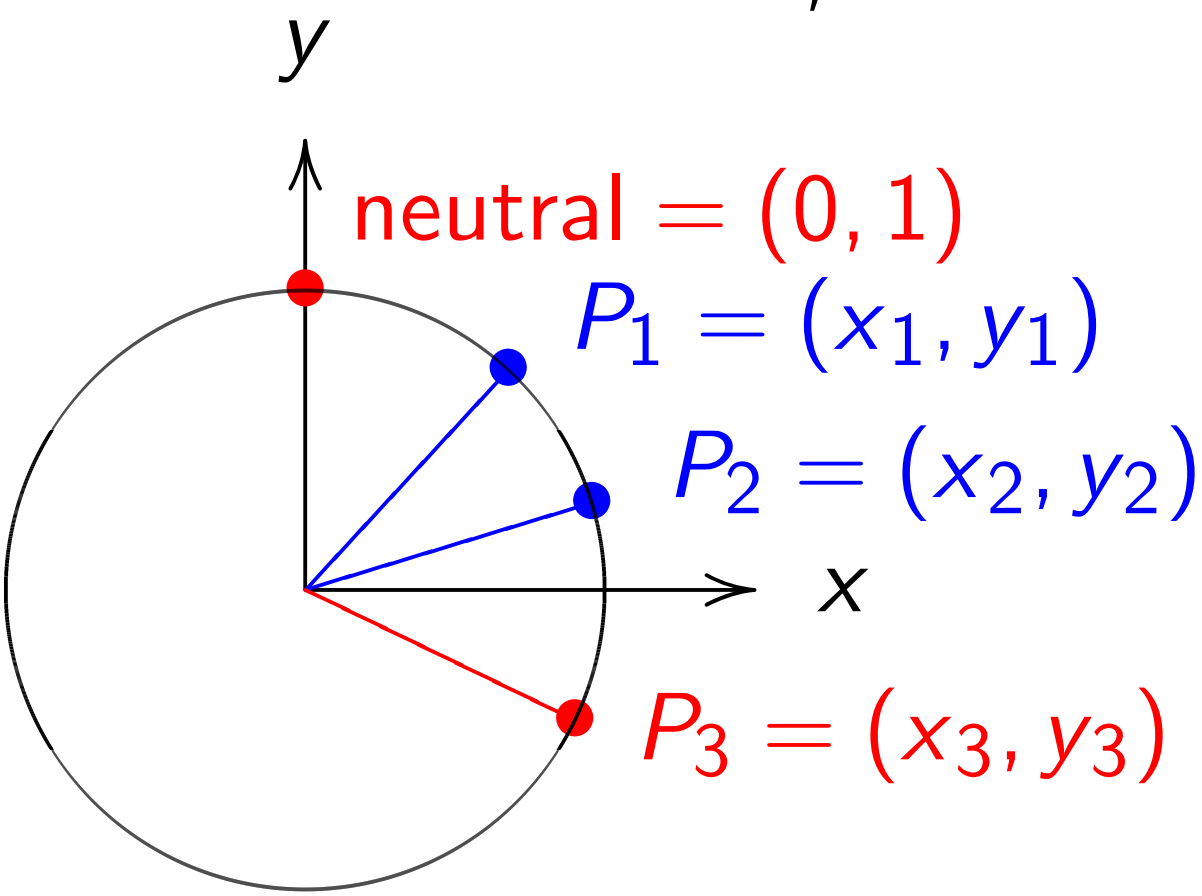
Clock addition without sin, cos:



Use Cartesian coordinates for addition. Addition formula for the clock $x^2 + y^2 = 1$:

sum $(x_1, y_1) + (x_2, y_2) = (x_3, y_3)$

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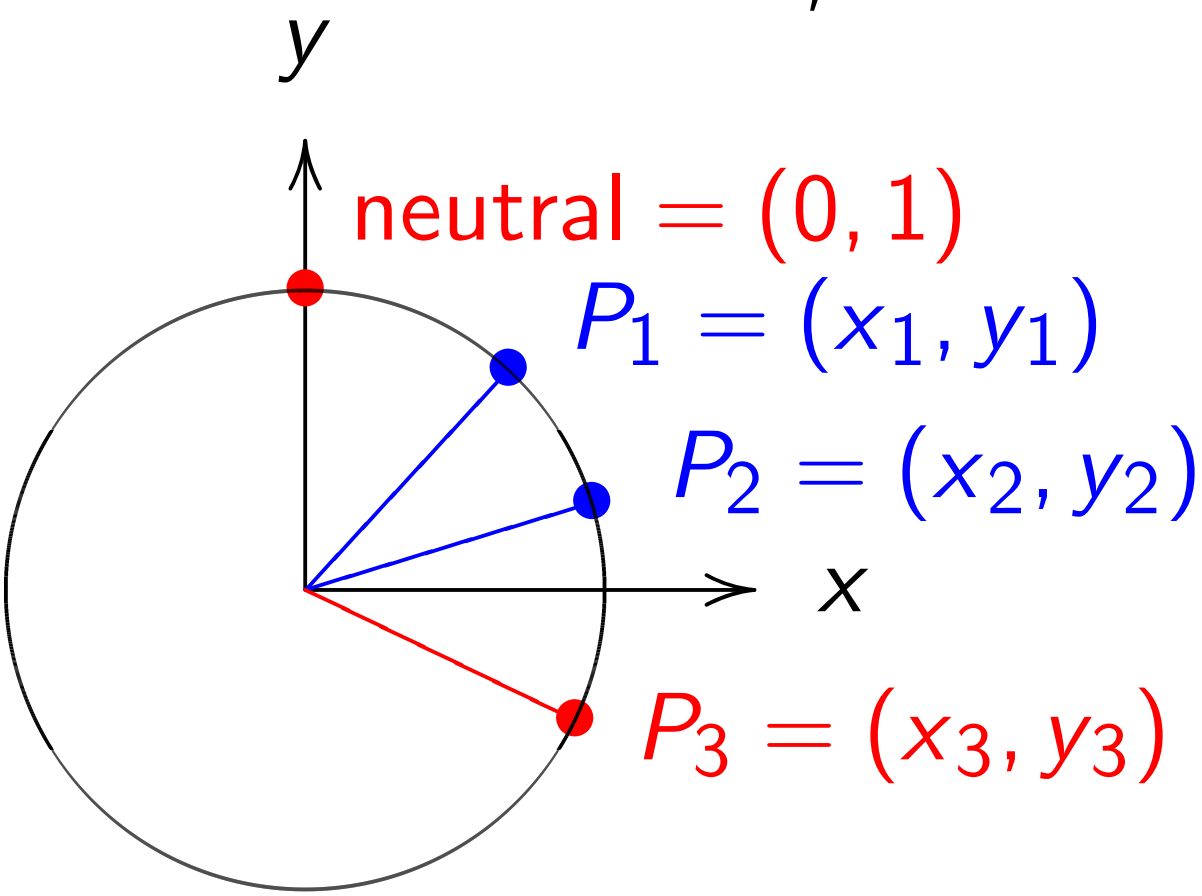

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$= (x_1y_2 + y_1x_2, y_1y_2 - x_1x_2)$.

Note $(x_1, y_1) + (-x_1, y_1) = (0, 1)$.

$kP = \underbrace{P + P + \cdots + P}_{k \text{ copies}}$ for $k \geq 0$.

Examples of clock addition:

$$\text{“2:00”} + \text{“5:00”}$$
$$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3/4})$$
$$= (-1/2, -\sqrt{3/4}) = \text{“7:00”}.$$

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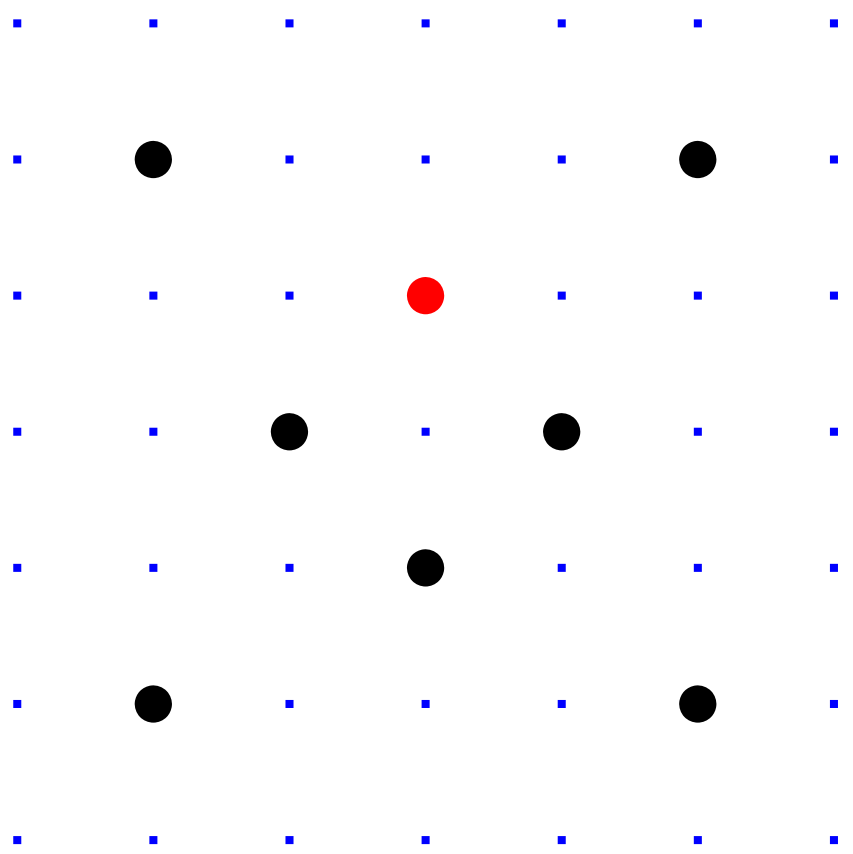
Clocks over finite fields



$\mathrm{Clock}(\mathbf{F}_7) =$

$\{(x, y) \in \mathbf{F}_7 \times \mathbf{F}_7 : x^2 + y^2 = 1\}.$

Here $\mathbf{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

$= \{0, 1, 2, 3, -3, -2, -1\}$

with $+, -, \times$ modulo 7.

E.g. $2 \cdot 5 = 3$ and $3/2 = 5$ in $\mathbf{F}_7$.

```
>>> for x in range(7):
...   for y in range(7):
...     if (x*x+y*y) % 7 == 1:
...       print (x,y)
...
(0, 1)
(0, 6)
(1, 0)
(2, 2)
(2, 5)
(5, 2)
(5, 5)
(6, 0)
>>>
```

```
>>> class F7:
...   def __init__(self,x):
...     self.int = x % 7
...   def __str__(self):
...     return str(self.int)
...   __repr__ = __str__
...
>>> print F7(2)
2
>>> print F7(6)
6
>>> print F7(7)
0
>>> print F7(10)
3
```

```
>>> F7.__eq__ = lambda a,b: \
...    a.int == b.int
>>>
>>> print F7(7) == F7(0)
True
>>> print F7(10) == F7(3)
True
>>> print F7(-3) == F7(4)
True
>>> print F7(0) == F7(1)
False
>>> print F7(0) == F7(2)
False
>>> print F7(0) == F7(3)
False
```

```
>>> F7.__add__ = lambda a,b: \
...   F7(a.int + b.int)
>>> F7.__sub__ = lambda a,b: \
...   F7(a.int - b.int)
>>> F7.__mul__ = lambda a,b: \
...   F7(a.int * b.int)
>>>
>>> print F7(2) + F7(5)
0
>>> print F7(2) - F7(5)
4
>>> print F7(2) * F7(5)
3
>>>
```

Larger example: Clock($\mathbf{F}_{1000003}$).

```
p = 1000003

class Fp:

  ...


def clockadd(P1,P2):

  x1,y1 = P1

  x2,y2 = P2

  x3 = x1*y2+y1*x2

  y3 = y1*y2-x1*x2

  return x3,y3
```

```
>>> P = (Fp(1000),Fp(2))
>>> P2 = clockadd(P,P)
>>> print P2
(4000, 7)
>>> P3 = clockadd(P2,P)
>>> print P3
(15000, 26)
>>> P4 = clockadd(P3,P)
>>> P5 = clockadd(P4,P)
>>> P6 = clockadd(P5,P)
>>> print P6
(780000, 1351)
>>> print clockadd(P3,P3)
(780000, 1351)
>>>
```

```
>>> def scalarmult(n,P):
...     if n == 0: \
...     return (Fp(0),Fp(1))
...     if n == 1: return P
...     Q = scalarmult(n//2,P)
...     Q = clockadd(Q,Q)
...     if n % 2: Q = clockadd(P,Q)
...     return Q
...
>>> n = oursixdigitsecret
>>> scalarmult(n,P)
(947472, 736284)
>>>
```

Can you figure out our secret n?

Clock cryptography

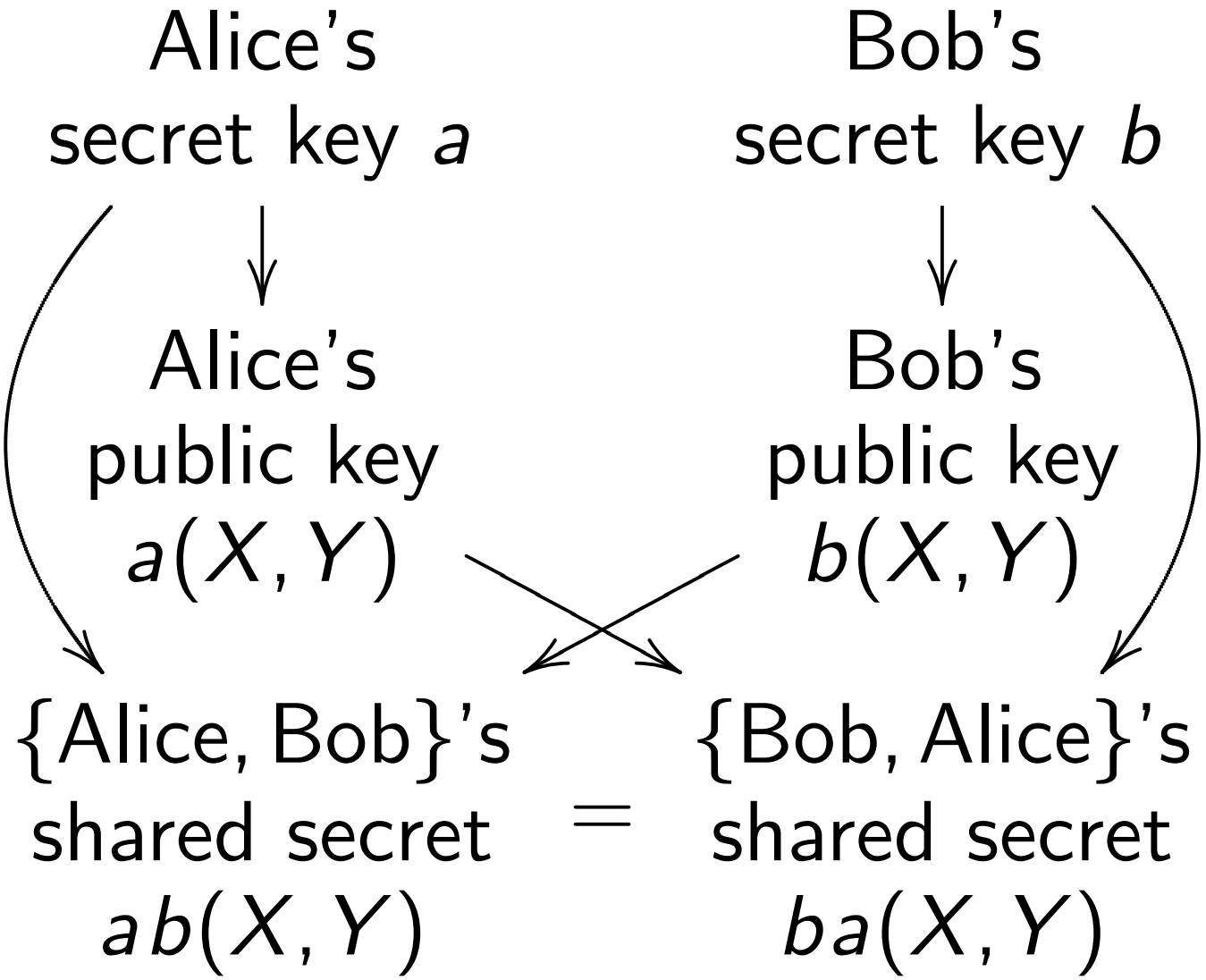
The "Clock Diffie–Hellman protocol":

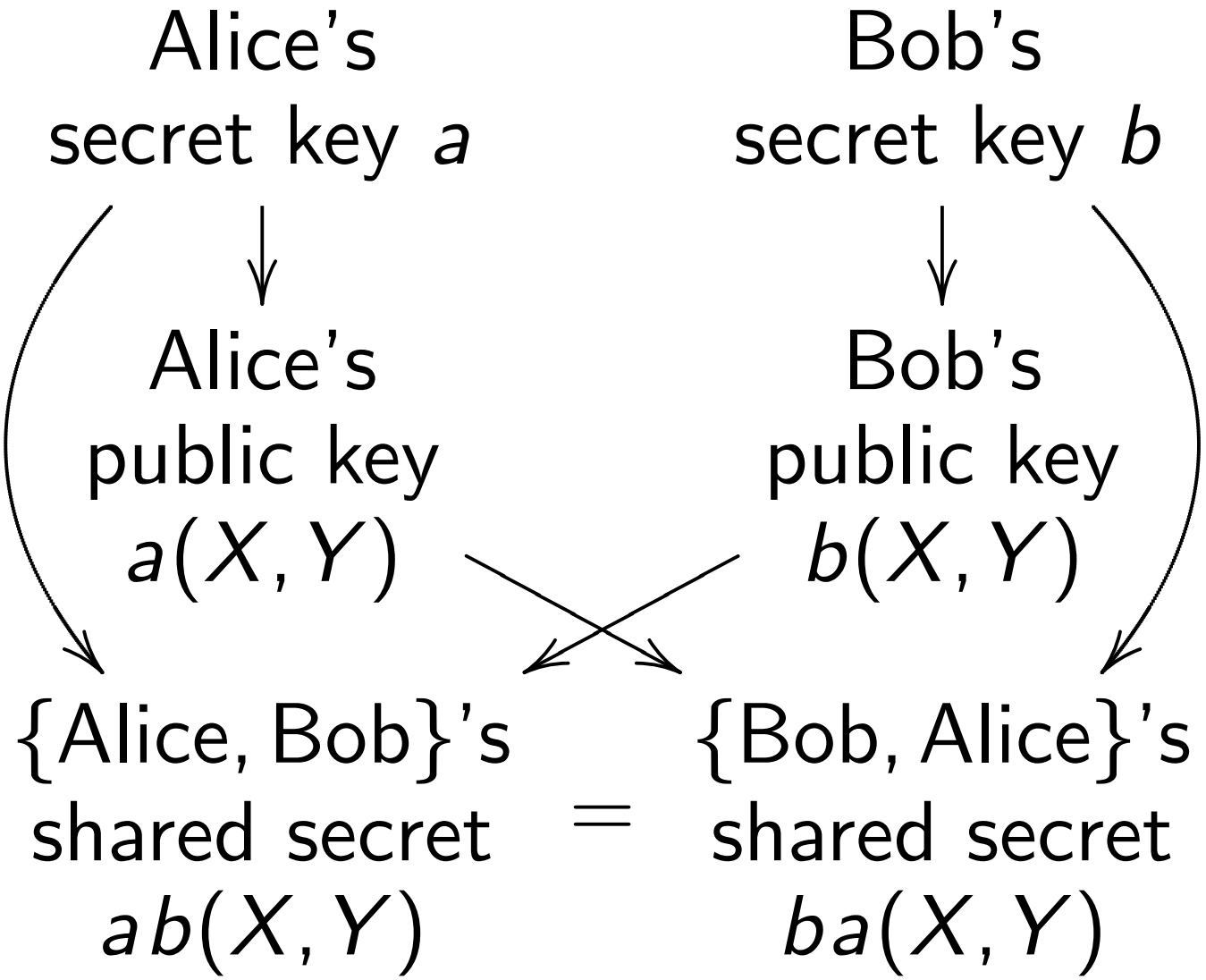
Standardize large prime p & **base point** $(x, y) \in \mathrm{Clock}(\mathbf{F}_p)$.

Alice chooses big secret a, computes her public key $a(x, y)$.

Bob chooses big secret b, computes his public key $b(x, y)$.

Alice computes $a(b(x, y))$.
Bob computes $b(a(x, y))$.
They use this shared secret to encrypt with AES-GCM etc.


Alice's
secret key a
Bob's
secret key b
Alice's
public key
$a(X, Y)$
Bob's
public key
$b(X, Y)$
{Alice, Bob}'s
shared secret
$ab(X, Y)$
=
{Bob, Alice}'s
shared secret
$ba(X, Y)$



Warning #1:

Many p are unsafe!

Warning #2:

Clocks aren't elliptic!

To match RSA-3072 security need $p \approx 2^{1536}$.

Warning #3:
Attacker sees more than
public keys $a(x, y)$ and $b(x, y)$.

Attacker sees how much *time*
Alice uses to compute $a(b(x, y))$.
Often attacker can see time for
each operation performed by
Alice, not just total time.
This reveals secret scalar a.

Break by timing attacks, e.g.,
2011 Brumley–Tuveri.

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Fix: **constant-time** code,
performing same operations
no matter what scalar is.

Exercise

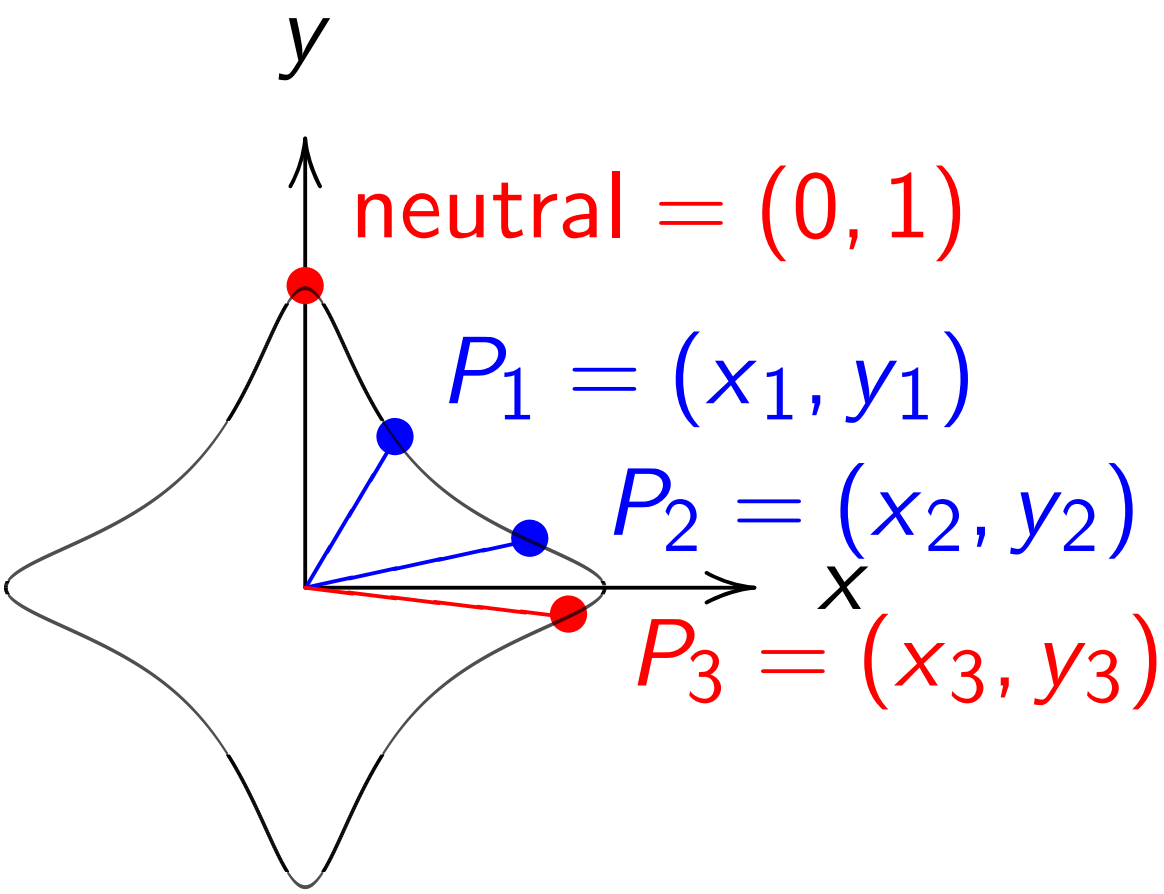
How many multiplications
do you need to compute
$(x_1y_2 + y_1x_2, y_1y_2 - x_1x_2)$?

How many multiplications
do you need to double a point,
i.e. to compute
$(x_1y_1 + y_1x_1, y_1y_1 - x_1x_1)$?

How can you optimize the
computation if squarings are
cheaper than multiplications?
Assume $\mathbf{S} < \mathbf{M} < 2\mathbf{S}$.

Addition on an Edwards curve

Change the curve on which Alice and Bob work.

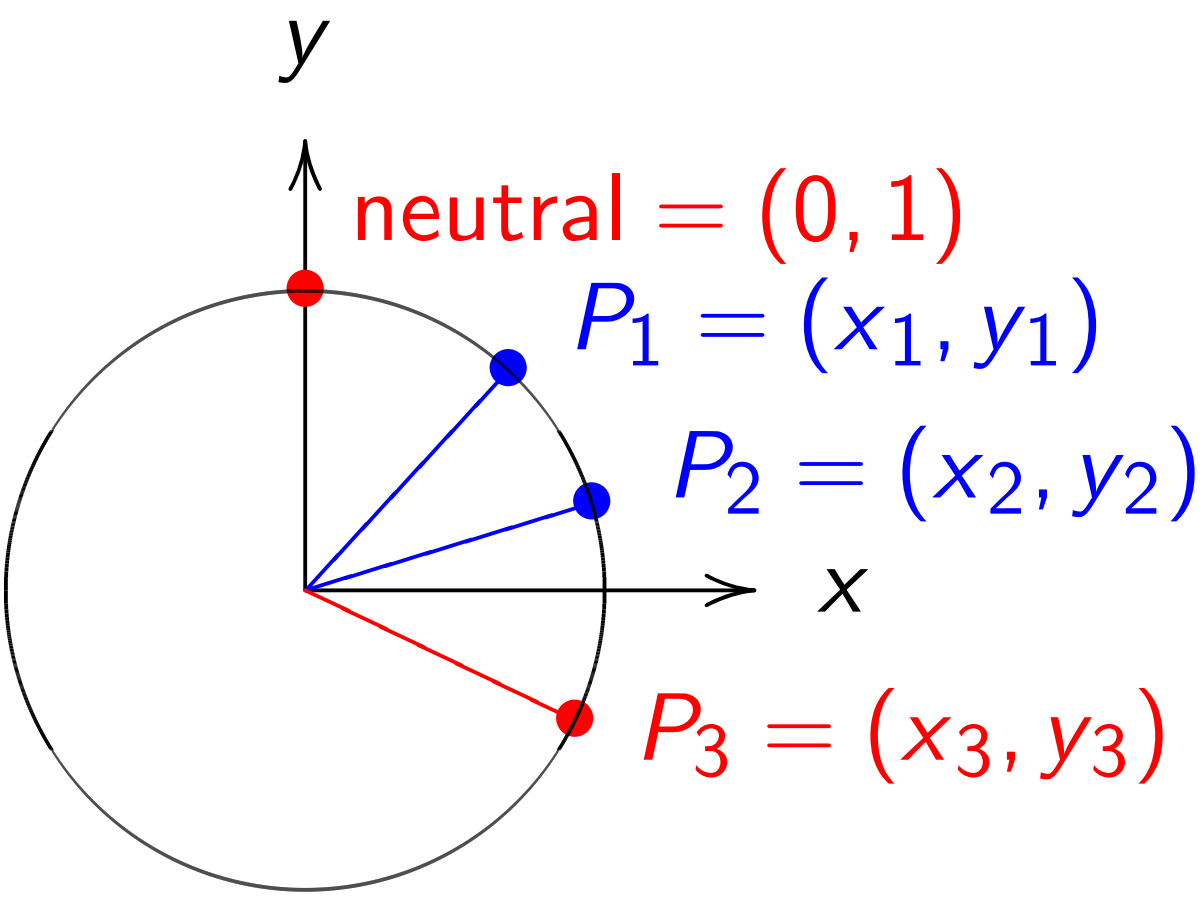


$x^2 + y^2 = 1 - 30x^2y^2$.

Sum of (x_1, y_1) and (x_2, y_2) is

$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2),$
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$.

The clock again, for comparison:



$x^2 + y^2 = 1$.

Sum of (x_1, y_1) and (x_2, y_2) is

$(x_1y_2 + y_1x_2,$

$y_1y_2 - x_1x_2)$.

"Hey, there were divisions
in the Edwards addition law!
What if the denominators are 0?"
Answer: They aren't!
If $x_i = 0$ or $y_i = 0$ then
$1 \pm 30x_1x_2y_1y_2 = 1 \neq 0$.
If $x^2 + y^2 = 1 - 30x^2y^2$
then $30x^2y^2 < 1$
so $\sqrt{30}\ |xy| < 1$.

"Hey, there were divisions
in the Edwards addition law!
What if the denominators are 0?"
Answer: They aren't!
If $x_i = 0$ or $y_i = 0$ then
$1 \pm 30x_1x_2y_1y_2 = 1 \neq 0$.
If $x^2 + y^2 = 1 - 30x^2y^2$
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so $\sqrt{30}\,|xy| < 1$.

If $x_1^2 + y_1^2 = 1 - 30x_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 - 30x_2^2y_2^2$
then $\sqrt{30}\,|x_1y_1| < 1$
and $\sqrt{30}\,|x_2y_2| < 1$

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If $x_1^2 + y_1^2 = 1 - 30x_1^2y_1^2$
and $x_2^2 + y_2^2 = 1 - 30x_2^2y_2^2$
then $\sqrt{30}\,|x_1y_1| < 1$
and $\sqrt{30}\,|x_2y_2| < 1$
so $30\,|x_1y_1x_2y_2| < 1$
so $1 \pm 30x_1x_2y_1y_2 > 0$.

The Edwards addition law
$(x_1, y_1) + (x_2, y_2) =$
$((x_1y_2+y_1x_2)/(1-30x_1x_2y_1y_2),$
$(y_1y_2-x_1x_2)/(1+30x_1x_2y_1y_2))$
is a group law for the curve
$x^2 + y^2 = 1 - 30x^2y^2$.

Some calculation required:

addition result is on curve;

addition law is associative.

Other parts of proof are easy:

addition law is commutative;

$(0, 1)$ is neutral element;

$(x_1, y_1) + (-x_1, y_1) = (0, 1)$.

Edwards curves mod p

Choose an odd prime p.

Choose a *non-square* $d \in \mathbf{F}_p$.

$\{(x, y) \in \mathbf{F}_p \times \mathbf{F}_p :$

$$x^2 + y^2 = 1 + dx^2y^2\}$$

is a "complete Edwards curve".

Roughly $p + 1$ pairs (x, y).

```
def edwardsadd(P1,P2):
  x1,y1 = P1
  x2,y2 = P2
  x3 = (x1*y2+y1*x2)/ \
  (1+d*x1*x2*y1*y2)
  y3 = (y1*y2-x1*x2)/ \
  (1-d*x1*x2*y1*y2)
  return x3,y3
```

Denominators are never 0.

But need different proof;

"$x^2 + y^2 > 0$" doesn't work.

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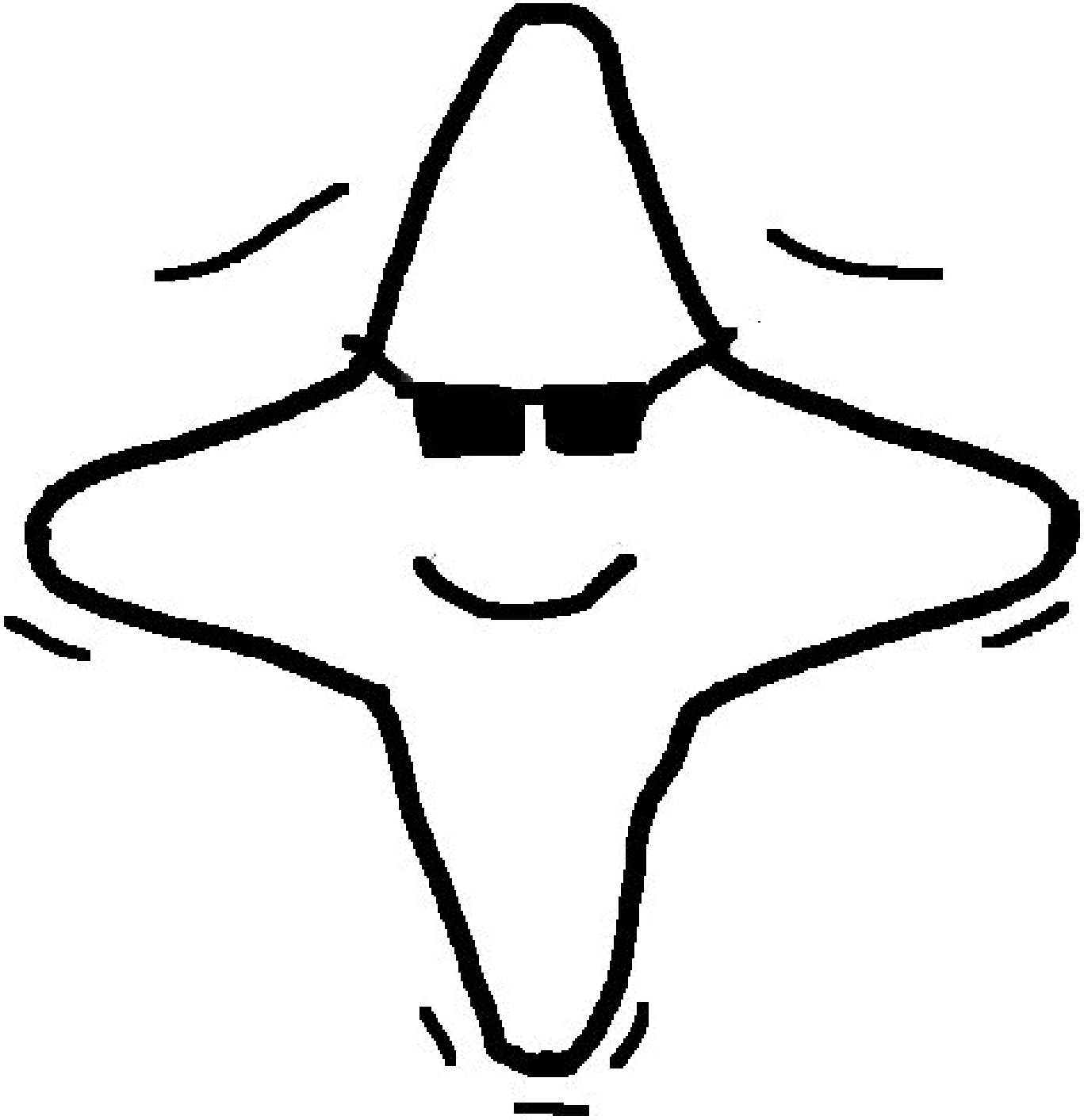
Answer: Can prove that
the denominators are never 0.
Addition law is **complete**.

This proof relies on
choosing *non-square* d.

If we instead choose square d:
curve is still elliptic, and
addition *seems to work*,
but there are failure cases,
often exploitable by attackers.
Safe code is more complicated.

Edwards curves are cool



ECDSA

Users can sign messages
using Edwards curves.

Take a point P on an Edwards
curve modulo a prime $p > 2$.

ECDSA signer needs to know
the *order of* P.

There are only finitely many other
points; about p in total.
Adding P to itself will eventually
reach $(0, 1)$; let ℓ be the smallest
integer > 0 with $\ell P = (0, 1)$.
This ℓ is the order of P.

The signature scheme has as system parameters a curve E; a base point P; and a hash function h with output length at least $\lfloor \log_2 \ell \rfloor + 1$.
Alice's secret key is an integer a and her public key is $P_A = aP$.

To sign message m,
Alice computes $h(m)$;
picks random k;
computes $R = kP = (x_1, y_1)$;
puts $r \equiv y_1 \bmod \ell$; computes $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.

The signature on m is (r, s).

Anybody can verify signature
given m and (r, s):
Compute $w_1 \equiv s^{-1} h(m) \bmod \ell$
and $w_2 \equiv s^{-1} \cdot r \bmod \ell$.
Check whether the y-coordinate
of $w_1 P + w_2 P_A$ equals r modulo ℓ
and if so, accept signature.

Alice's signatures are valid:

$$w_1 P + w_2 P_A =$$
$$(s^{-1} h(m)) P + (s^{-1} \cdot r) P_A =$$
$$(s^{-1} (h(m) + r a)) P = kP$$

and so the y-coordinate of this
expression equals r,
the y-coordinate of kP.

Attacker's view on signatures

Anybody can produce an $R = kP$.
Alice's private key is only used in
$s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.

Can fake signatures if one can break the DLP, i.e., if one can compute a from P_A.
Most of this course deals with methods for breaking DLPs.
Sometimes attacks are easier...

If k is known for some $m, (r, s)$ then $a \equiv (sk - h(m))/r \bmod \ell$.

If two signatures $m_1, (r, s_1)$ and $m_2, (r, s_2)$ have the same value for r: assume $k_1 = k_2$; observe $s_1 - s_2 = k_1^{-1}(h(m_1) + ra - (h(m_2) + ra))$; compute $k = (s_1 - s_2)/(h(m_1) - h(m_2))$. Continue as above.

If bits of many k's are known (biased PRNG) can attack $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$ as hidden number problem using lattice basis reduction.

Malicious signer

Alice can set up her public key so that two messages of her choice share the same signature, i.e., she can claim to have signed m_1 or m_2 at will:
$R = (x_1, y_1)$ and $-R = (-x_1, y_1)$ have the same y-coordinate.
Thus, (r, s) fits $R = kP$, $s \equiv k^{-1}(h(m_1) + r\,a) \bmod \ell$ and $-R = (-k)P$, $s \equiv -k^{-1}(h(m_2) + r\,a) \bmod \ell$ if $a \equiv -(h(m_1)+h(m_2))/2r \bmod \ell$.

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$a \equiv -(h(m_1)+h(m_2))/2r \bmod \ell$.

(Easy tweak: include bit of x_1.)

More elliptic curves

Edwards curves are elliptic.
Easiest way to understand
elliptic curves is Edwards.

Geometrically, all elliptic curves
are Edwards curves.

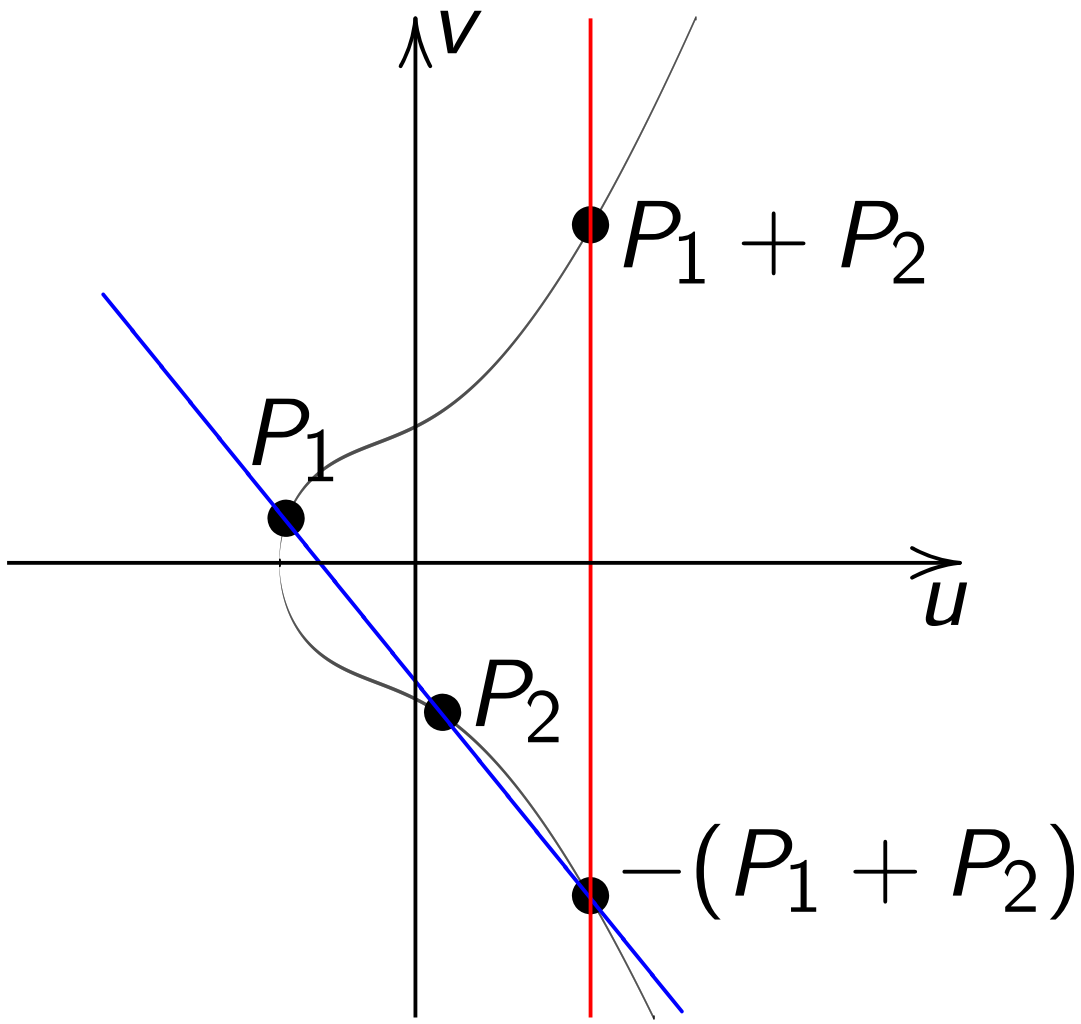
Algebraically,
more elliptic curves exist
(not always point of order 4).

Every odd-char curve can be
expressed as Weierstrass curve
$v^2 = u^3 + a_2u^2 + a_4u + a_6$.

Warning: "Weierstrass" has
different meaning in char 2.

Addition on Weierstrass curve

$$v^2 = u^3 + u^2 + u + 1$$

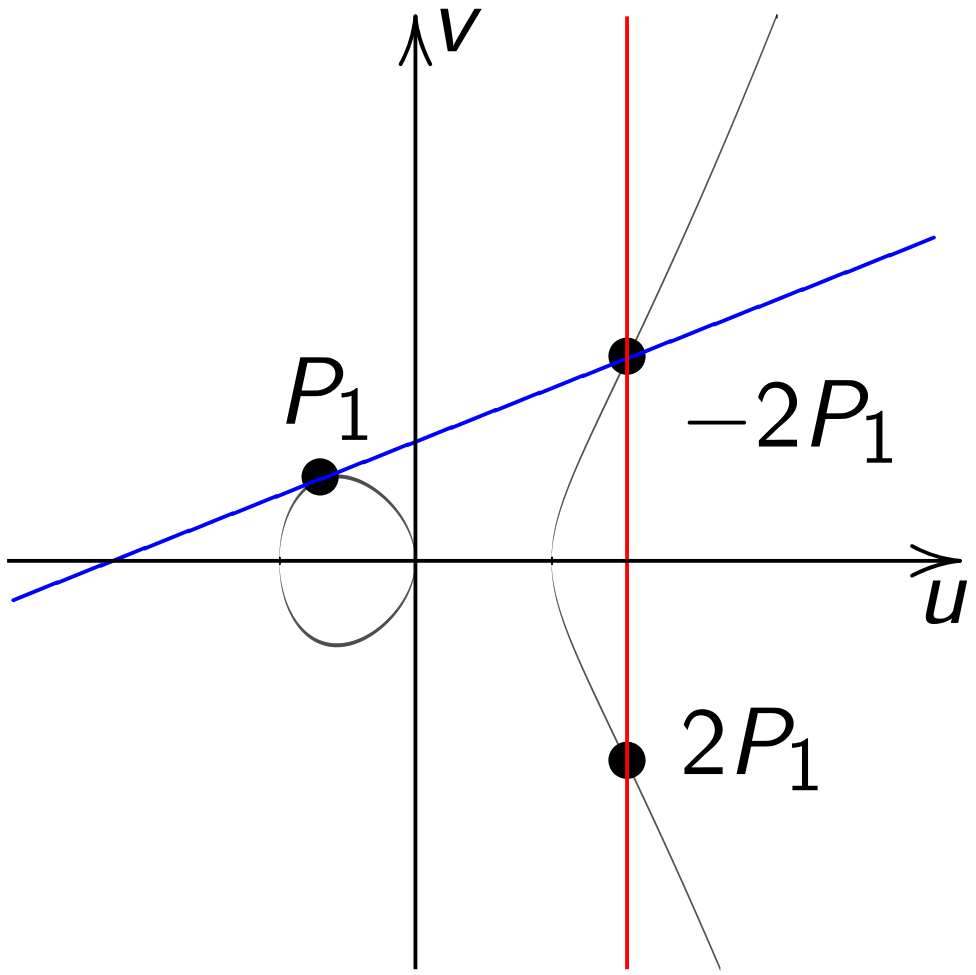


Slope $\lambda = (v_2 - v_1)/(u_2 - u_1)$.

Note that $u_1 \neq u_2$.

Doubling on Weierstrass curve

$v^2 = u^3 - u$



Slope $\lambda = (3u_1^2 - 1)/(2v_1)$.

In most cases
$(u_1, v_1) + (u_2, v_2) =$
(u_3, v_3) where $(u_3, v_3) =$
$(\lambda^2 - u_1 - u_2, \lambda(u_1 - u_3) - v_1)$.

$u_1 \neq u_2$, "addition" (alert!):
$\lambda = (v_2 - v_1)/(u_2 - u_1)$.
Total cost $1\mathbf{I} + 2\mathbf{M} + 1\mathbf{S}$.

$(u_1, v_1) = (u_2, v_2)$ and $v_1 \neq 0$,
"doubling" (alert!):
$\lambda = (3u_1^2 + 2a_2u_1 + a_4)/(2v_1)$.
Total cost $1\mathbf{I} + 2\mathbf{M} + 2\mathbf{S}$.

Also handle some exceptions:
$(u_1, v_1) = (u_2, -v_2)$; ∞ as input.
Messy to implement and test.

Birational equivalence

Starting from point (x, y)
on $x^2 + y^2 = 1 + dx^2y^2$:

Define $A = 2(1 + d)/(1 - d)$,
$B = 4/(1 - d)$;
$u = (1 + y)/(B(1 - y))$,
$v = u/x = (1 + y)/(Bx(1 - y))$.
(Skip a few exceptional points.)

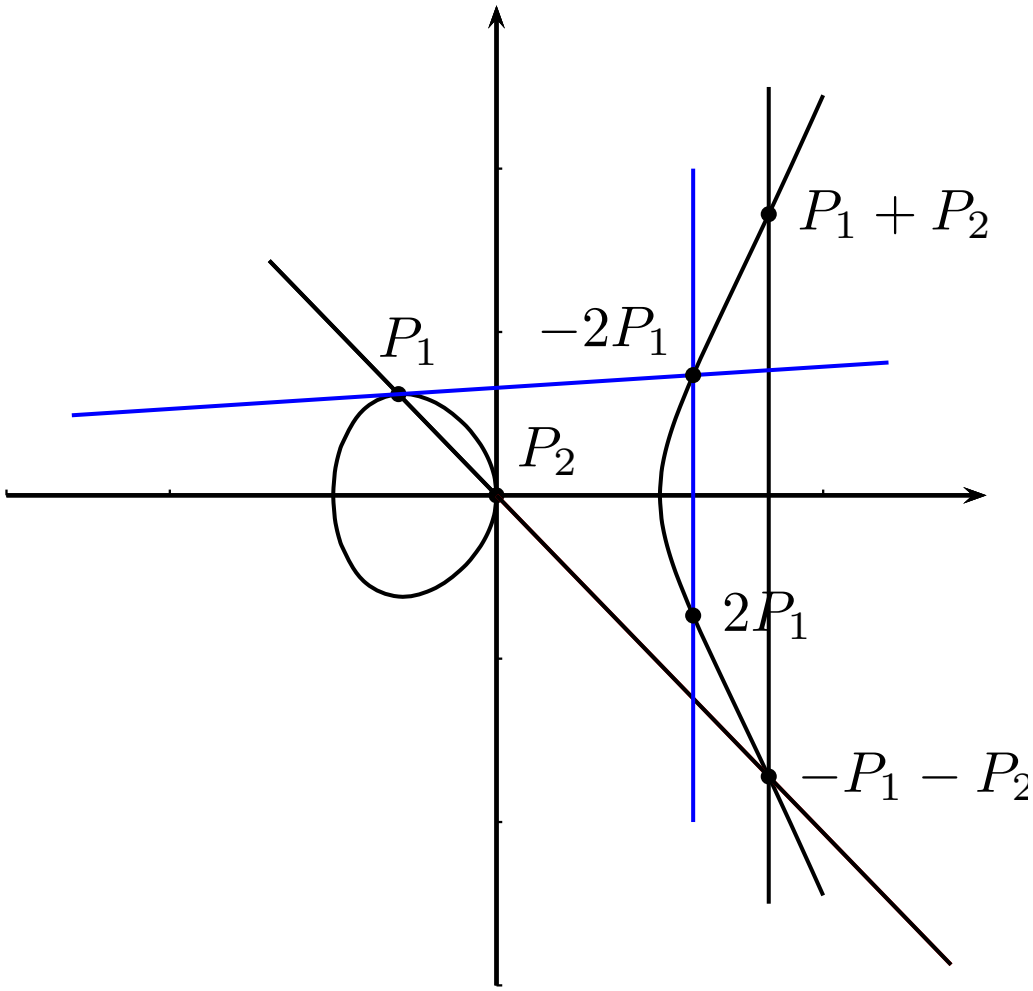
Then (u, v) is a point on
a Weierstrass curve:
$v^2 = u^3 + (A/B)u^2 + (1/B^2)u$.

Easily invert this map:
$x = u/v$, $y = (Bu - 1)/(Bu + 1)$.

Attacker can transform Edwards curve to Weierstrass curve and vice versa; $n(x, y) \mapsto n(u, v)$.
$\Rightarrow$ Same discrete-log security!
Can choose curve representation so that implementation of attack is faster/easier.

System designer can choose curve representation so that protocol runs fastest; no need to worry about security degradation.

Elliptic-curve groups



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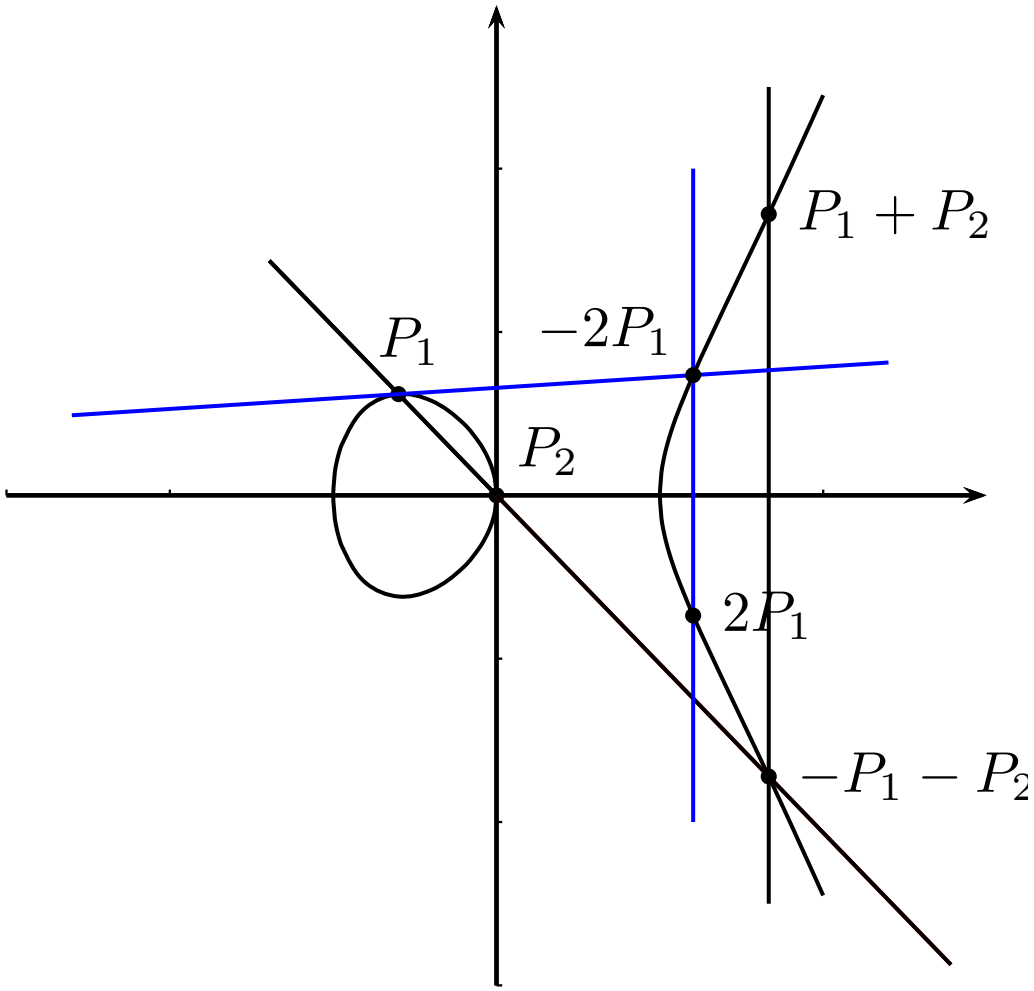

Following algorithms will need a unique representative per point. For that Weierstrass curves are the speed leader.

The discrete-logarithm problem

Define $p = 1000003$ and consider the Weierstrass curve $y^2 = x^3 - x$ over $\mathbf{F}_p$.

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points and $P = (101384, 614510)$
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points and $P = (101384, 614510)$
is a point of order $2 \cdot 53^2 \cdot 89$.
In general, point counting over $\mathbf{F}_p$
runs in time polynomial in $\log p$.
Number of points in
$[p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$.
The group is isomorphic to
$\mathbf{Z}/n \times \mathbf{Z}/m$, where $n \mid m$ and
$n \mid (p - 1)$.

Can we find an integer
$n \in \{1, 2, 3, \ldots, 500001\}$
such that $nP = (670366, 740819)$?

This point was generated as a multiple of P; could also be outside cyclic group.

Could find n by brute force.
Is there a faster way?

Understanding brute force

Can compute successively
$1P = (101384, 614510)$,
$2P = (102361, 628914)$,
$3P = (77571, 87643)$,
$4P = (650289, 31313)$,
$500001P = -P$.
$500002P = \infty$.

At some point we'll find n
with $nP = (670366, 740819)$.

Maximum cost of computation:
≤ 500001 additions of P;
≤ 500001 nanoseconds on a CPU
that does 1 ADD/nanosecond.

This is negligible work
for $p \approx 2^{20}$.

But users can
standardize a larger p,
making the attack slower.

Attack cost scales linearly:
$\approx 2^{50}$ ADDs for $p \approx 2^{50}$,
$\approx 2^{100}$ ADDs for $p \approx 2^{100}$, etc.

(Not exactly linearly:
cost of ADDs grows with p.
But this is a minor effect.)

Computation has a good chance
of finishing earlier.
Chance scales linearly:
1/2 chance of 1/2 cost;
1/10 chance of 1/10 cost; etc.

"So users should choose large n."

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"So users should choose large n."

That's pointless. We can apply
"random self-reduction":
choose random r, say 69961;
compute $rP = (593450, 987590)$;
compute $(r + n)P$ as
$(593450, 987590)+(670366, 740819)$;
compute discrete log;
subtract r mod 500002; obtain n.

Computation can be parallelized.

One low-cost chip can run
many parallel searches.
Example, 2^6 €: one chip,
2^{10} cores on the chip,
each 2^{30} ADDs/second?
Maybe; see SHARCS workshops
for detailed cost analyses.

Attacker can run
many parallel chips.
Example, 2^{30} €: 2^{24} chips,
so 2^{34} cores,
so 2^{64} ADDs/second,
so 2^{89} ADDs/year.

Multiple targets and giant steps

Computation can be applied
to many targets at once.

Given 100 DL targets n_1P,
n_2P, ..., $n_{100}P$:
Can find *all* of $n_1, n_2, \ldots, n_{100}$
with ≤ 500002 ADDs.

Simplest approach: First build
a sorted table containing
n_1P, ..., $n_{100}P$.
Then check table for
$1P$, $2P$, etc.

Interesting consequence #1:
Solving all 100 DL problems
isn't much harder than
solving one DL problem.

Interesting consequence #2:
Solving *at least one*
out of 100 DL problems
is much easier than
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When did this computation
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When did this computation
find its *first* n_i?
Typically $\approx 500002/100$ mults.

Can use random self-reduction
to turn a single target
into multiple targets.
Let ℓ be the order of P.

Given nP:
Choose random $r_1, r_2, \ldots, r_{100}$.
Compute $r_1P + nP$,
$r_2P + nP$, etc.

Solve these 100 DL problems.
Typically $\approx \ell/100$ mults
to find *at least one*
$r_i + n \bmod \ell$,
immediately revealing n.

Also spent some ADDs
to compute each $r_i P$:
$\approx \lg p$ ADDs for each i.

Faster: Choose $r_i = i\,r_1$
with $r_1 \approx \ell/100$.
Compute $r_1 P$;
$r_1 P + n P$;
$2r_1 P + n P$;
$3r_1 P + n P$; etc.
Just 1 ADD for each new i.

$\approx 100 + \lg \ell + \ell/100$ ADDs
to find n given $n P$.

Faster: Increase 100 to $\approx \sqrt{\ell}$.
Only $\approx 2\sqrt{\ell}$ ADDs
to solve one DL problem!
"Shanks baby-step-giant-step
discrete-logarithm algorithm."

Example: $p = 1000003, \ell = 500002$, $P = (101384, 614510)$,
$Q = nP = (670366, 740819)$.
Compute $708P$=(393230, 421116).
Then compute 707 targets:
$708P + Q = (342867, 153817)$,
$2 \cdot 708P + nP = (430321, 994742)$,
$3 \cdot 708P + nP = (423151, 635197)$,
$\ldots, 706 \cdot 708P + nP = (534170, 450849)$.

Build a sorted table of targets:

$600 \cdot 708P + Q = (799978, 929249)$,

$27 \cdot 708P + Q = (785344, 831127)$,

$219 \cdot 708P + Q = (425475, 793466)$,

...

$242 \cdot 708P + Q = (262804, 347755)$,

...

$317 \cdot 708P + Q = (599785, 189116)$.

Look up P, $2P$, $3P$, etc. in table.

$620P = (950652, 688508)$; find

$596 \cdot 708P + Q = (950652, 688508)$

in the table of targets;

so $620 = 596 \cdot 708 + n \mod 500002$;

deduce $n = 78654$.

Factors of the group order

P has order $2 \cdot 53^2 \cdot 89$.

Given $Q = nP$, find $n = \log_P Q$:

$R = (53^2 \cdot 89)P$ has order 2, and
$S = (53^2 \cdot 89)Q$ is multiple of R.
Compute $n_1 = \log_R S \equiv n \bmod 2$.

$R = (2 \cdot 53 \cdot 89)P$ has order 53,
and
$S = (2 \cdot 53 \cdot 89)Q$ is multiple of R.
Compute
$n_2 = \log_R S \equiv n \bmod 53$.
This is a DLP in a group
of size 53.

$T = (2 \cdot 89)(Q - n_2 P)$ is also
a multiple of R, i.e., has order 53.
Compute
$n_3 = \log_R T \equiv n \bmod 53$.
Now $n_2 + 53n_3 \equiv n \bmod 53^2$.

$R = (2 \cdot 53^2)P$ has order 89, and
$S = (2 \cdot 53^2)Q$ is multiple of R.
Compute
$n_4 = \log_R S \equiv n \bmod 89$.

Use Chinese Remainder Theorem
$n \equiv n_1 \bmod 2$,
$n \equiv n_2 + 53n_3 \bmod 53^2$,
$n \equiv n_4 \bmod 89$,
to determine n modulo $2 \cdot 53^2 \cdot 89$.

This "Pohlig-Hellman method" converts an order-ab DL into an order-a DL, an order-b DL, and a few scalar multiplications.

Here $(53^2 \cdot 89)P = (1, 0)$ and $(53^2 \cdot 89)Q = \infty$, thus $n_1 = 0$.

$(2 \cdot 53 \cdot 89)P = (539296, 488875)$,
$(2 \cdot 53 \cdot 89)Q = (782288, 572333)$.
A search quickly finds $n_2 = 2$.
$(2 \cdot 89)(Q - 2P) = \infty$, thus $n_3 = 0$ and $n_2 + 53n_3 = 2$.

$(2 \cdot 53^2)P = (877560, 947848)$ and
$(2 \cdot 53^2)Q = (822491, 118220)$.
Compute $n_4 = 67$,
e.g. using BSGS.

Use Chinese Remainder Theorem
$n \equiv 0 \bmod 2$,
$n \equiv 2 \bmod 53^2$,
$n \equiv 67 \bmod 89$,
to determine $n = 78654$.

Pohlig-Hellman method reduces security of discrete logarithm problem in group generated by P to security of largest prime order subgroup.

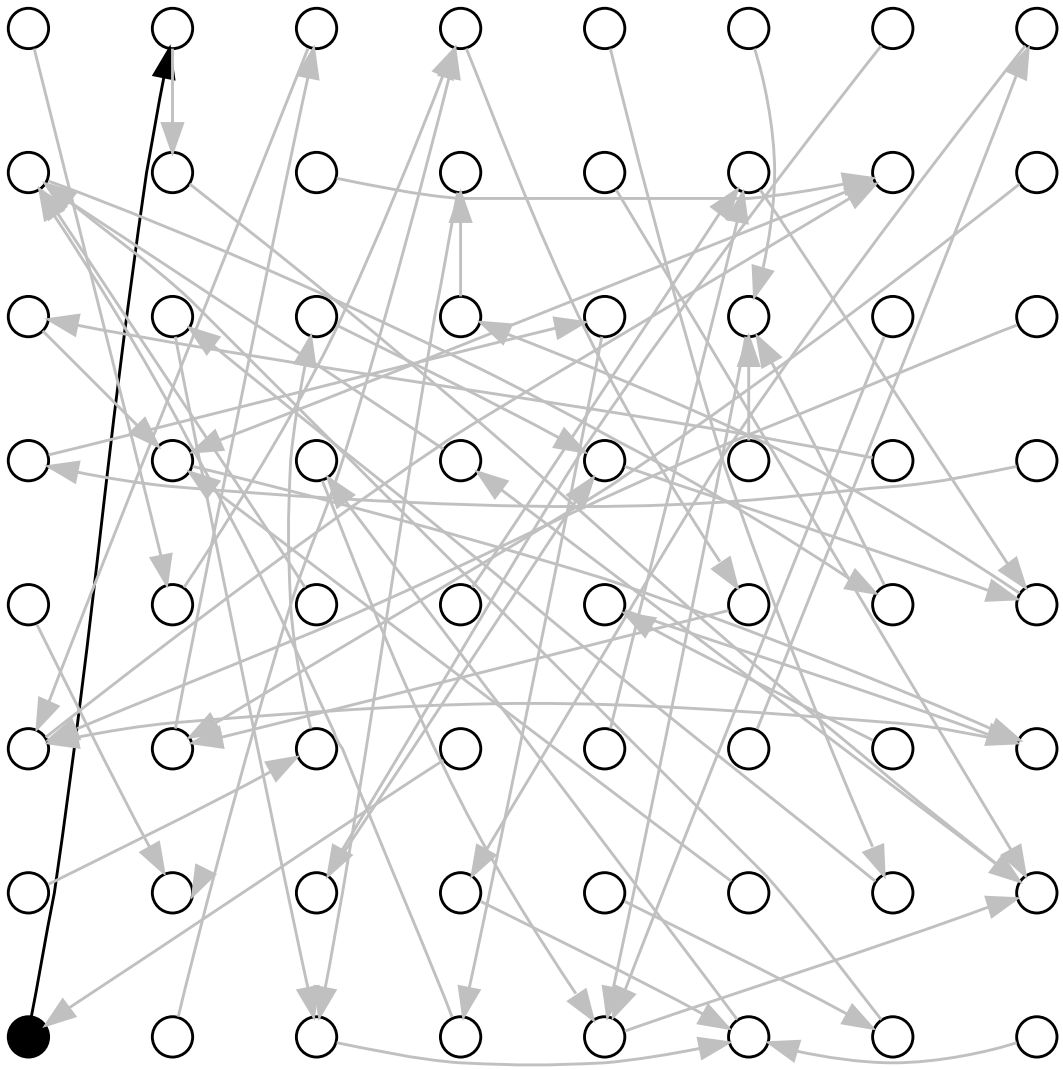
The rho method

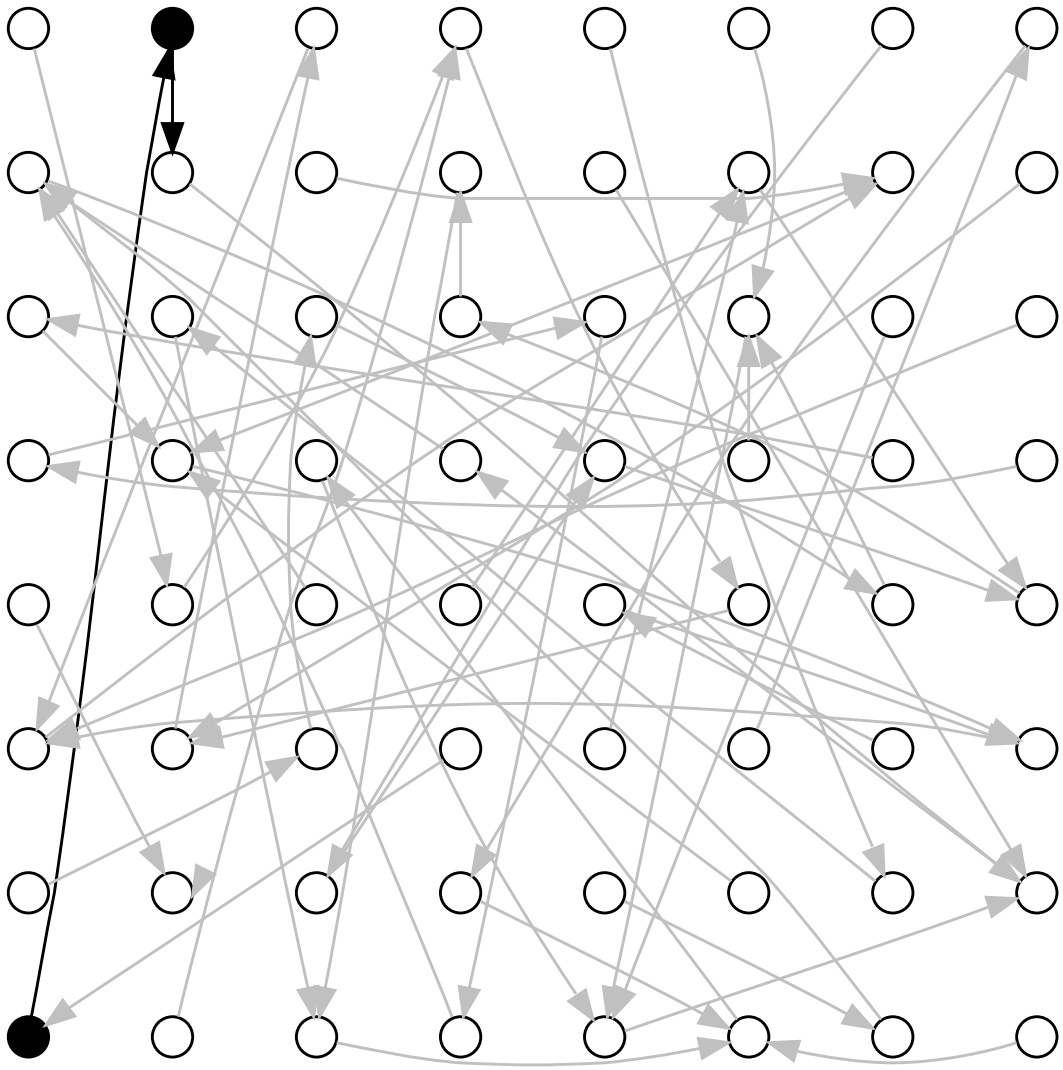
Simplified, non-parallel rho:

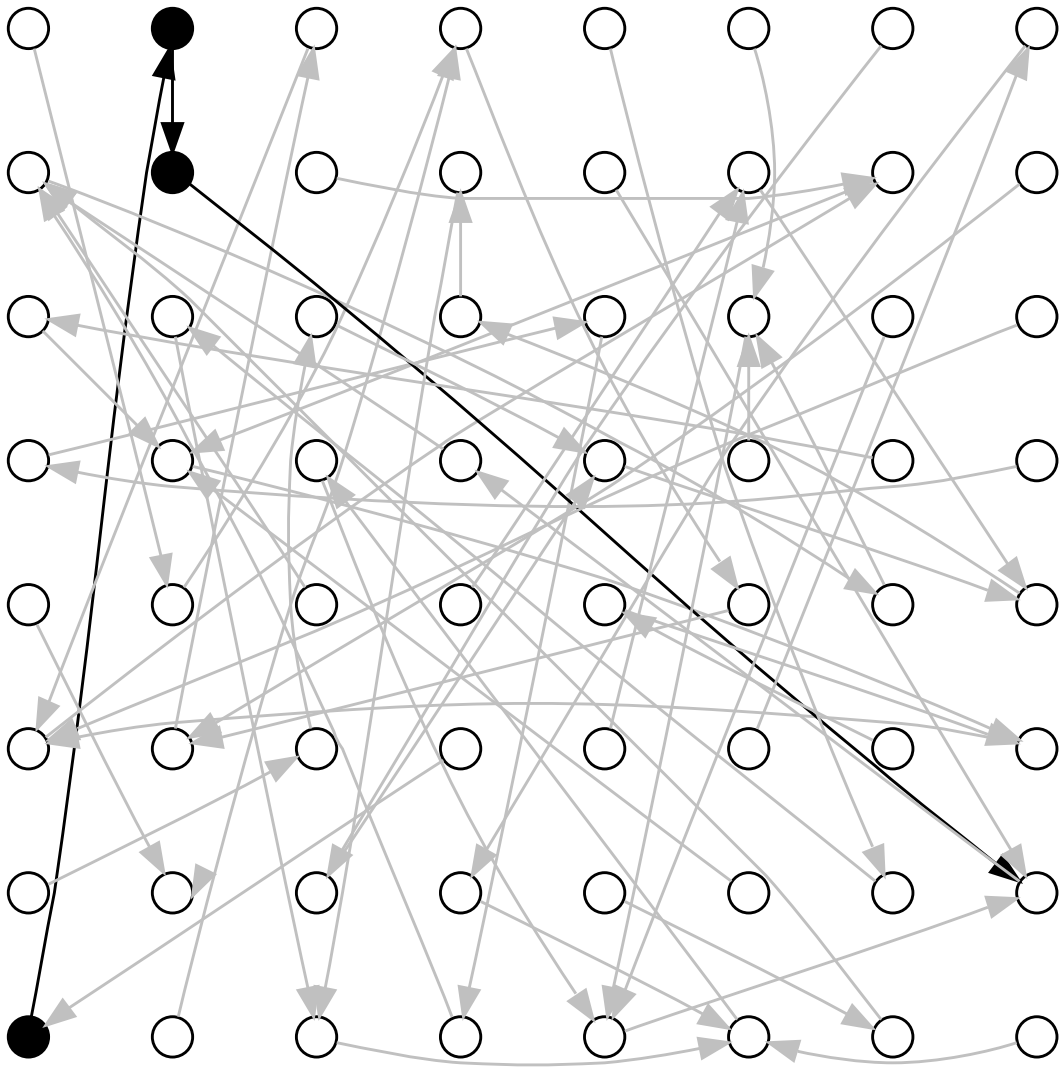
Make a pseudo-random walk
in the group $\langle P \rangle$,
where the next step depends
on current point: $W_{i+1} = f(W_i)$.

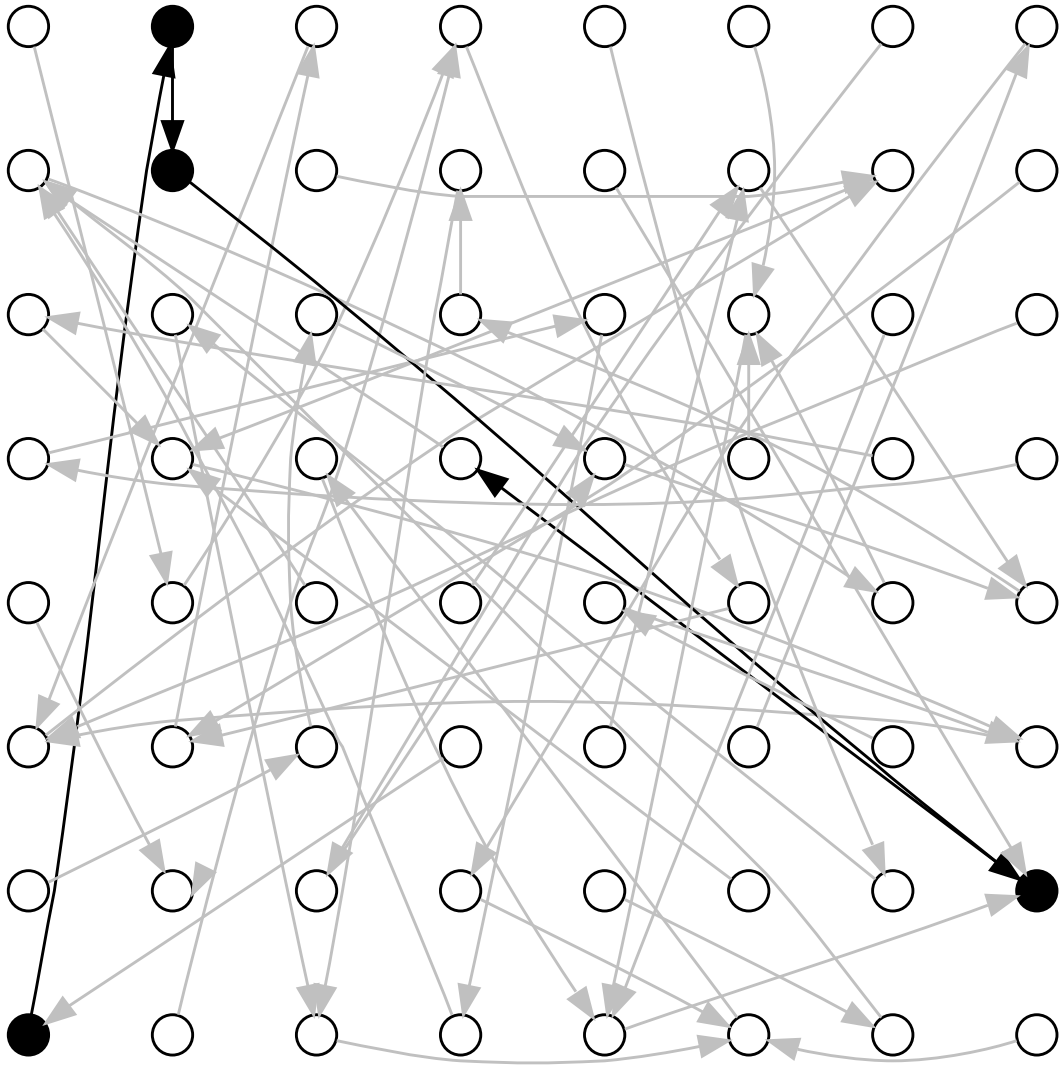
Birthday paradox:
Randomly choosing from ℓ
elements picks one element twice
after about $\sqrt{\pi \ell / 2}$ draws.

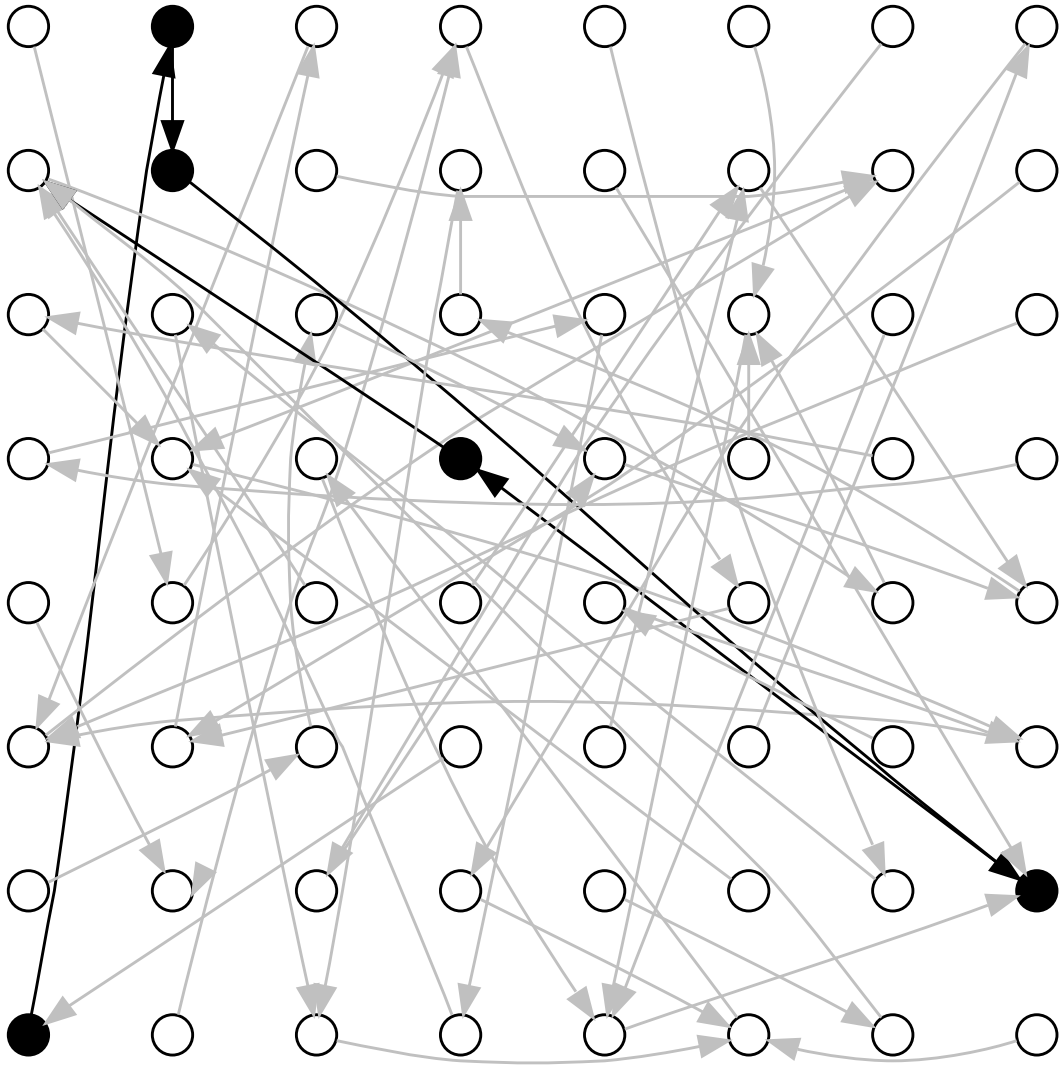
The walk now enters a cycle.
Cycle-finding algorithm
(e.g., Floyd) quickly detects this.

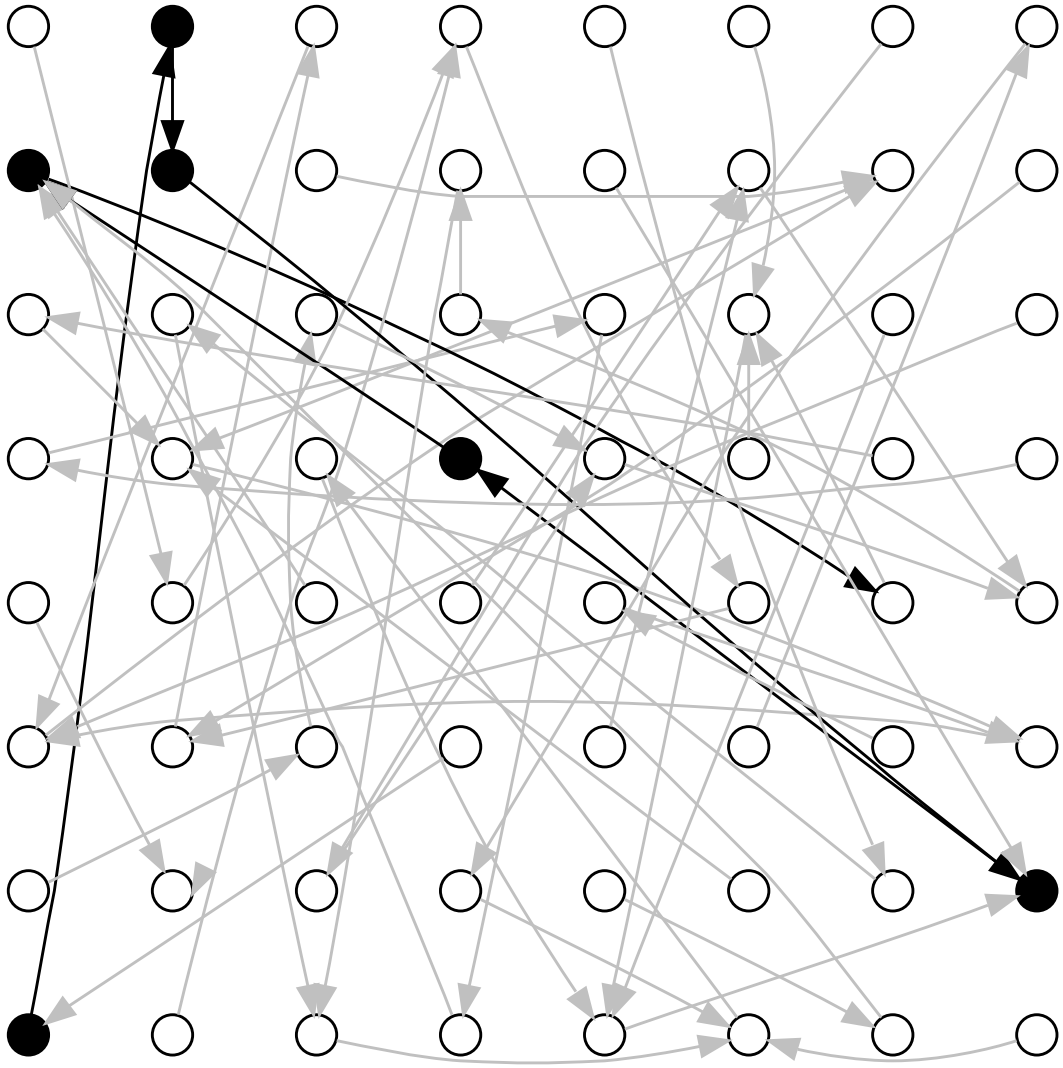


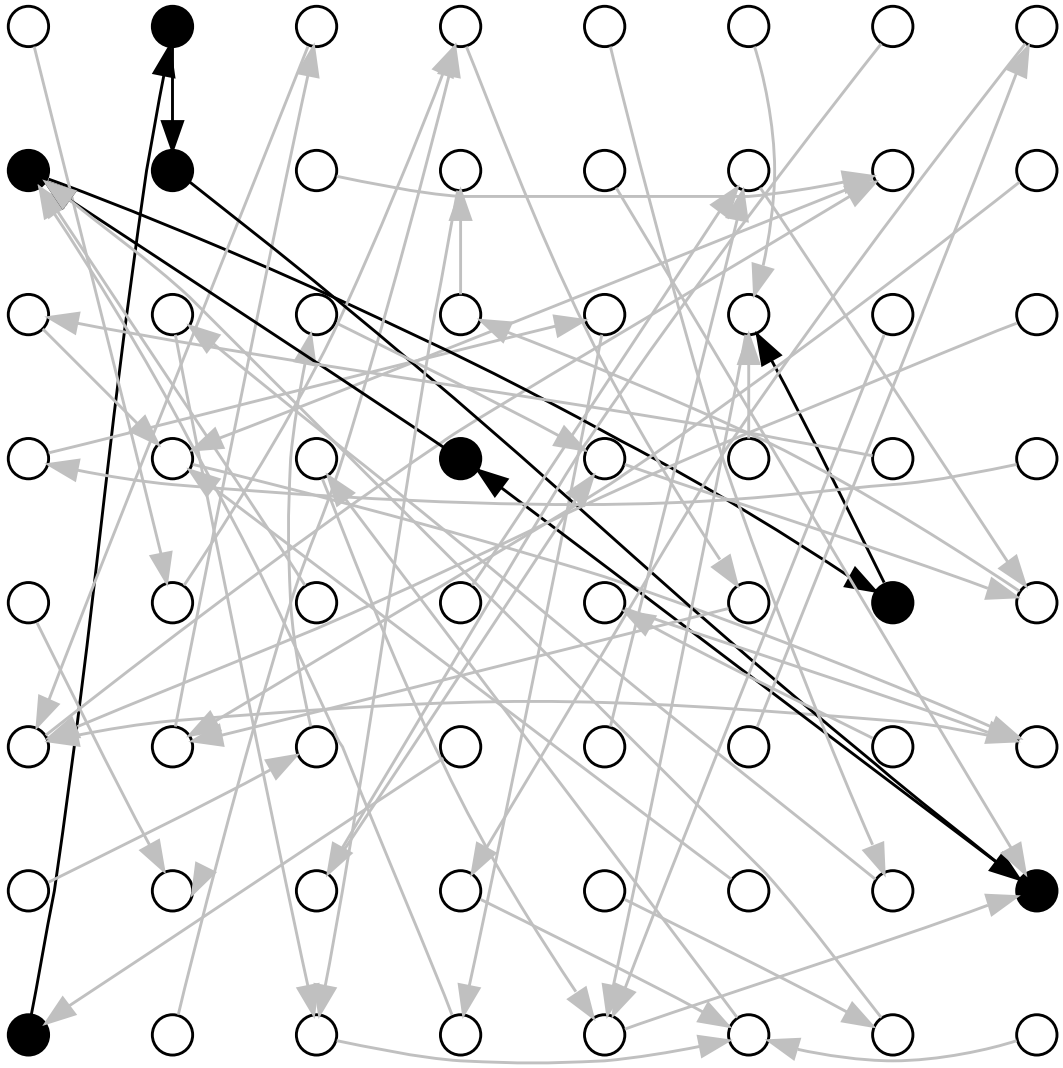


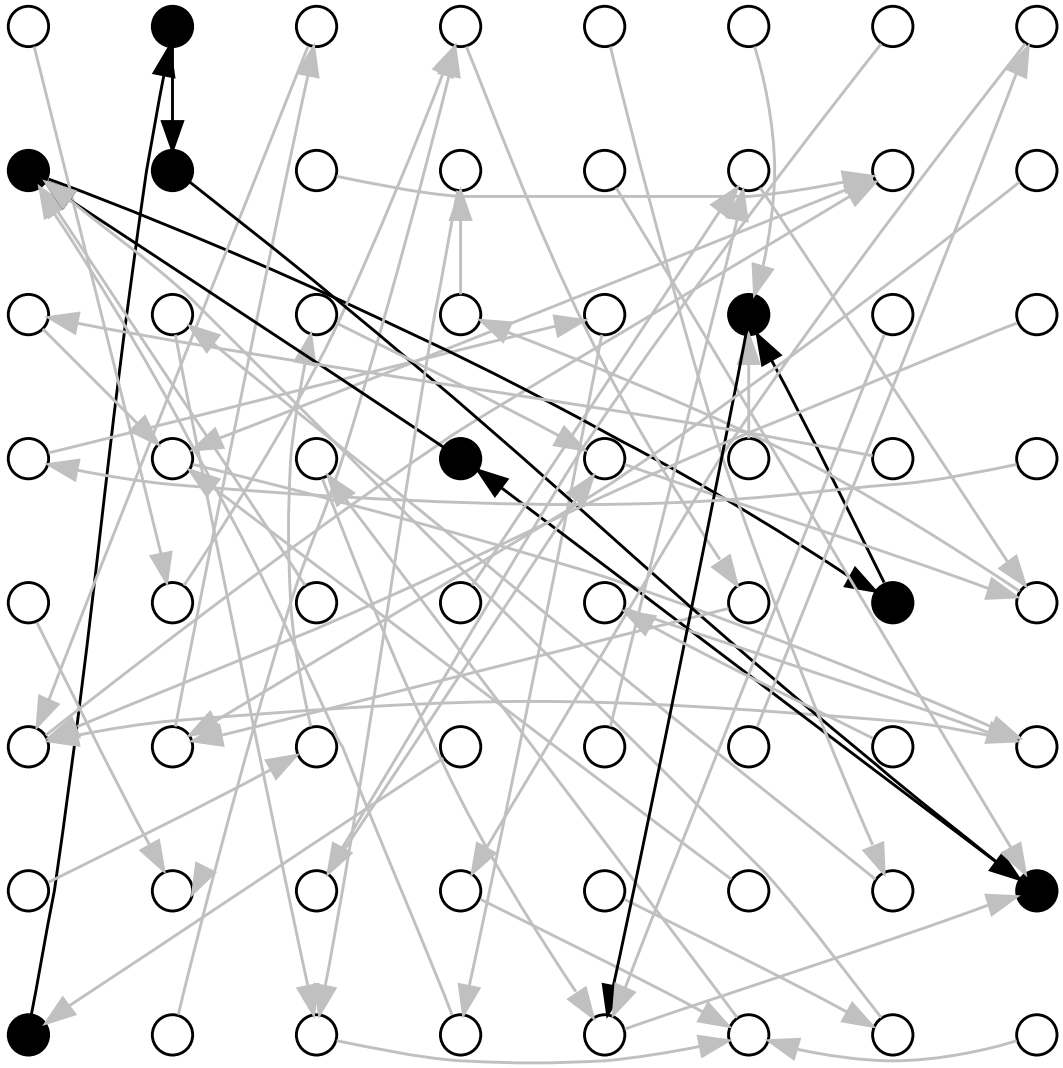


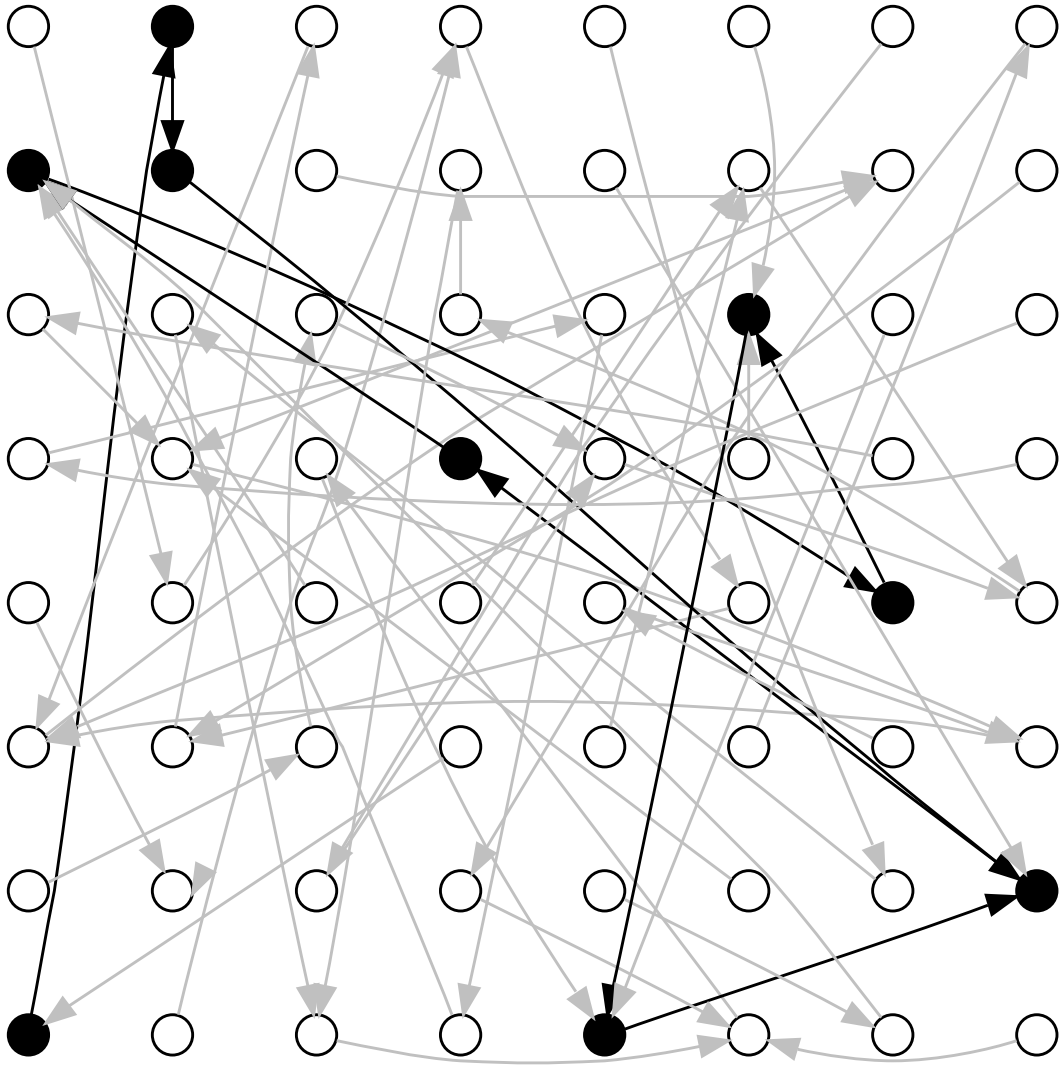


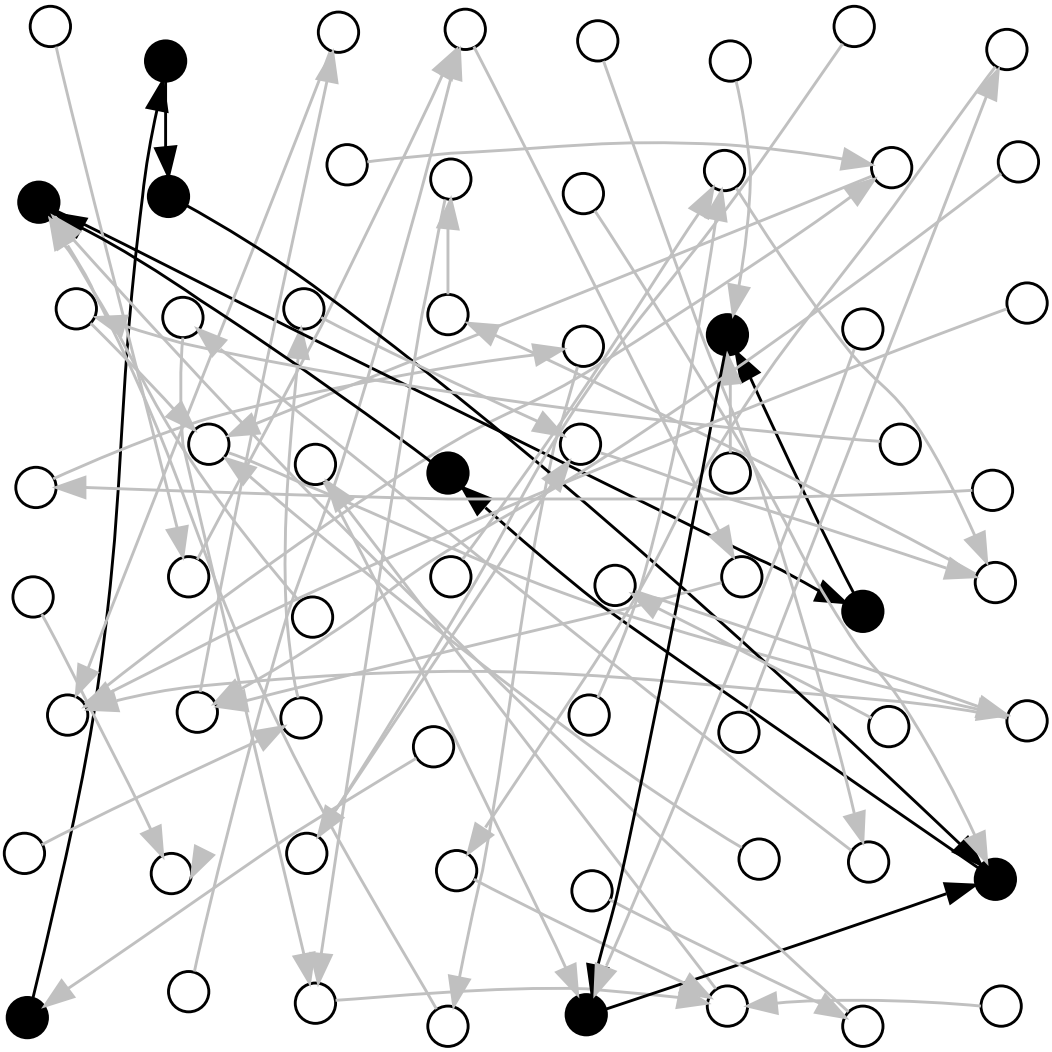


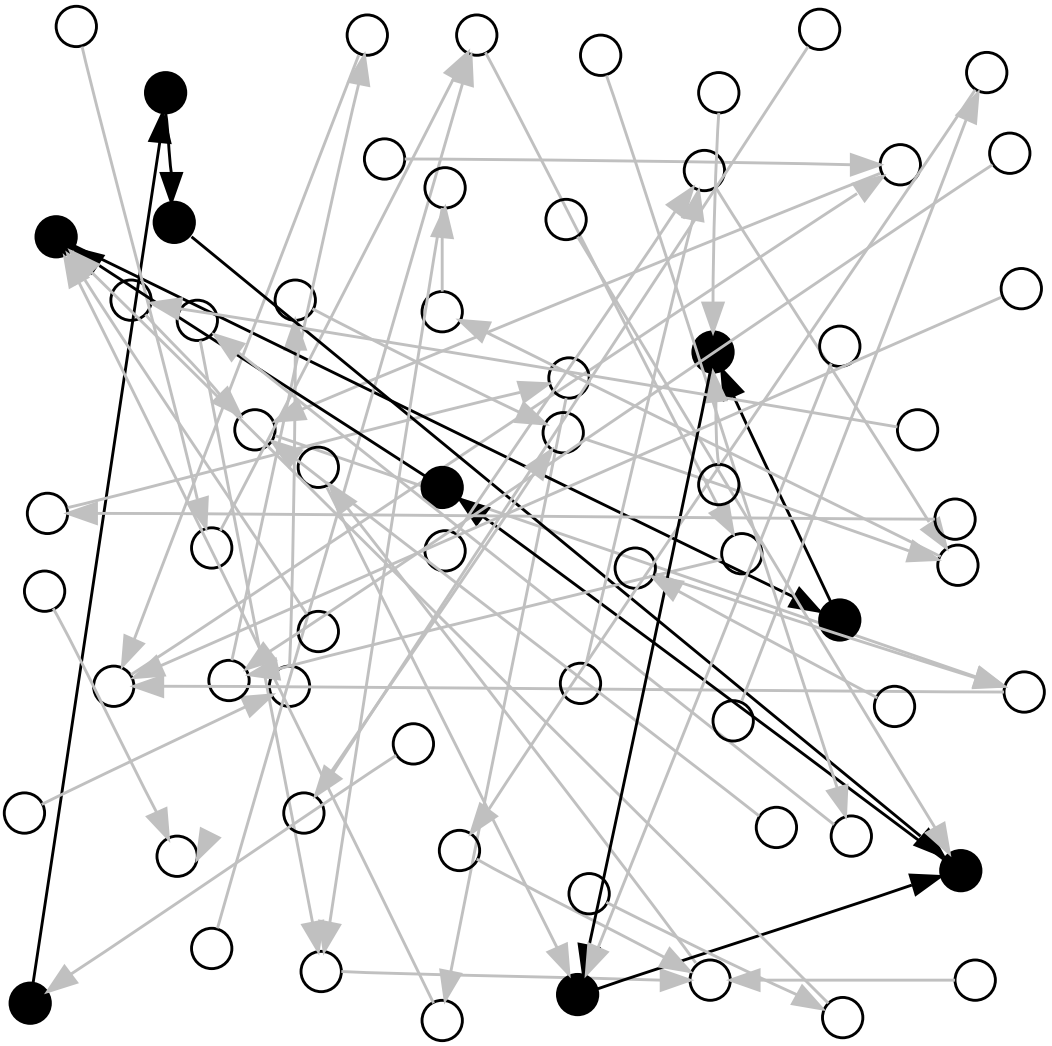


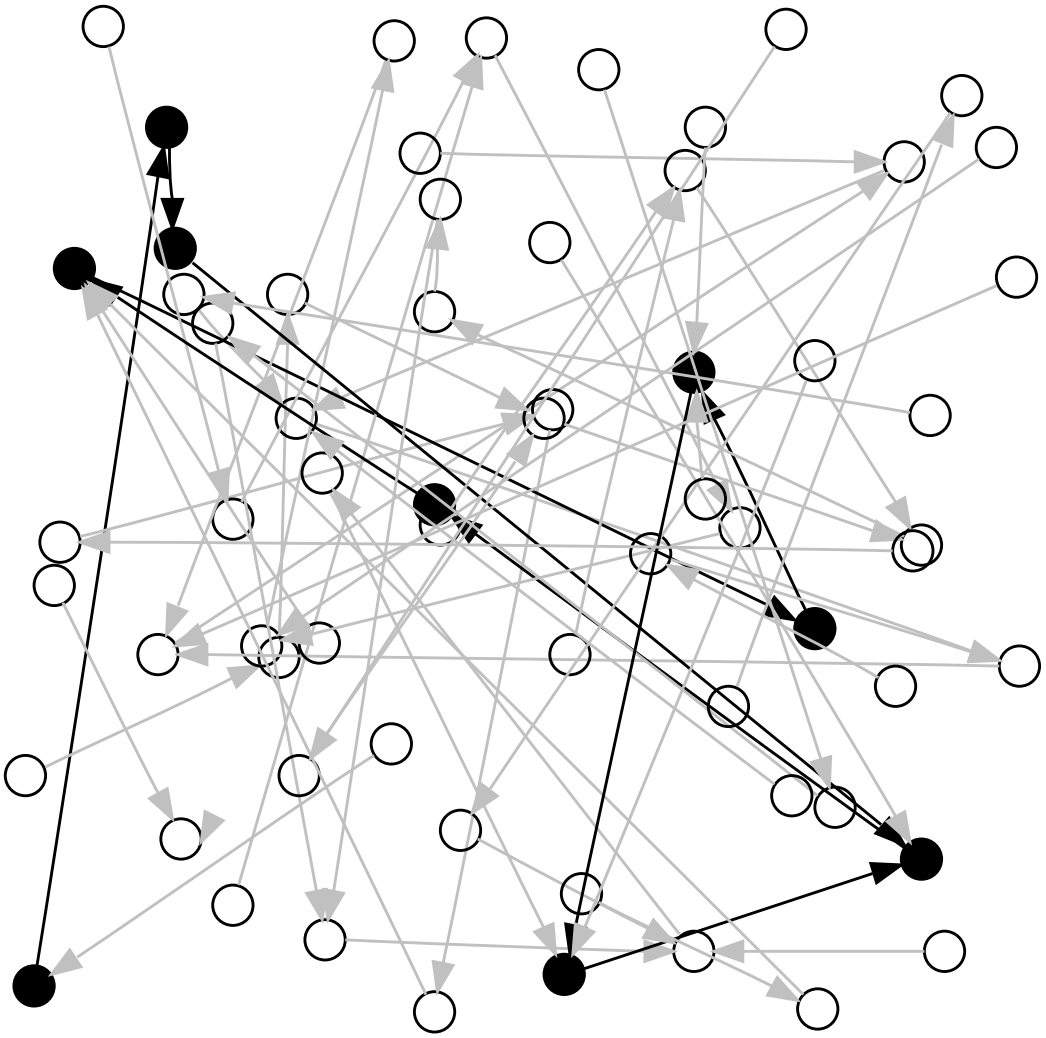


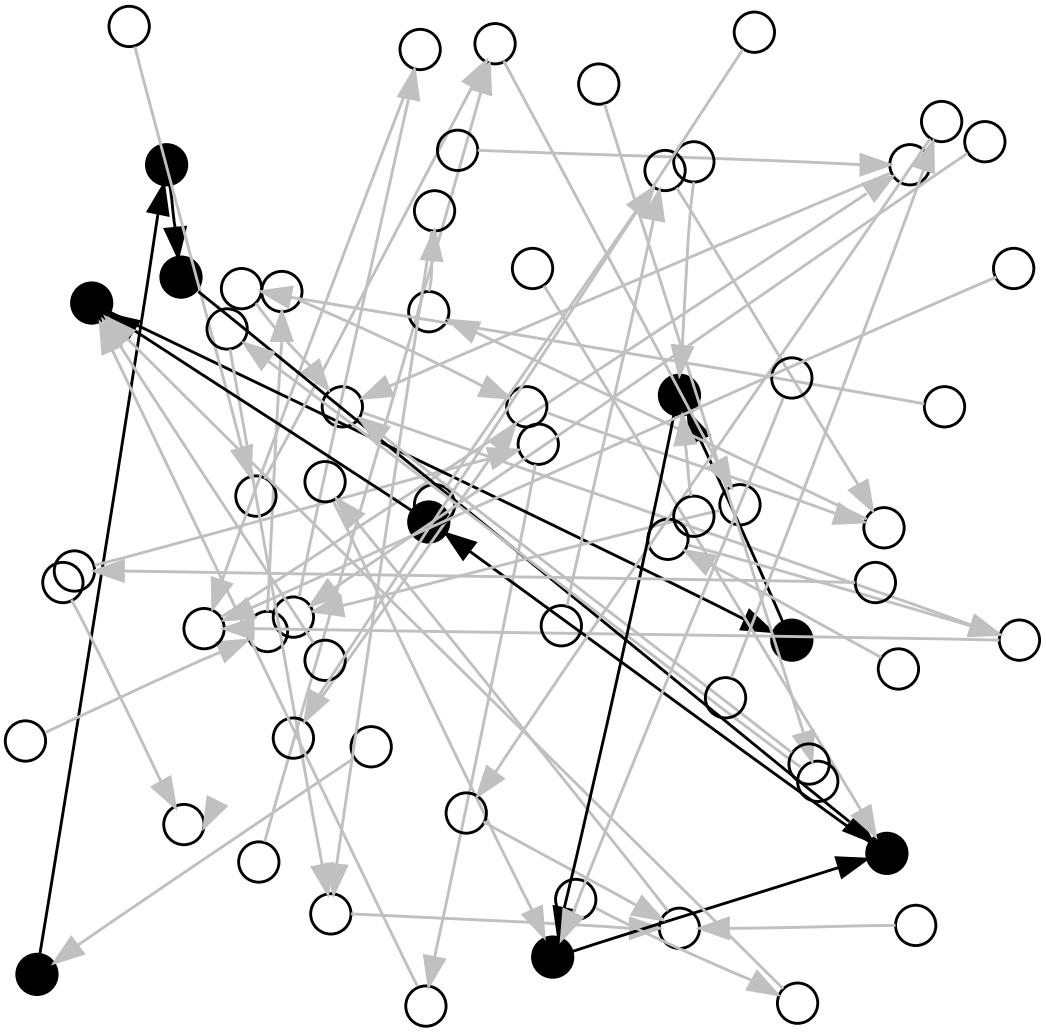


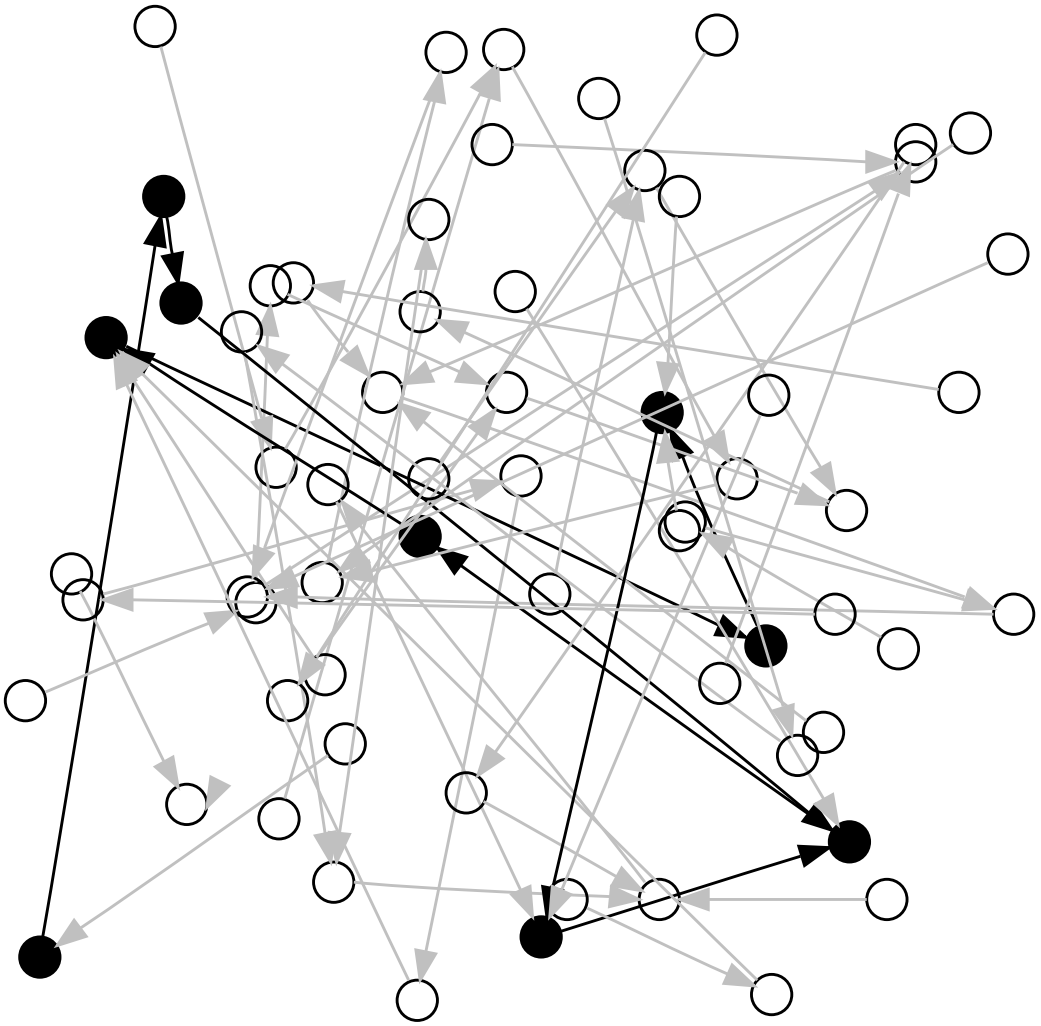


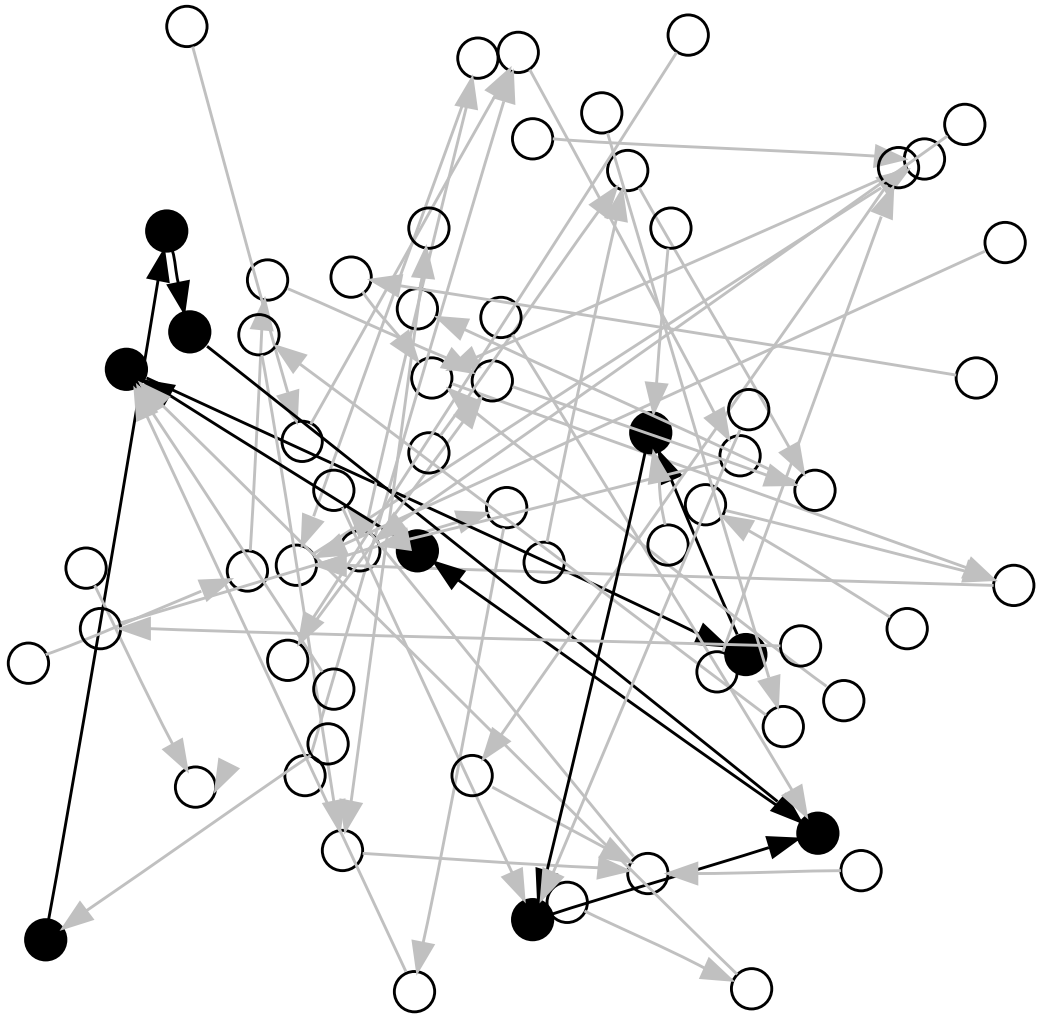


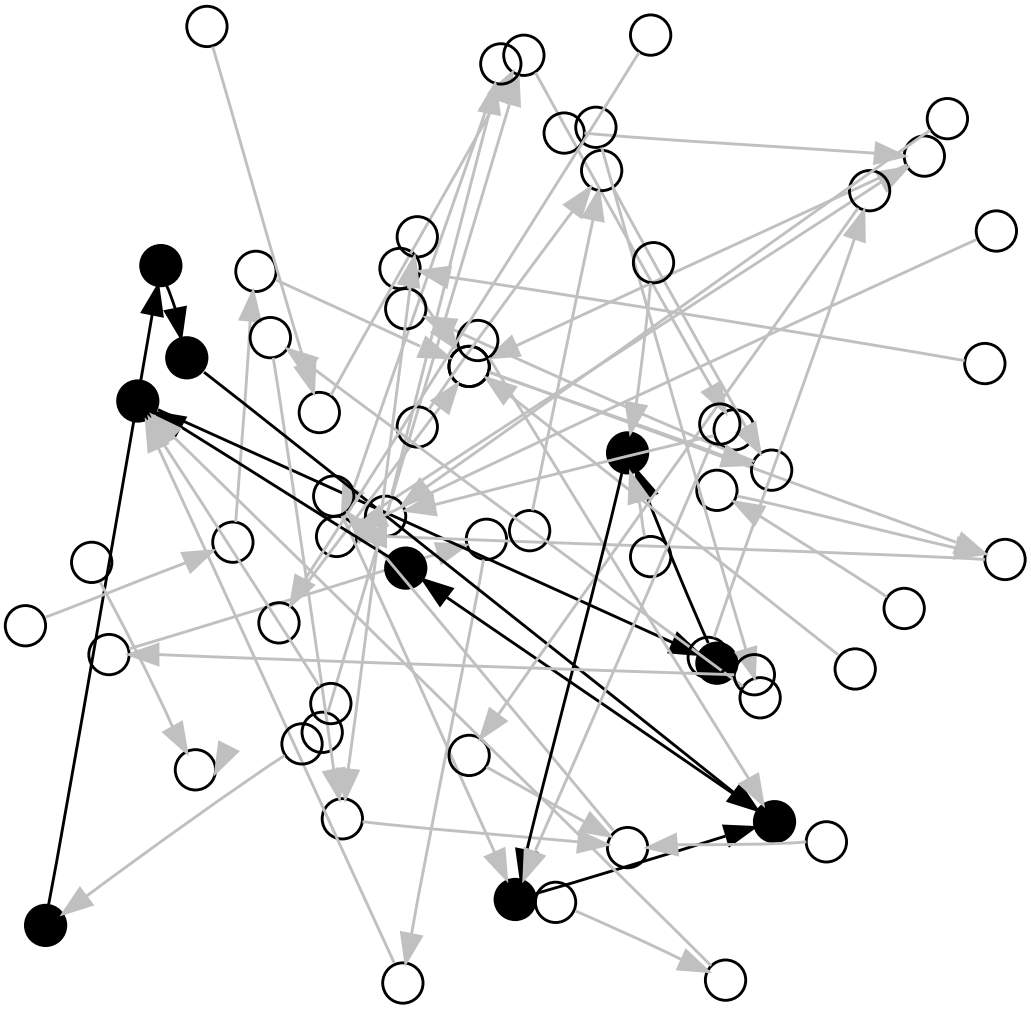


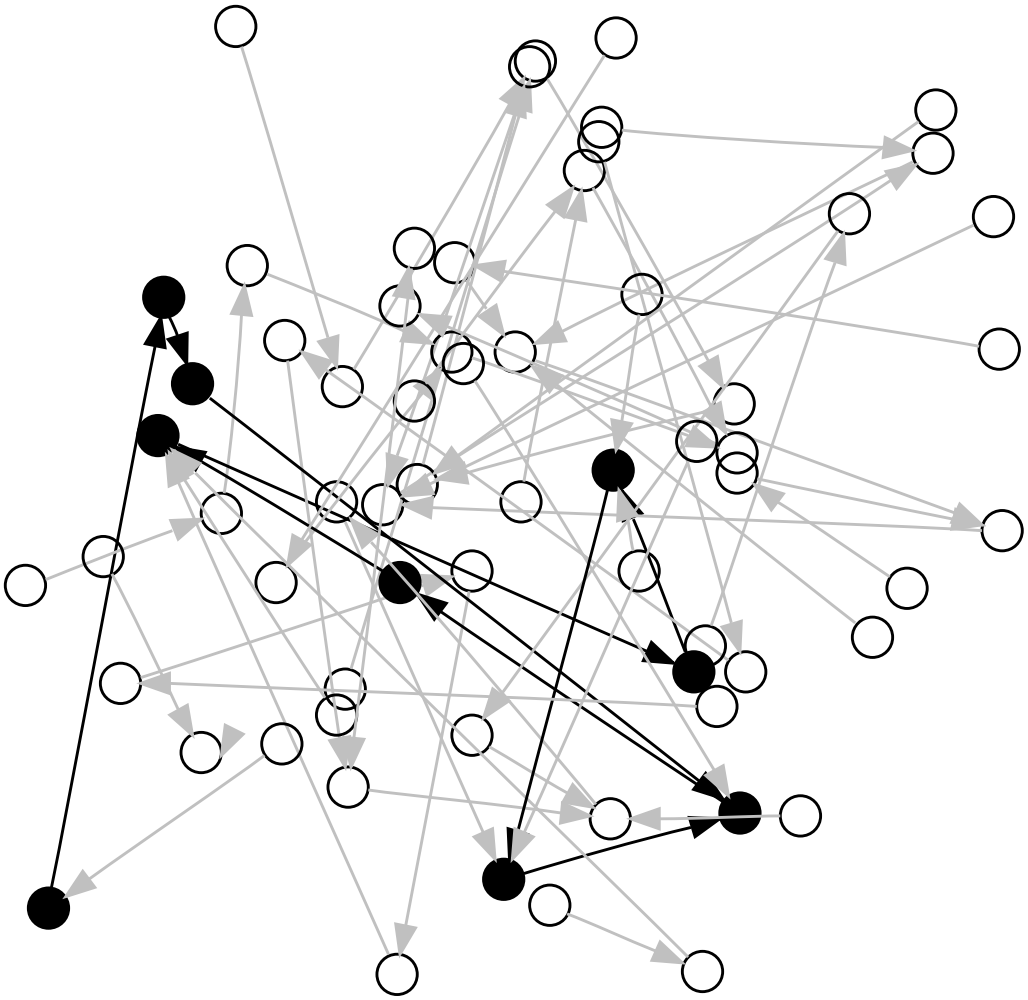


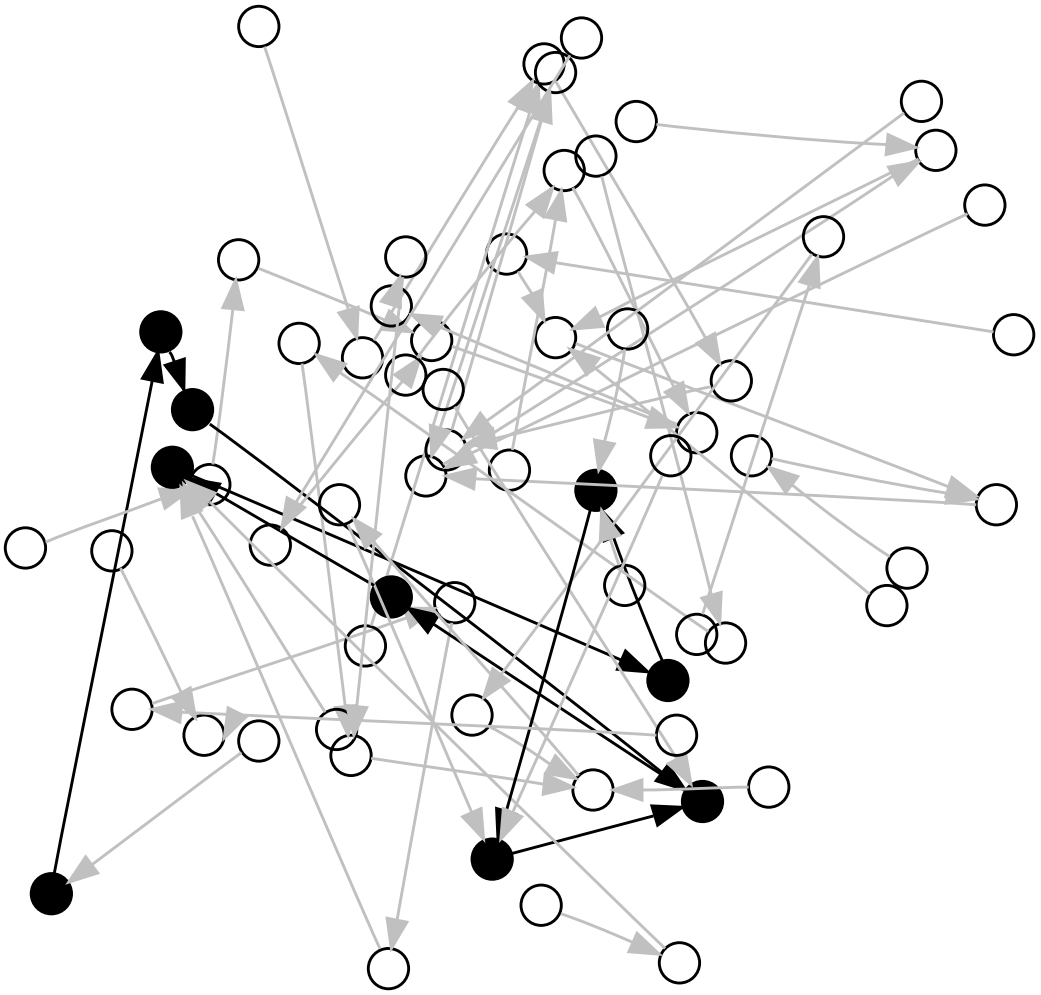


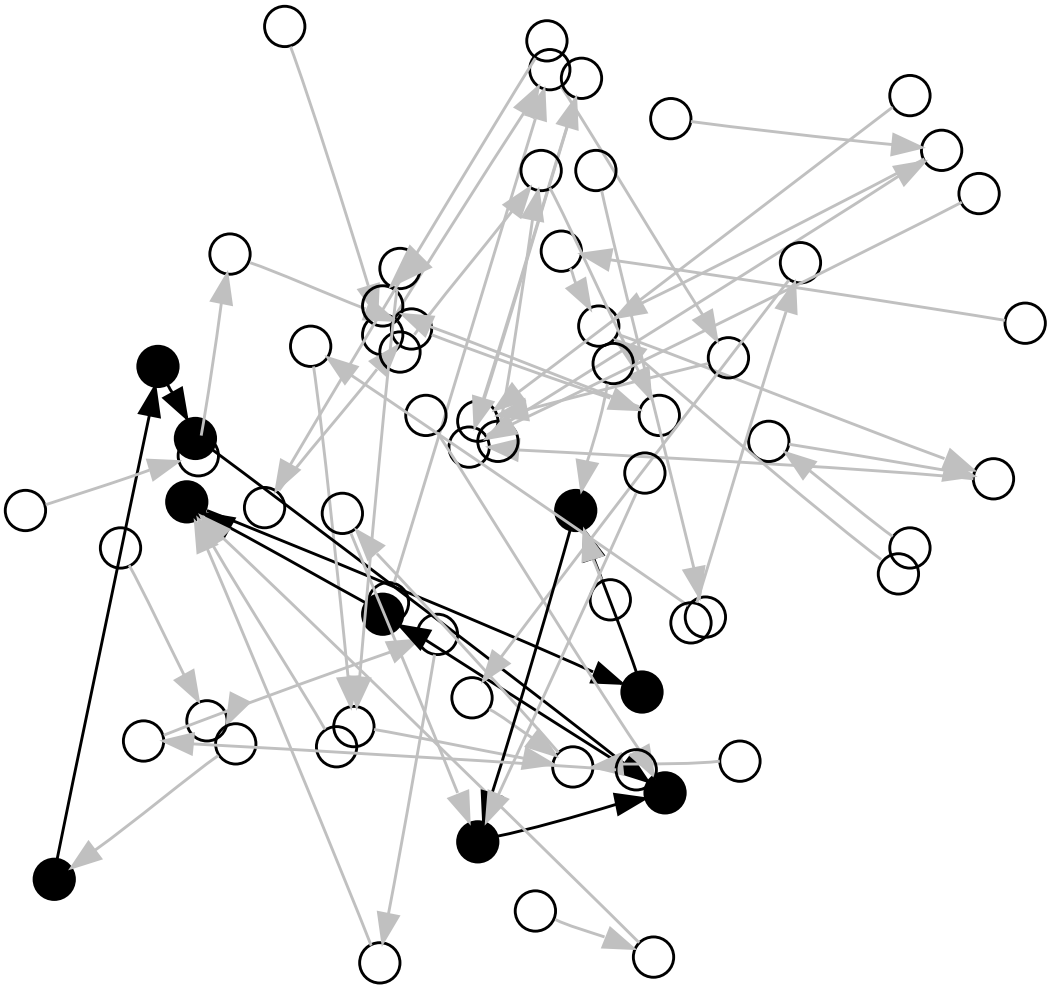


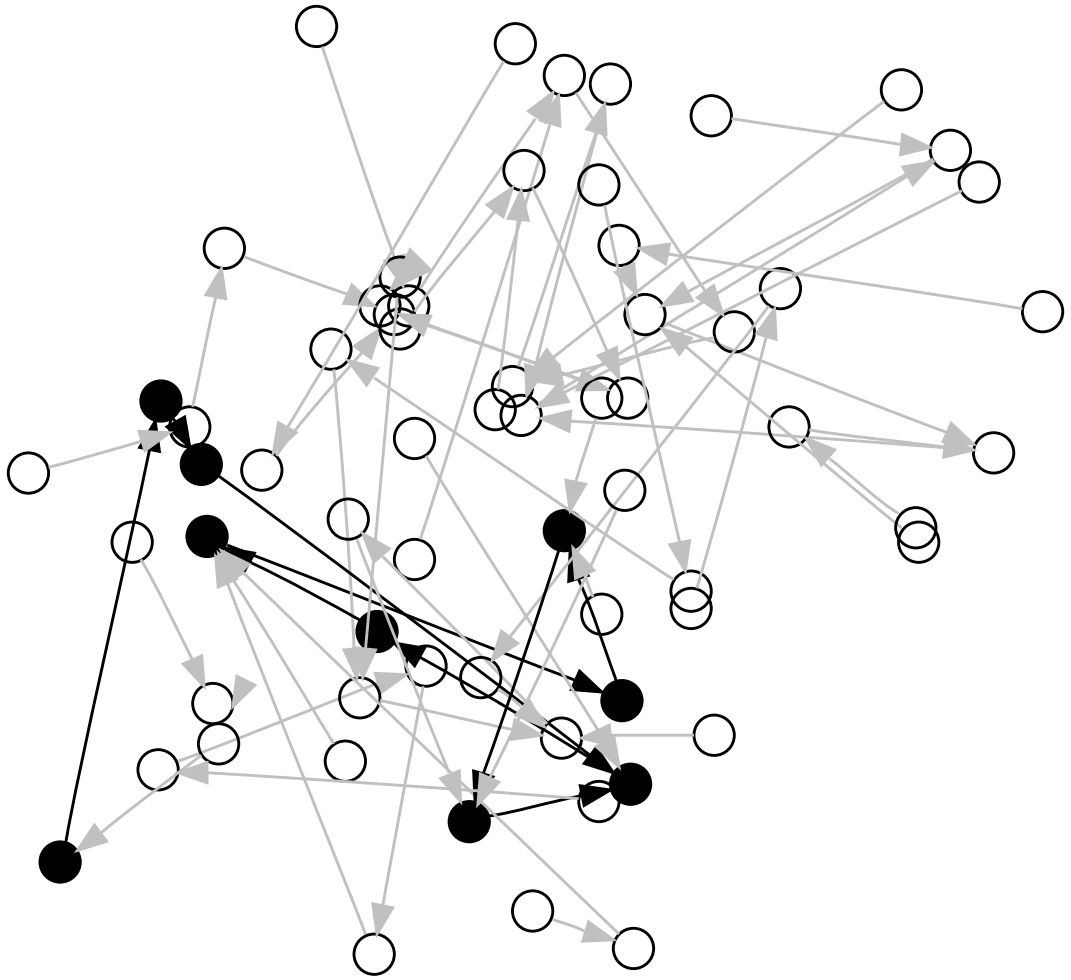


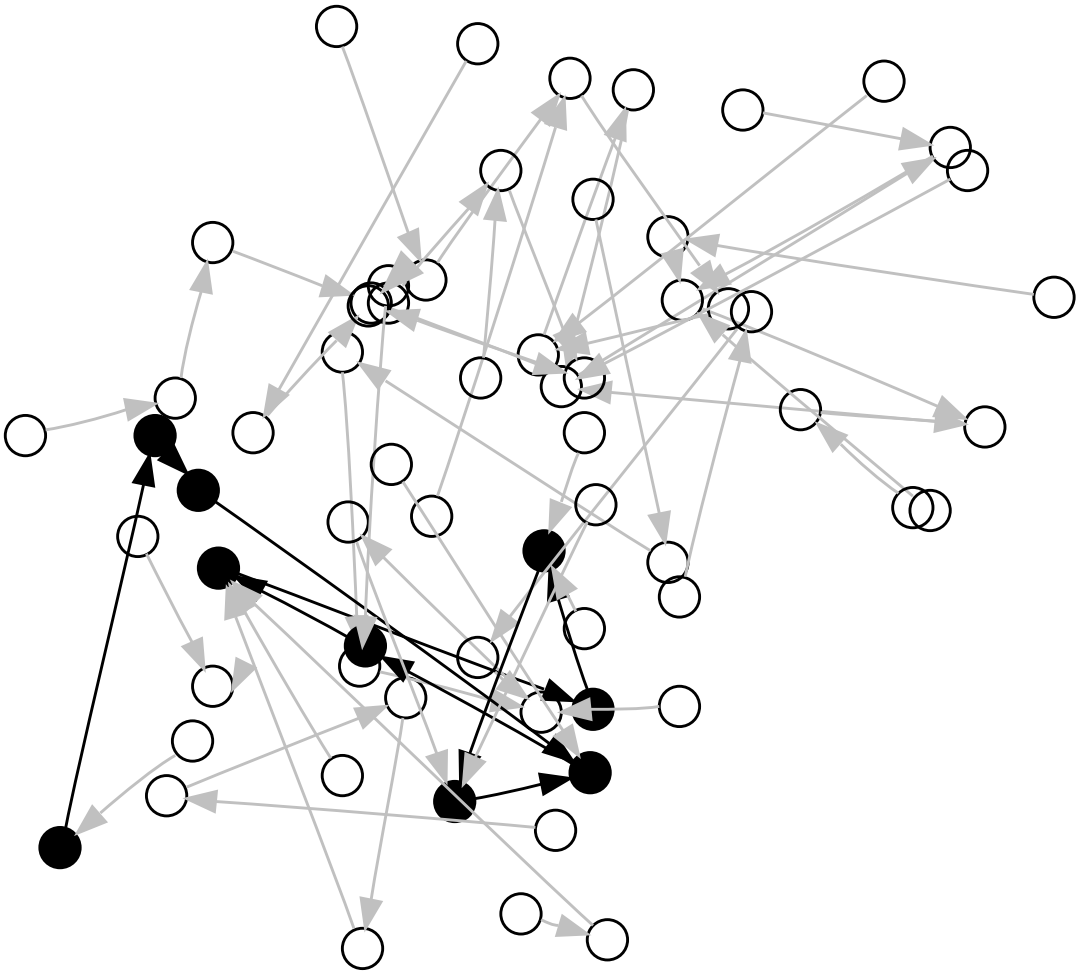


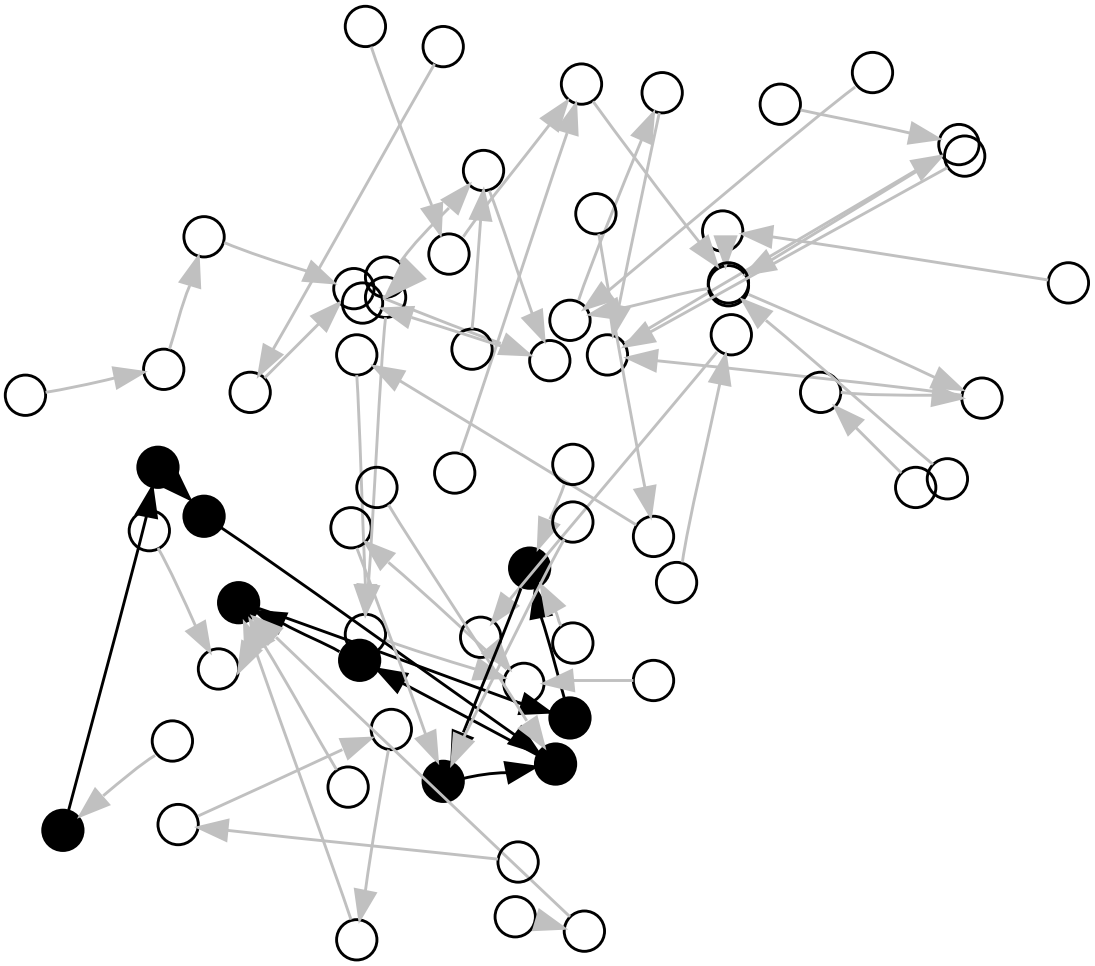


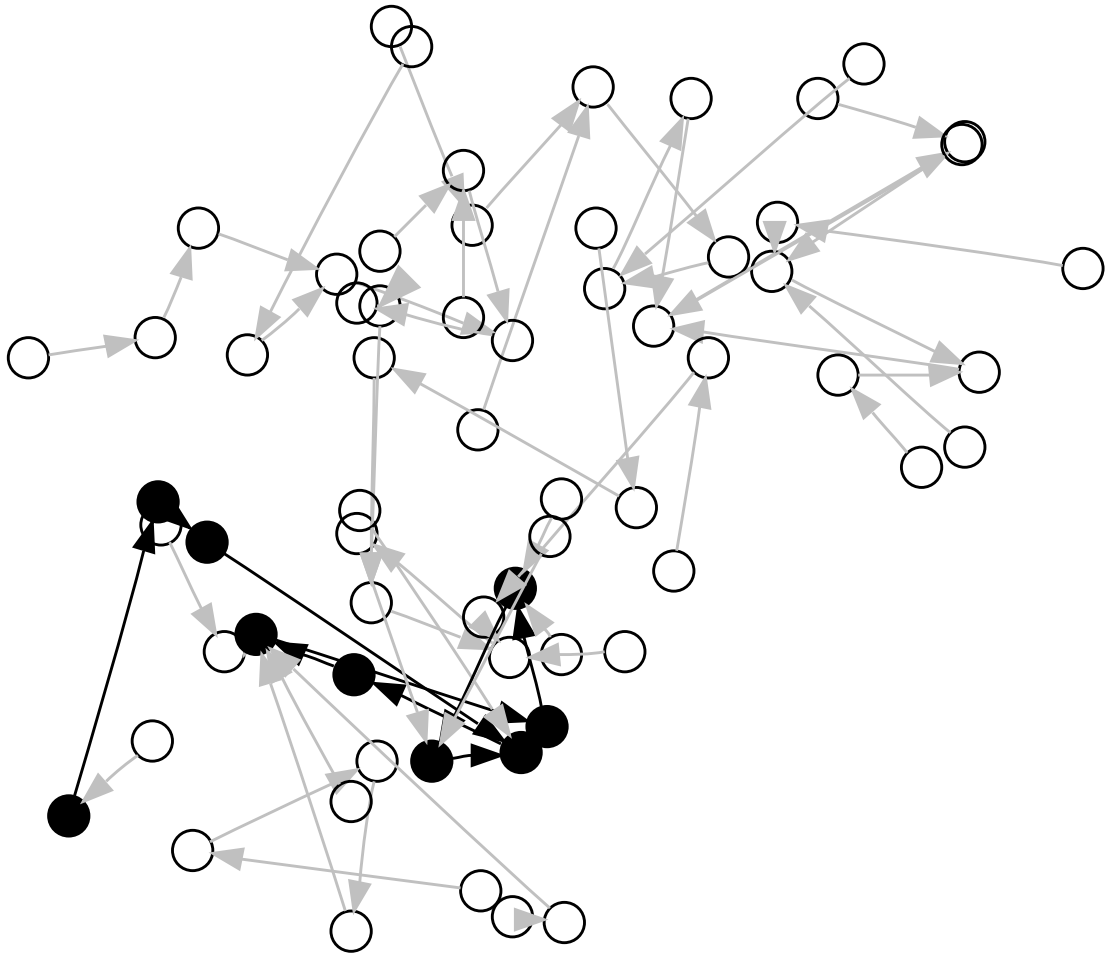


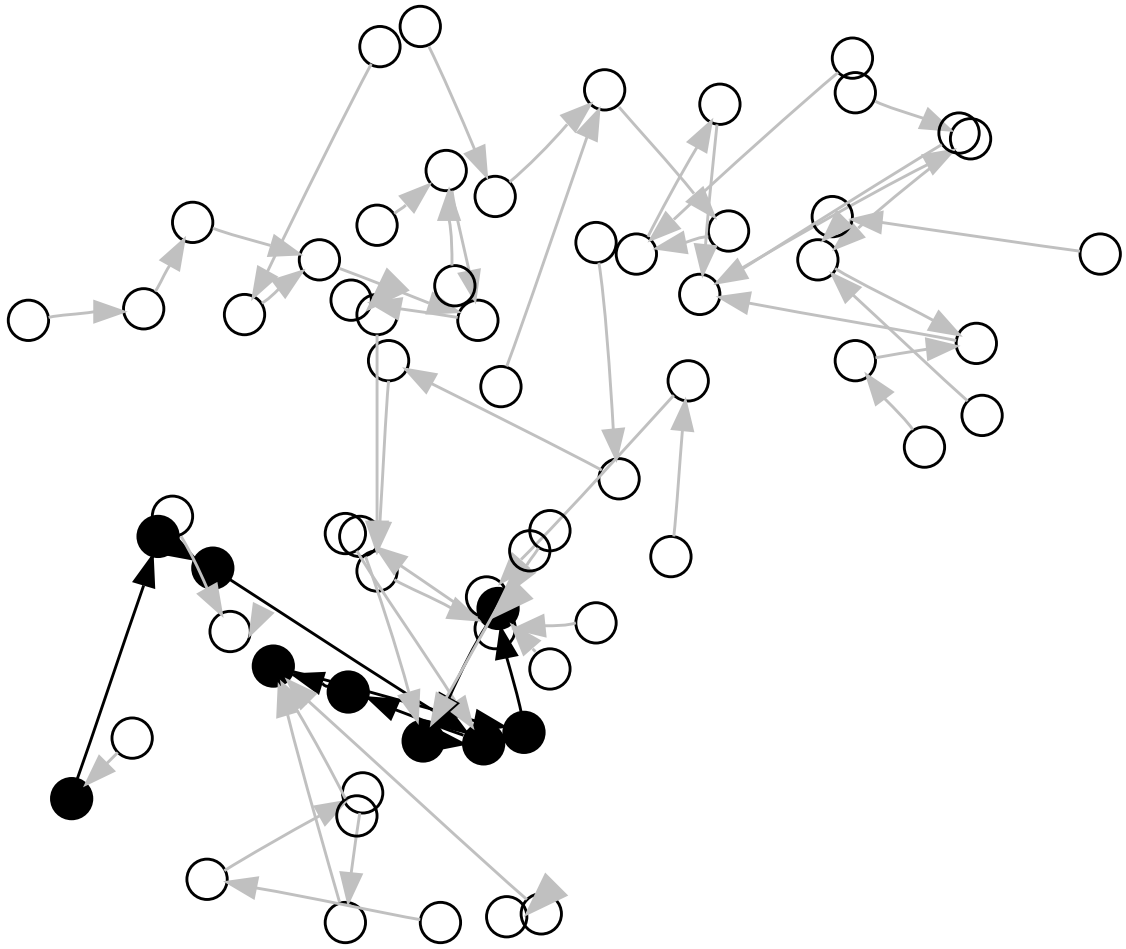


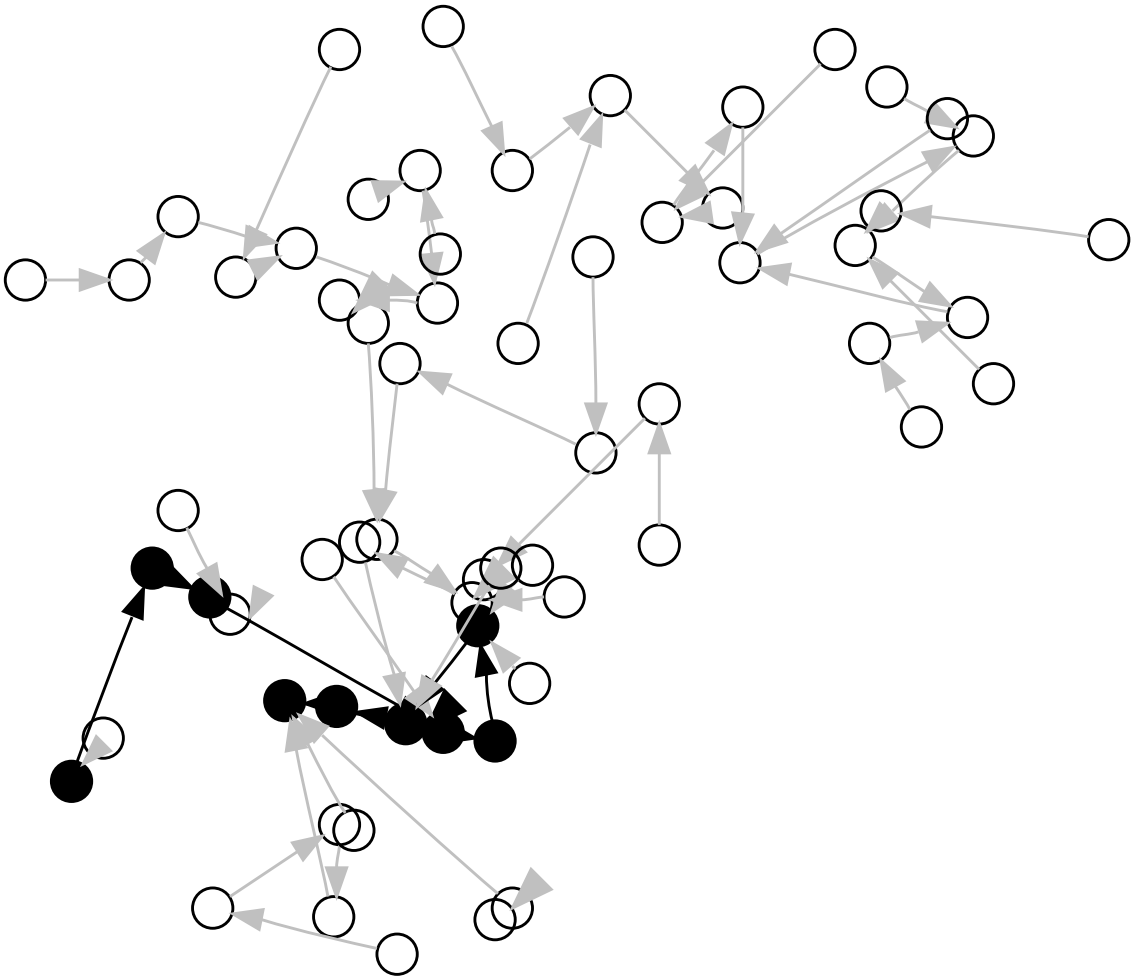


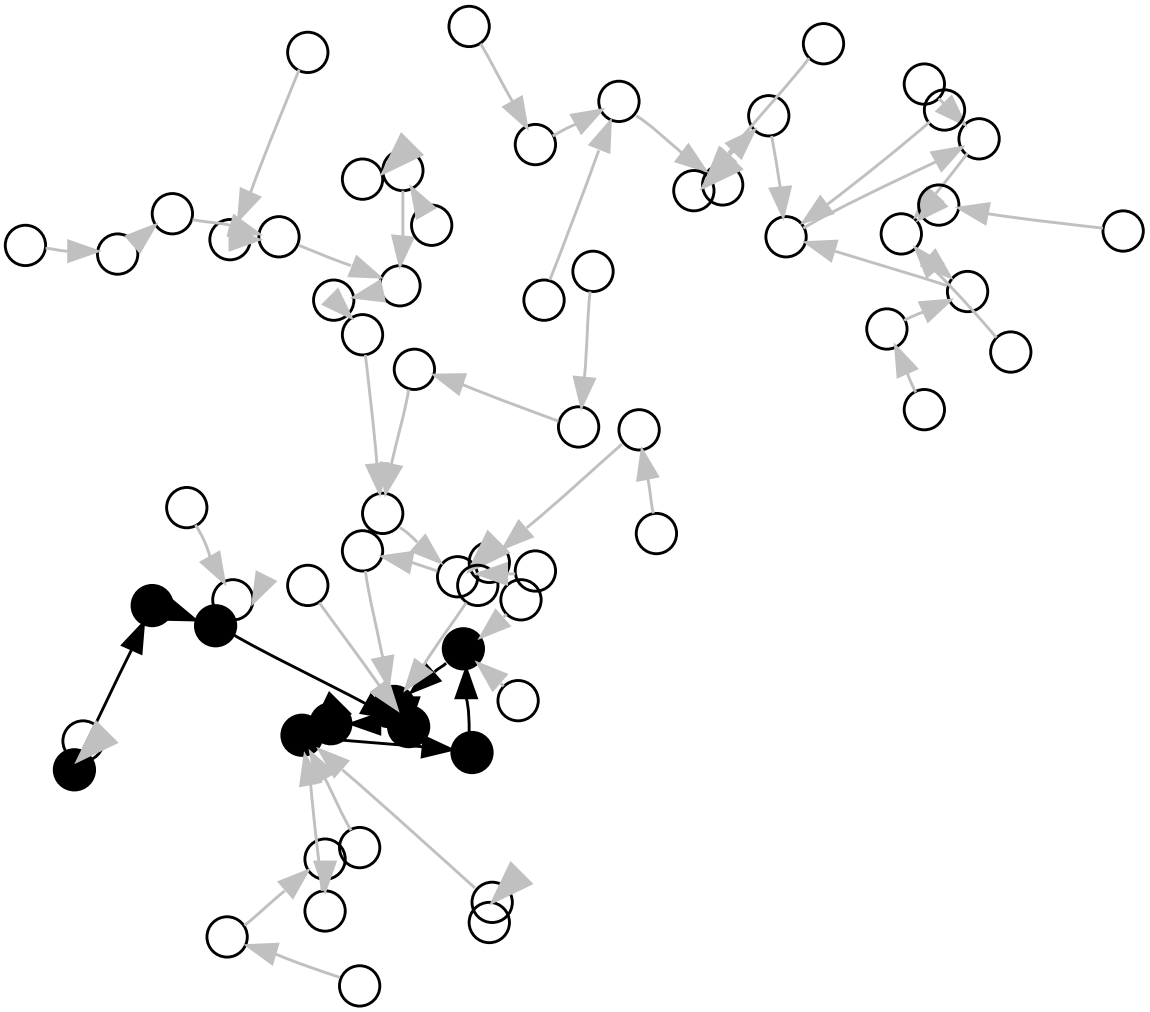


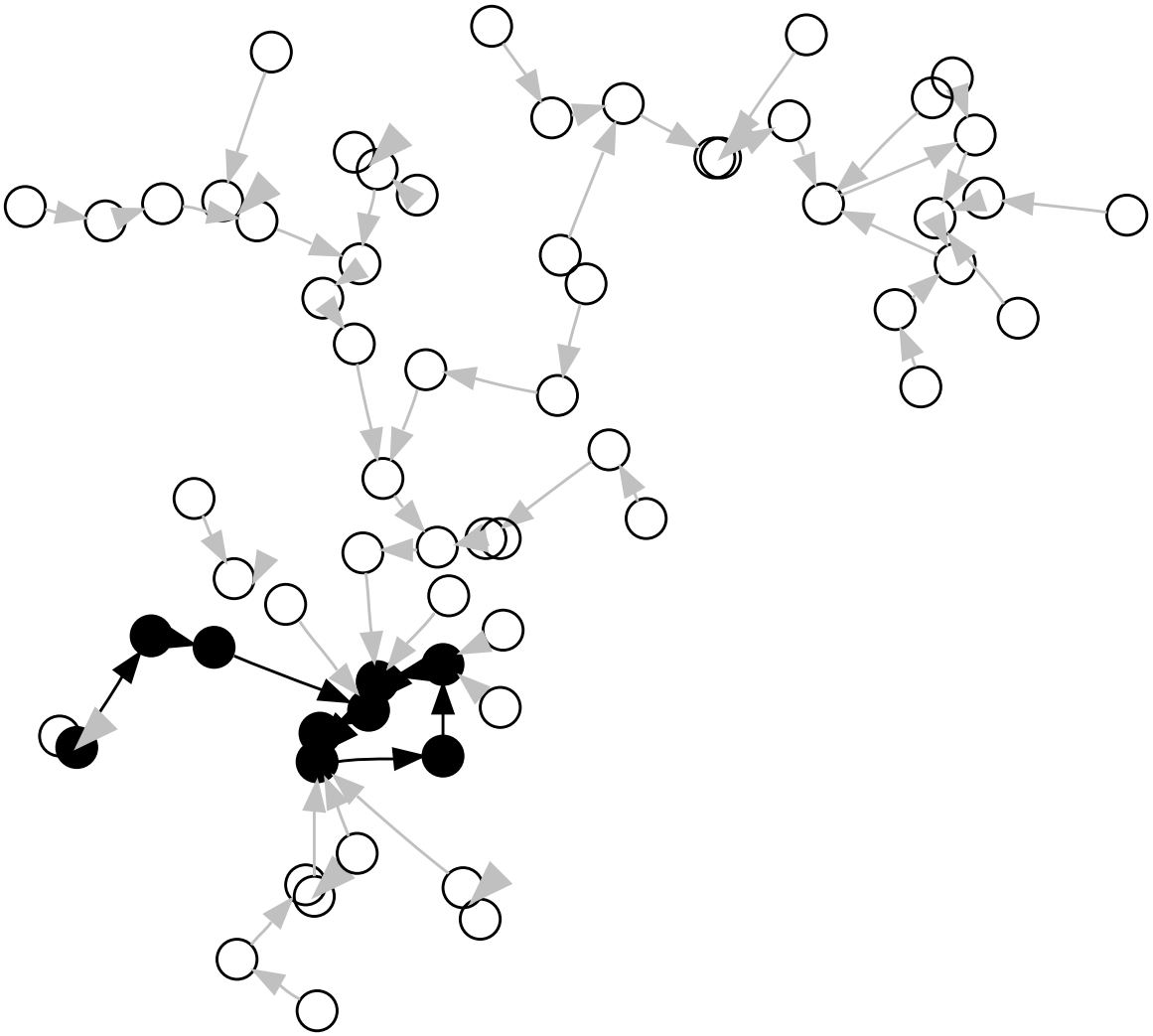


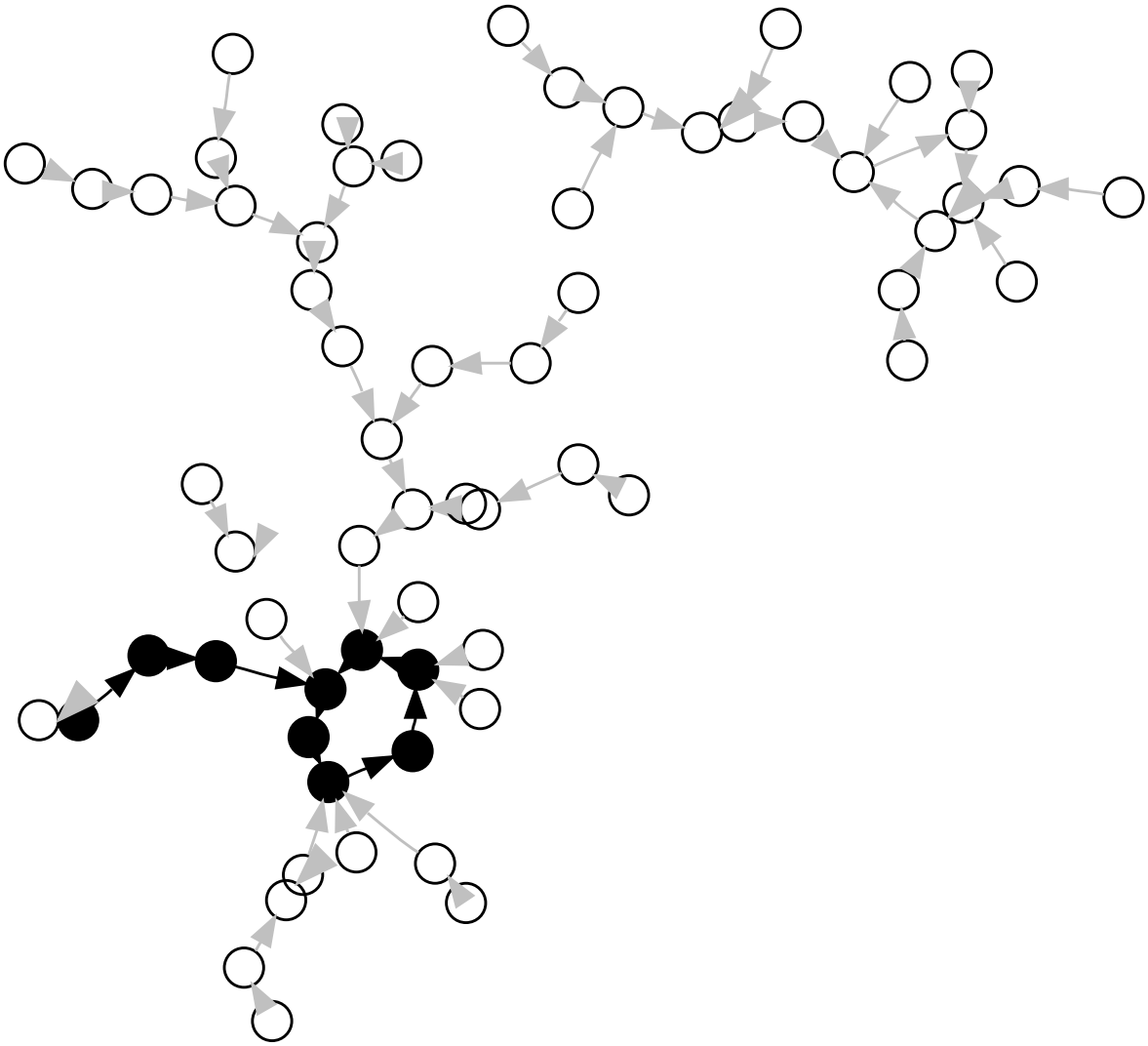


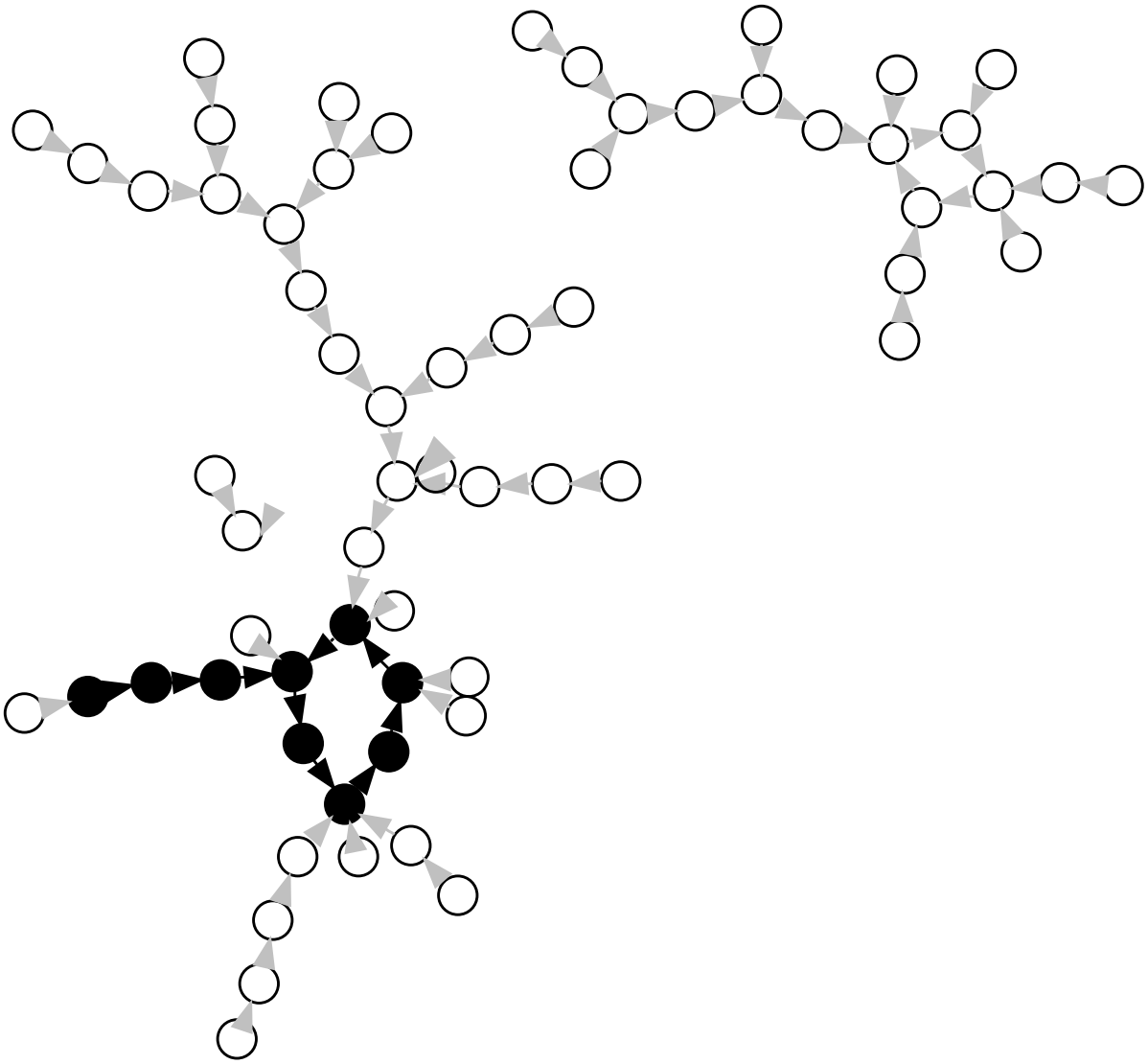












Assume that for each point
we know $a_i, b_i \in \mathbf{Z}/\ell\mathbf{Z}$
so that $W_i = a_i P + b_i Q$.

Then $W_i = W_j$ means that
$a_i P + b_i Q = a_j P + b_j Q$
so $(b_i - b_j)Q = (a_j - a_i)P$.
If $b_i \neq b_j$ the DLP is solved:
$n = (a_j - a_i)/(b_i - b_j)$.

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e.g. $f(W_i) = a(W_i)P + b(W_i)Q$,
starting from some initial
combination $W_0 = a_0 P + b_0 Q$.
If any W_i and W_j collide then
$W_{i+1} = W_{j+1}$, $W_{i+2} = W_{j+2}$,
etc.

If functions $a(W)$ and $b(W)$ are random modulo ℓ, iterations perform a random walk in $\langle P \rangle$.
If a and b are chosen such that $f(W_i) = f(-W_i)$ then the walk is defined on *equivalence classes* under $\pm$.
There are only $\lceil \ell/2 \rceil$ different classes. This reduces the average number of iterations by a factor of almost exactly $\sqrt{2}$.

In general, Pollard's rho method can be combined with any easily computed group automorphism of small order.

Parallel collision search

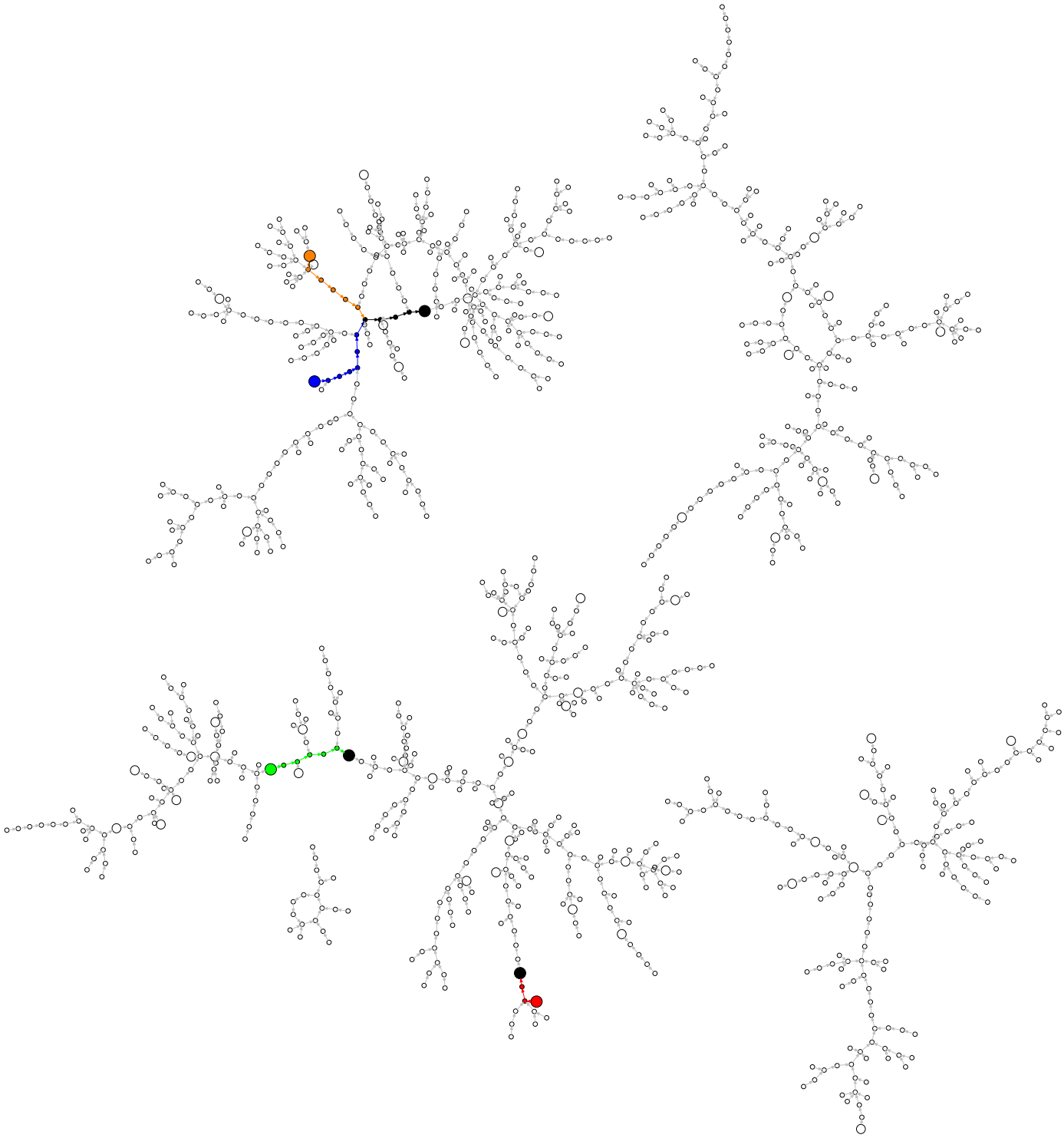
Running Pollard's rho method on N computers gives speedup of $\approx \sqrt{N}$ from increased likelihood of finding collision.

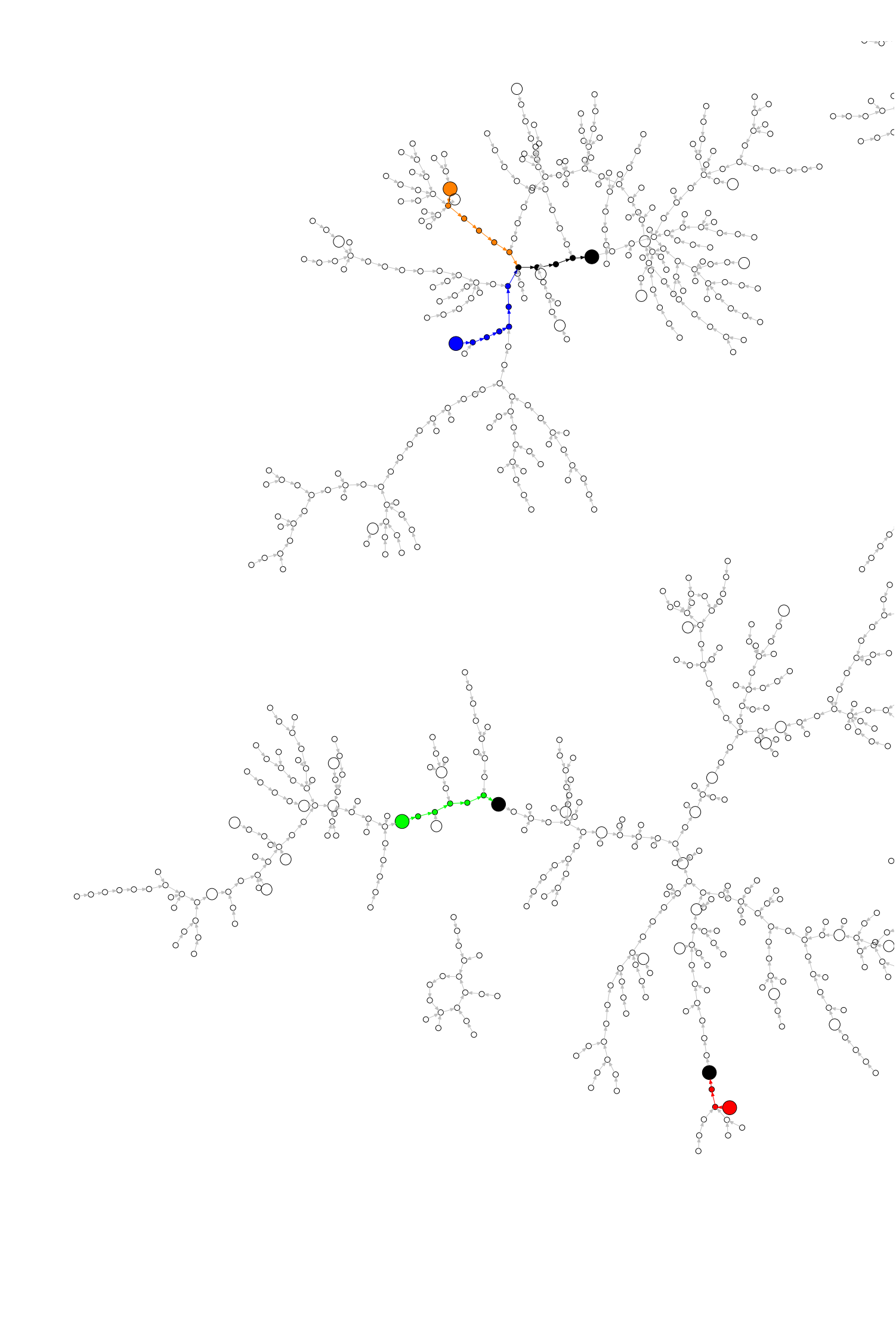
Want better way to spread computation across clients. Want to find collisions between walks on *different* machines, without frequent synchronization!

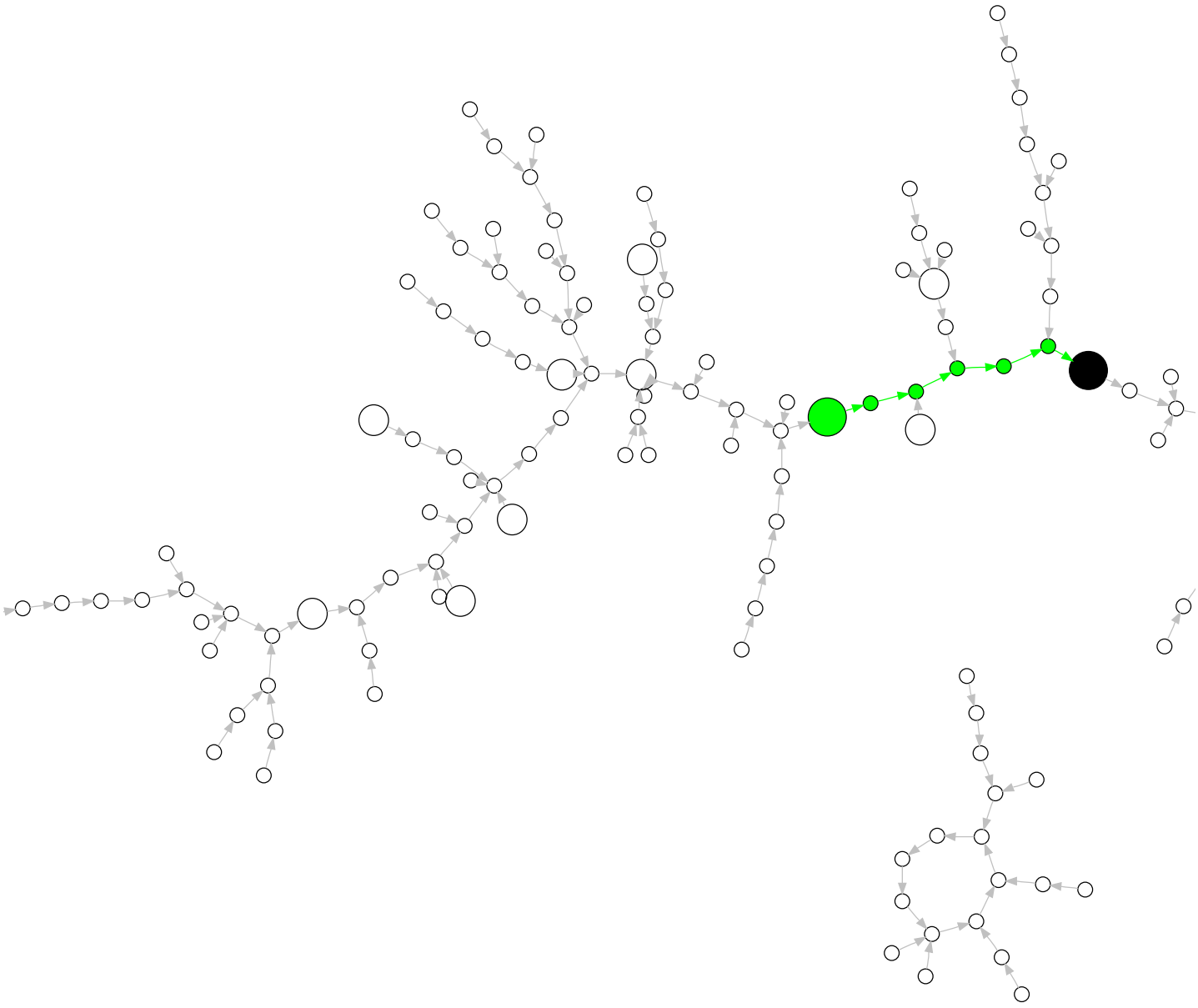
Better method due to van Oorschot and Wiener (1999). Declare some subset of $\langle P \rangle$ to be *distinguished points*.

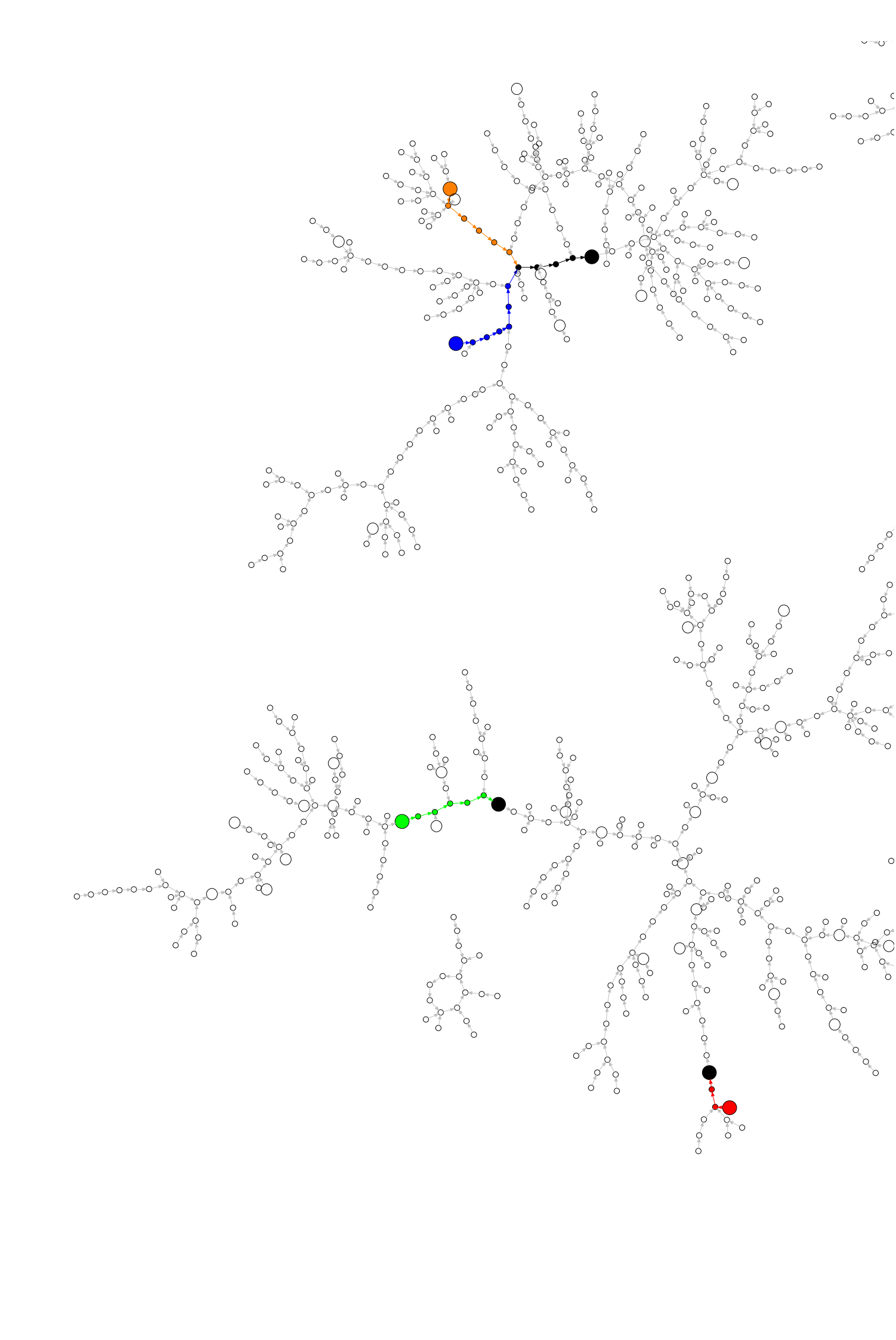
Parallel rho: Perform many walks
with different starting points
but same update function f.
If two different walks
find the same point then
their subsequent steps will match.

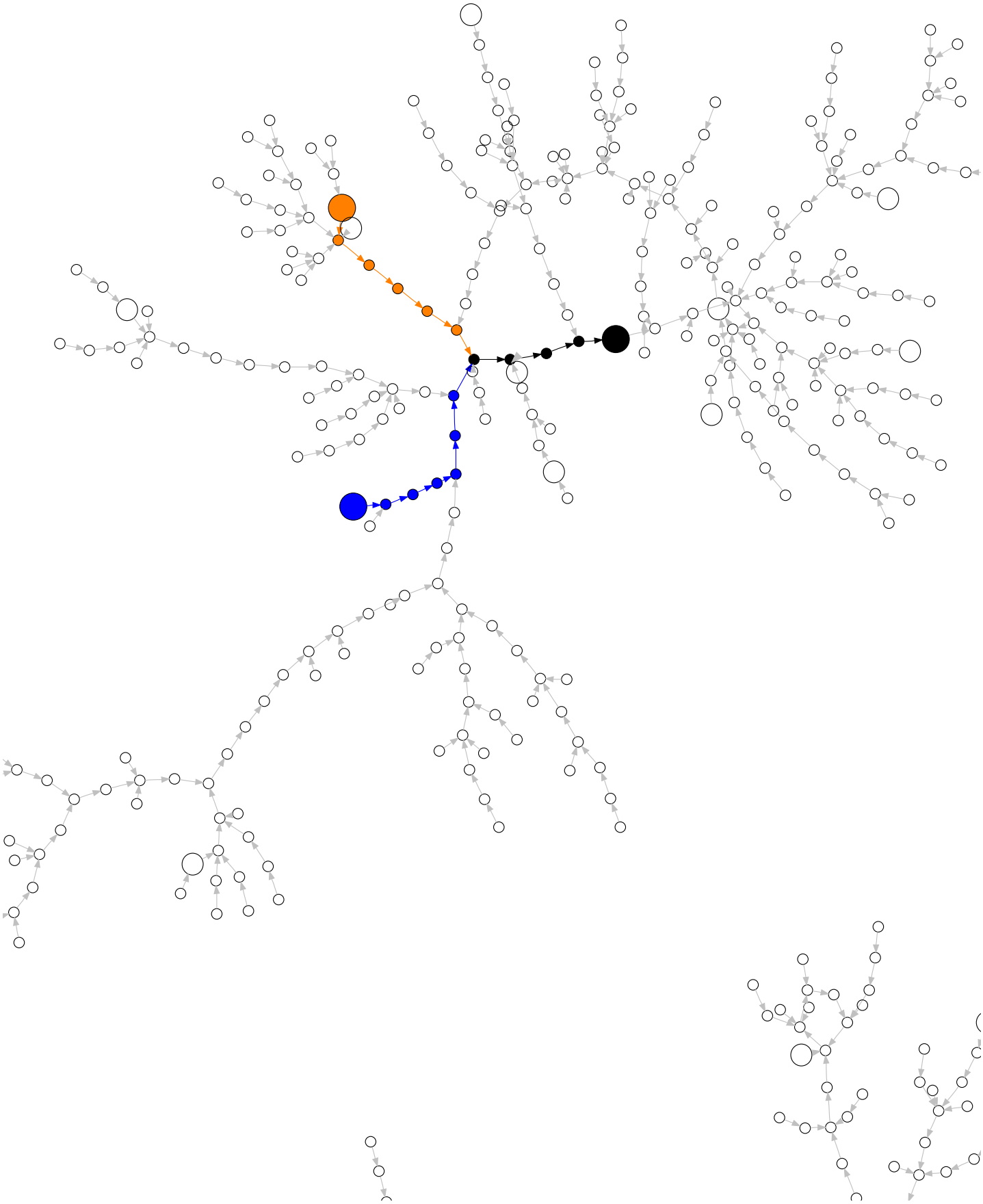
Terminate each walk once it hits
a distinguished point and report
the point along with a_i and b_i to
server.
Server receives, stores, and sorts
all distinguished points.
Two walks reaching same
distinguished point give collision.
This collision solves the DLP.



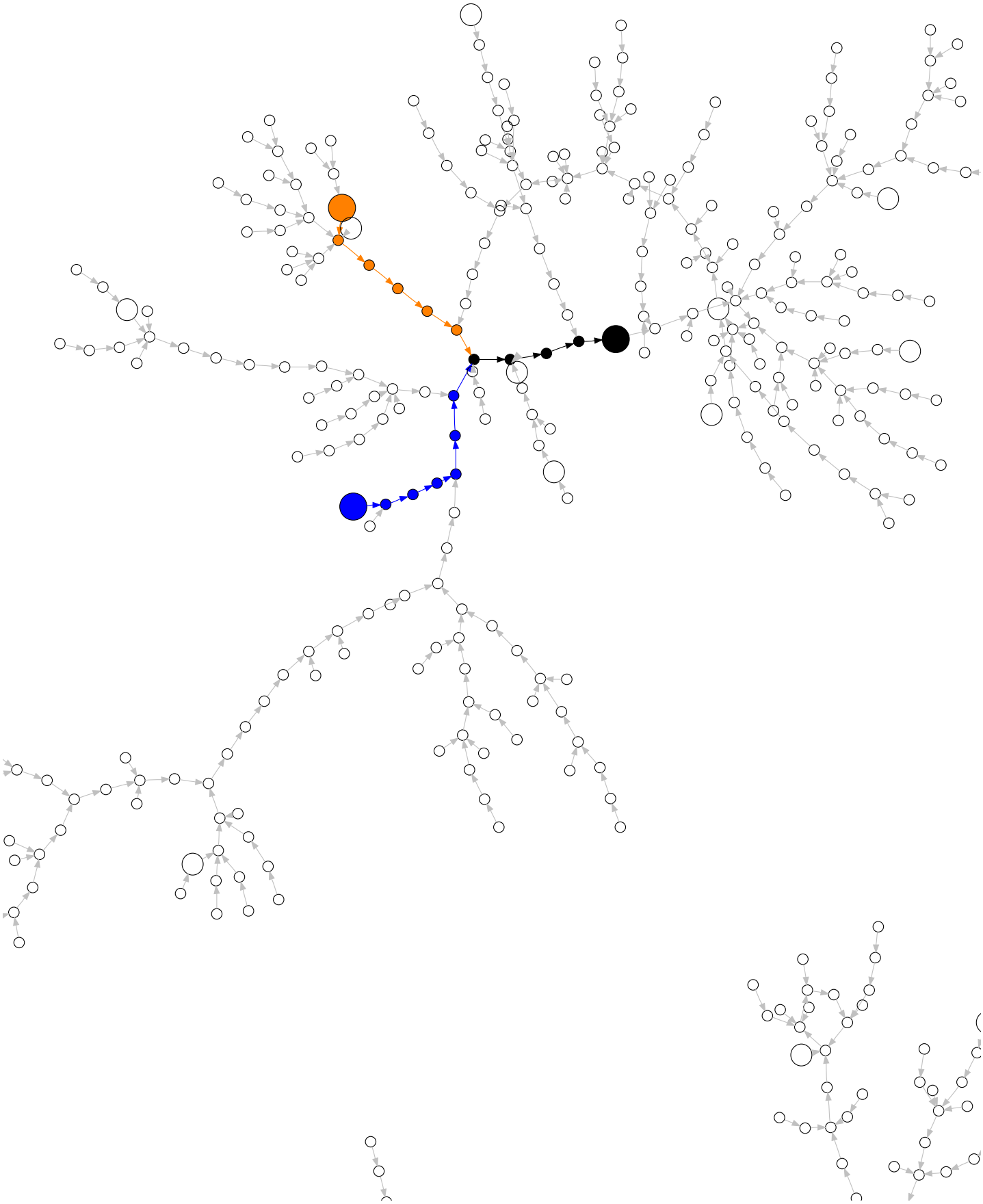








Attacker chooses frequency and definition of distinguished points.
Tradeoffs are possible:
If distinguished points are rare, a small number of very long walks will be performed. This reduces the number of distinguished points sent to the server but increases the delay before a collision is recognized.
If distinguished points are frequent, many shorter walks will be performed.
In any case do not wait for cycle.
Total # of iterations unchanged.



Additive walks

Generic rho method requires two scalar multiplications for each iteration.
Could replace by double-scalar multiplication; could further merge the 2-scalar multiplications across several parallel iterations.

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More efficient: use *additive walk*:
Start with $W_0 = a_0 P$ and put
$f(W_i) = W_i + c_j P + d_j Q$
where $j = h(W_i)$.

Pollard's initial proposal:

Use $x(W_i) \bmod 3$ as h

and update:

$$W_{i+1} = \begin{cases} W_i + P \text{ for } x(W_i) \bmod 3 = 0 \\ 2W_i \quad \text{for } x(W_i) \bmod 3 = 1 \\ W_i + Q \text{ for } x(W_i) \bmod 3 = 2 \end{cases}$$

Easy to update a_i and b_i.

$$(a_{i+1}, b_{i+1}) = \begin{cases} (a_i + 1, b_i) \text{ for } x(W_i) \bmod 3 = 0 \\ (2a_i, 2b_i) \quad \text{for } x(W_i) \bmod 3 = 1 \\ (a_i, b_i + 1) \text{ for } x(W_i) \bmod 3 = 2 \end{cases}$$

Additive walk requires only one addition per iteration.
h maps from $\langle P \rangle$ to $\{0, 1, \ldots, r-1\}$, and $R_j = c_j P + d_j Q$ are precomputed for each $j \in \{0, 1, \ldots, r-1\}$.

Easy coefficient update:
$W_i = a_i P + b_i Q$,
where a_i and b_i are defined recursively as follows:
$a_{i+1} = a_i + c_{h(W_i)}$ and
$b_{i+1} = b_i + d_{h(W_i)}$.

Additive walks have disadvantages:
The walks are noticeably nonrandom; this means they need more iterations than the generic rho method to find a collision.

This effect disappears as r grows, but but then the precomputed table $R_0, \ldots, R_{r-1}$ does not fit into fast memory. This depends on the platform, e.g. trouble for GPUs.
More trouble with adding walks later.

Randomness of adding walks

Let $h(W) = i$ with probability p_i.

Fix a point T, and let W and W' be two independent uniform random points.

Let $W \neq W'$ both map to T.

This event occurs if

Randomness of adding walks

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Fix a point T, and let W and W' be two independent uniform random points.

Let $W \neq W'$ both map to T.

This event occurs if simultaneously for $i \neq j$:
$T = W + R_i = W' + R_j$;
$h(W) = i$; $h(W') = j$.

These conditions have probability $1/\ell^2$, p_i, and p_j respectively.

Summing over all (i, j)
gives the overall probability
$\left(\sum_{i \neq j} p_i p_j\right) / \ell^2 \quad =$
$\left(\sum_{i,j} p_i p_j - \sum_i p_i^2\right) / \ell^2 \; =$
$\left(1 - \sum_i p_i^2\right) / \ell^2$.

This means that the probability of an immediate collision from W and W' is $\left(1 - \sum_i p_i^2\right) / \ell$, where we added over the ℓ choices of T. In the simple case that all the p_i are $1/r$, the difference from the optimal $\sqrt{\pi \ell / 2}$ iterations is a factor of $1/\sqrt{1 - 1/r} \approx 1 + 1/(2r)$.

Various heuristics leading to standard $\sqrt{1-1/r}$ formula in different ways:

1981 Brent–Pollard;

2001 Teske;

2009 ECC2K-130 paper, eprint 2009/541.

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2010 Bernstein–Lange:
Standard formula is wrong!
There is a further slowdown
from higher-order anti-collisions:
e.g. $W + R_i + R_k \neq W' + R_j + R_l$
if $R_i + R_k = R_j + R_l$.
$\approx 1\%$ slowdown for ECC2K-130.

Eliminating storage

Usual description: each walk
keeps track of a_i and b_i
with $W_i = a_i P + b_i Q$.

This requires each client to
implement arithmetic modulo ℓ
or at least keep track of
how often each R_j is used.

For distinguished points
these values are
transmitted to server (bandwidth)
which stores them as
e.g. (W_i, a_i, b_i) (space).

2009 ECC2K-130 paper:
Remember where you started.
If $W_i = W_j$ is the collision of distinguished points,
can recompute these walks with a_i, b_i, a_j, and b_j;
walk is deterministic!
Server stores 2^{45} distinguished points; only needs to know coefficients for 2 of them.

Our setup: Each walk remembers seed; server stores distinguished point and seed.
Saves time, bandwidth, space.

Negation and rho

$W = (x, y)$ and $-W = (x, -y)$
have same x-coordinate.
Search for x-coordinate collision.

Search space for collisions is
only $\lceil \ell/2 \rceil$; this gives factor $\sqrt{2}$
speedup . . . if $f(W_i) = f(-W_i)$.

To ensure $f(W_i) = f(-W_i)$:
Define $j = h(|\, W_i \,|)$ and
$f(W_i) = |\, W_i \,| + c_j P + d_j Q$,
with, e.g., $|\, W_i \,|$ the lexicographic
minimum of $W_i, -W_i$.
This negation speedup
is textbook material.

Problem: this walk can
run into fruitless cycles!
Example: If $\mid W_{i+1} \mid = -W_{i+1}$
and $h(\mid W_{i+1} \mid) = j = h(\mid W_i \mid)$
then $W_{i+2} = f(W_{i+1}) =$
$-W_{i+1} + c_j P + d_j Q =$
$-(\mid W_i \mid + c_j P + d_j Q) + c_j P +$
$d_j Q = - \mid W_i \mid$ so $\mid W_{i+2} \mid = \mid W_i \mid$
so $W_{i+3} = W_{i+1}$
so $W_{i+4} = W_{i+2}$ etc.

If h maps to r different values
then expect this example to occur
with probability $1/(2r)$
at each step.
Known issue, not quite textbook.

Eliminating fruitless cycles

Issue of fruitless cycles is known and several fixes are proposed. See appendix of full version ePrint 2011/003 for even more details and historical comments.

Summary: most of them got it wrong.

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Summary: most of them got it wrong.

So what to do?
Choose a big r, e.g. $r = 2048$.
$1/(2r) = 1/4096$ small;
cycles infrequent.

Define $|(x, y)|$ to mean
(x, y) for $y \in \{0, 2, 4, \ldots, p-1\}$
or
$(x, -y)$ for $y \in \{1, 3, 5, \ldots, p-2\}$.

Precompute points
$R_0, R_1, \ldots, R_{r-1}$ as known
random multiples of P.

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Precompute points
$R_0, R_1, \ldots, R_{r-1}$ as known
random multiples of P.
Can do full scalar multiplication in
inversion-free coordinates!
Start each walk at a point
$W_0 = |\, b_0 Q \,|$,
where b_0 is chosen randomly.
Compute $W_1, W_2, \ldots$ as
$W_{i+1} = |\, W_i + R_{h(W_i)} \,|$.

Occasionally, every w iterations,
check for fruitless cycles
of length 2.
For those cases change the
definition of W_i as follows:
Compute W_{i-1} and check
whether $W_{i-1} = W_{i-3}$.
If $W_{i-1} \neq W_{i-3}$, put $W_i = W_{i-1}$.
If $W_{i-1} = W_{i-3}$, put
$W_i = |\, 2 \min\{W_{i-1}, W_{i-2}\} \,|$,
where min means
lexicographic minimum.
Doubling the point
makes it escape the cycle.

Cycles of length 4, 6, or 12
occur far less frequently.
Cycles of length 4, or 6
are detected when checking
for cycles of length 12;
so skip individual ones.

Same way of escape:
define $W_i =$
$| \, 2\min\{W_{i-1}, W_{i-2}, W_{i-3}, W_{i-4},$
$W_{i-5}, W_{i-6}, W_{i-7}, W_{i-8},$
$W_{i-9}, W_{i-10}, W_{i-11}, W_{i-12}\} \, |$
if trapped
and $W_i = W_{i-1}$ otherwise.

Do not store all these points!

When checking for cycle,
store only potential entry point
W_{i-13} (one coordinate, for
comparison) and the
smallest point encountered since
(to escape).

For large DLP
look for larger cycles;
in general, look for
fruitless cycles of even lengths
up to $\approx (\log \ell)/(\log r)$.

How to choose w?

Fruitless cycles of length 2 appear
with probability $\approx 1/(2r)$.
These cycles persist
until detected.
After w iterations,
probability of cycle $\approx w/(2r)$,
wastes $\approx w/2$ iterations
(on average) if it does appear.

Do not choose w
as small as possible!
If a cycle has *not* appeared then
the check wastes an iteration.

The overall loss is approximately
$1 + w^2/(4r)$ iterations out of w.
To minimize the quotient
$1/w + w/(4r)$ we take $w \approx 2\sqrt{r}$.

Cycles of length $2c$ appear with
probability $\approx 1/r^c$,
optimal checking frequency is
$\approx 1/r^{c/2}$.
Loss rapidly disappears
as c increases.
Can use lcm of cycle lengths
to check.

Concrete example: 112-bit DLP

Use $r = 2048$. Check for 2-cycles every 48 iterations.
Check for larger cycles much less frequently.
Unify the checks for 4-cycles and 6-cycles into a check for 12-cycles every 49152 iterations.

Choice of r has big impact!
$r = 512$ calls for checking
for 2-cycles every 24 iterations.
In general, negation overhead
$\approx$ doubles when table size
is reduced by factor of 4.

Bernstein, Lange, Schwabe
(PKC 2011):

Our software solves
random ECDL on the same curve
(with no precomputation)
in 35.6 PS3 years on average.

For comparison:
Bos–Kaihara–Kleinjung–Lenstra–
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First big speedup:
We use the negation map.
Second speedup: Fast arithmetic.

Why are we confident this works?

We only have 1 PlayStation-3,
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Don't want to wait for 36 years
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Can produced scaled versions:
Use *same* prime field
(so that we can compare the field
arithmetic) and same curve shape
$y^2 = x^3 - 3x + b$
but vary b to get curves with
small subgroups.

This produces other curves, and many of those have smaller order subgroups.
Specify DLP in subgroup of size 2^{50}, or 2^{55}, or 2^{60} and show that the actual running time matches the expectation.
And that DLP is correct.

We used same property for a point to be distinguished as in big attack; probability is 2^{-20}.
Need to watch out that walks do not run into rho-type cycles (artefact of small group order).
We aborted overlong walks.

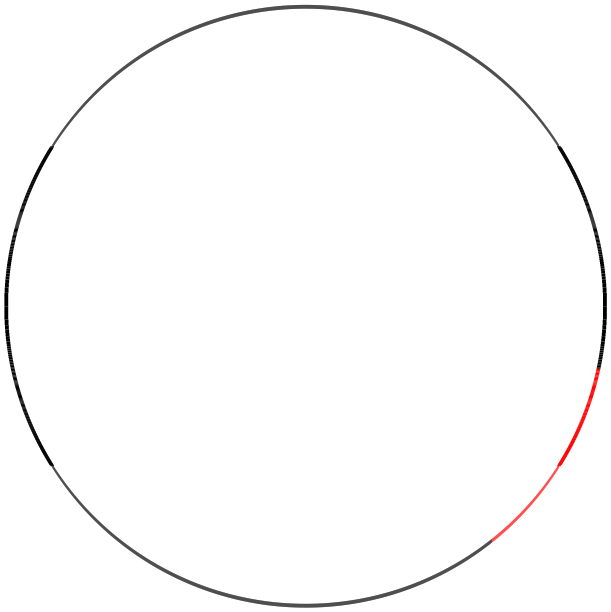
New record

Announced 29 Nov 2016,
most work by Ruben Niederhagen
(`@cryptocephaly` on twitter).

Elliptic curve over $\mathbf{F}_{2^{127}}$,
DLP in subgroup of order $2^{117.35}$.
Used parallel Pollard rho,
DP criterion: 30 top bits equal 0.

Expected
$\sqrt{\pi 2^{117.35}/4}/2^{30} \sim 379\,821\,956$
DPs, but ended up needing
968 531 433.
Computations ran on 64 to 576
FPGAs in parallel.

DLs in intervals



Want to use knowledge that DL is in a small interval $[a, b]$, much smaller than ℓ.

We can use this in baby-step giant-step algorithm.

How to use this in a memory-less algorithm?

Standard interval method:
Pollard's kangaroo method.

Pollard's kangaroos do small jumps around the interval.

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Real kangaroos sleep

Standard interval method:
Pollard's kangaroo method.

Pollard's kangaroos do small jumps around the interval.

Real kangaroos sleep

(at least outside Australia).

Kangaroo method

in Australia

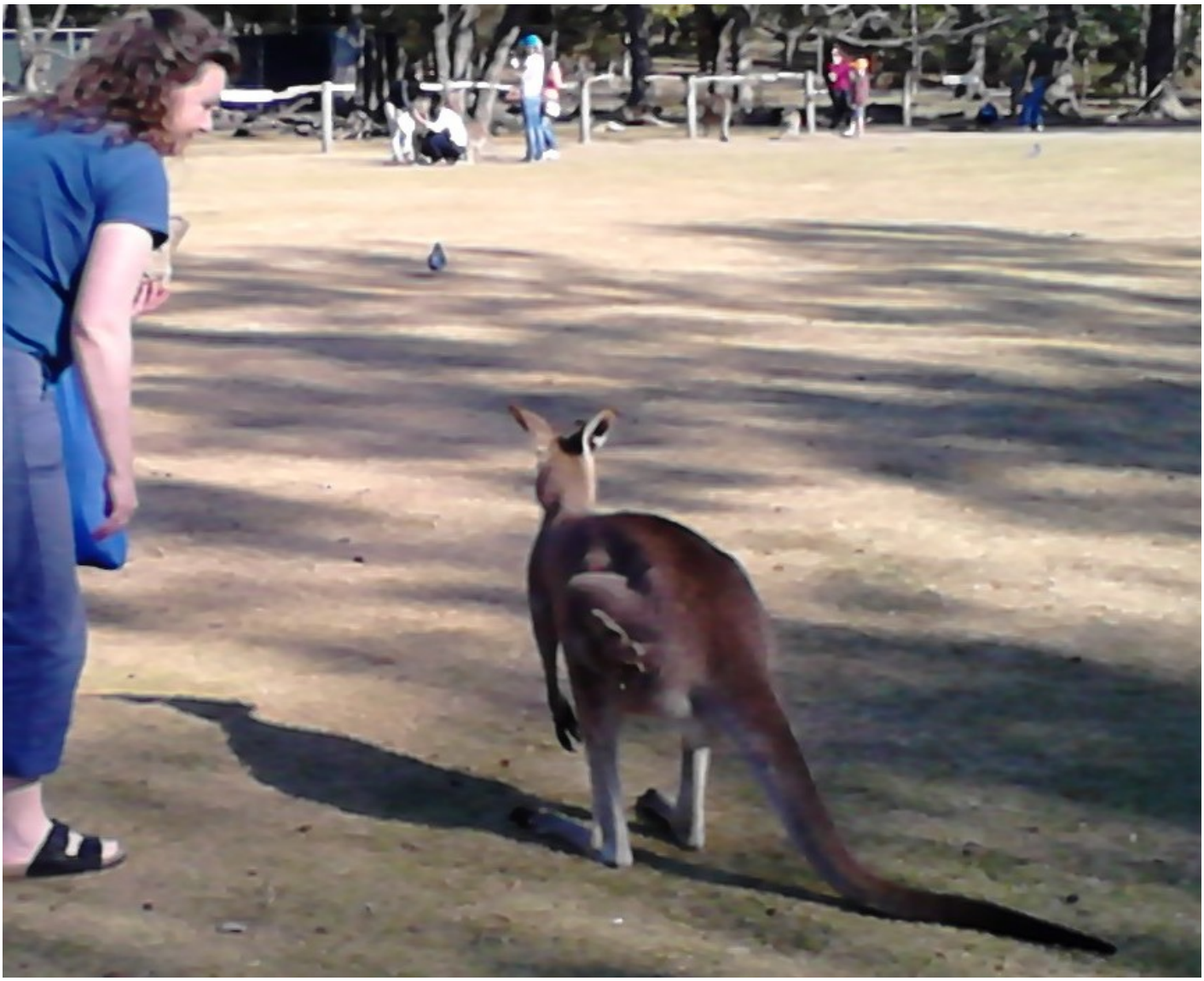
Main actor:

The tame kangaroo

starts at a known

multiple of P, e.g. bP.

The tame kangaroo jumps.



Jumps are determined
by current position.

The tame kangaroo jumps.

Jumps are determined by current position.

Average jump distance is $\sqrt{b-a}$.

The tame kangaroo jumps.

Jumps are determined by current position.

Average jump distance is $\sqrt{b-a}$.

The tame kangaroo jumps.

Jumps are determined by current position.

Average jump distance is $\sqrt{b-a}$.

The tame kangaroo stops

after a fixed number of jumps (about $\sqrt{b-a}$ many).

The tame kangaroo installs a trap and waits.

The wild kangaroo

starts at point Q.

Follows the same instructions for jumps.

But we don't know where
the starting point Q is.
Know $Q = nP$ with $n \in [a, b]$.

Hope that the paths of the tame
and wild kangaroo intersect.

Similar to the rho method the
kangaroos will hop on the same
path from that point onwards.

Eventually the wild kangaroo falls
into the trap.
(Or disappears in the distance if
paths have not intersected.
Start a fresh one
from $Q + P, Q + 2P, \ldots$)

Same story in math

Kangaroo = sequence $X_i \in \langle P \rangle$.

Starting point $X_0 = s_0 P$.

Distance $d_0 = 0$.

Step set: $S = \{s_1 P, \ldots, s_L P\}$,

with s_i on average

$s = \beta\sqrt{b-a}$.

Hash function

$H : \langle P \rangle \to \{1, 2, \ldots, L\}$.

Update function

$d_{i+1} = d_i + s_{H(X_i)}, \quad i = 0, 1, 2, \ldots,$

$X_{i+1} = X_i + s_{H(X_i)} P, \quad i = 0, 1, 2, \ldots$

Tame kangaroo starts at

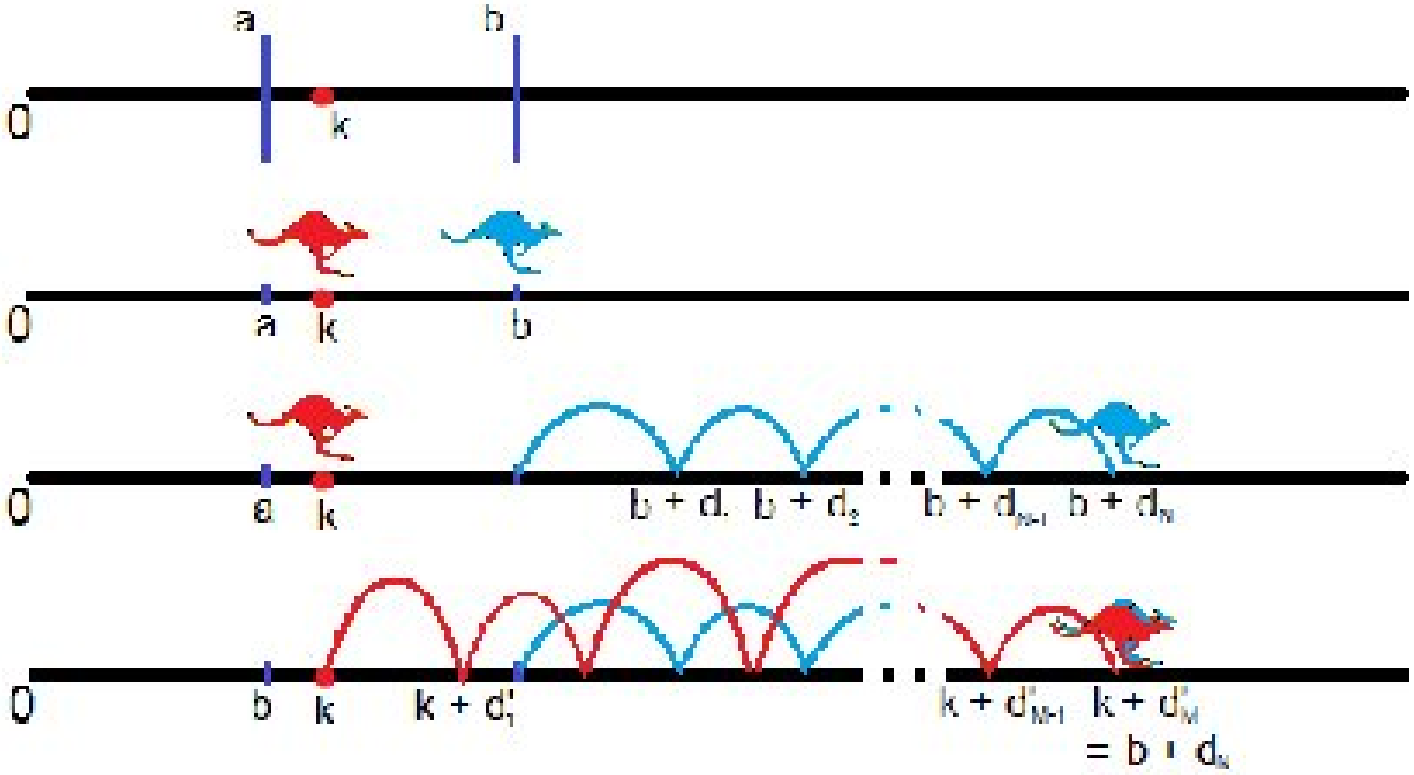
$X_0 = bP$,

wild kangaroo starts at

$X'_0 = Q = nP$.

Trap: distance d_N,

endpoint $X_N = (b + d_N)P$.



Picture credit:

Christine van Vredendaal.

Parallel kangaroo method

Use an entire herd

of tame kangaroos,

all starting

around $((b-a)/2)P$ ...

. . . and define certain spots as distinguished points

Also start a herd of
wild kangaroos around Q.
Hope that one wild and
one tame kangaroo
meet at one distinguished point.

Pairings

Let $(G_1, +), (G_2, +)$ and $(G_T, \cdot)$ be groups of prime order ℓ and let $e : G_1 \times G_2 \to G_T$ be a map satisfying
$e(P + Q, R') = e(P, R')e(Q, R')$,
$e(P, R' + S') = e(P, R')e(P, S')$.

Request further that e is non-degenerate in the first argument, i.e., if for some P $e(P, R') = 1$ for all $R' \in G_2$, then P is the identity in G_1

Such an e is called a *bilinear map* or *pairing*.

Consequences of pairings

Assume that $G_1 = G_2$,
in particular $e(P, P) \neq 1$.

Then for all triples
$(P_1, P_2, P_3) \in \langle P \rangle^3$
one can decide in time polynomial
in $\log \ell$ whether
$\log_P(P_3) = \log_P(P_1) \log_P(P_2)$
by comparing
$e(P_1, P_2)$ and $e(P, P_3)$.
This means that the decisional
Diffie-Hellman problem is easy.

The DL system G_1 is at most as secure as the system G_T.

Even if $G_1 \neq G_2$ one can transfer the DLP in G_1 to a DLP in G_T, provided one can find an element $P' \in G_2$ such that the map $P \to e(P, P')$ is injective.

Pairings are interesting attack tool if DLP in G_T is easier to solve; e.g. if G_T has index calculus attacks.

We want to define pairings
$G_1 \times G_2 \to G_T$
preserving the group structure.

The pairings we will use
map to the multiplicative group of
a finite extension field $\mathbf{F}_{q^k}$.
More precisely, $G_T \subset \mathbf{F}_{q^k}$, order ℓ.

To embed the points of order ℓ
into $\mathbf{F}_{q^k}$ there need to be ℓ-th
roots of unity are in $\mathbf{F}_{q^k}^*$.

The *embedding degree* k satisfies
k is minimal with $\ell \mid q^k - 1$.

E is supersingular if
for $| E(\mathbf{F}_q) | = q + 1 - t$, $q = p^r$,
it holds that $t \equiv 0 \bmod p$.

Otherwise it is ordinary.

Example:
$y^2 + y = x^3 + a_4 x + a_6$ over $\mathbf{F}_{2^r}$
is supersingular:
Each (x, y) point also gives
$(x, y + 1) \neq (x, y)$.

All points come in pairs,
except for ∞,
so $| E(\mathbf{F}_{2^r}) | = 1 + \text{even}$,
so $t \equiv 0 \bmod 2$.

Embedding degrees

Let E be supersingular and
$q = p \geq 5$, i.e $p > 2\sqrt{p}$.

Hasse's Theorem states
$| t | \leq 2\sqrt{p}$.
E supersingular implies
$t \equiv 0 \bmod p$, so $t = 0$ and
$| E(\mathbf{F}_p) | = p + 1$.

Obviously
$(p+1) \mid p^2 - 1 = (p+1)(p-1)$
so $k \leq 2$ for supersingular curves over prime fields.

Distortion maps

For supersingular curves there exist maps
$\phi : E(\mathbf{F}_q) \to E(\mathbf{F}_{q^k})$
i.e. maps $G_1 \to G_2$, giving $\tilde{e}(P, P) \neq 1$ for $\tilde{e}(P, P) = e(P, \phi(P))$.
Such a map is called a *distortion map*.

These maps are important since the only pairings we know how to compute are variants of *Weil pairing* and *Tate pairing* which have $e(P, P) = 1$.

Examples:

$y^2 = x^3 + a_4 x$,

for $p \equiv 3 \pmod 4$.

Distortion map

$(x, y) \mapsto (-x, \sqrt{-1}y)$.

$y^2 = x^3 + a_6$, for $p \equiv 2 \pmod 3$.

Distortion map $(x, y) \mapsto (jx, y)$

with $j^3 = 1, j \neq 1$.

In both cases, $\#E(\mathbf{F}_p) = p + 1$,

so $k = 2$.

Example from Tuesday:

$p = 1000003 \equiv 3 \bmod 4$ and

$y^2 = x^3 - x$ over $\mathbf{F}_p$.

Has $1000004 = p + 1$ points.

$P = (101384, 614510)$ is a point

of order 500002.

$nP = (670366, 740819)$.

Construct $\mathbf{F}_{p^2}$ as $\mathbf{F}_p(i)$.

$\phi(P) = (898619, 614510i)$.

Invoke magma and compute

$e(P, \phi(P)) = 387265 + 276048i$;

$e(Q, \phi(P)) = 609466 + 807033i$.

Solve with index calculus to get

$n = 78654$.

(Btw. this is the clock).

Summary of pairings

Menezes, Okamoto, and Vanstone for E supersingular:
For $p = 2$ have $k \leq 4$.
For $p = 3$ we $k \leq 6$
Over $\mathbf{F}_p$, $p \geq 5$ have $k \leq 2$.
These bounds are attained.

Not only supersingular curves:
MNT curves are non-supersingular curves with small k.
Other examples constructed for pairing-based cryptography – but small k unlikely to occur for random curve.

Summary of other attacks

Definition of embedding degree does not cover all attacks.
For $\mathbf{F}_{p^n}$ watch out that pairing can map to $\mathbf{F}_{p^{km}}$ with $m < n$.
Watch out for this when selecting curves over $\mathbf{F}_{p^n}$.

Anomalous curves:
If $E/\mathbf{F}_p$ has $\#E(\mathbf{F}_p) = p$
then transfer $E(\mathbf{F}_p)$ to $(\mathbf{F}_p, +)$.
Very easy DLP.
Not a problem for Koblitz curves, attack applies to order-p subgroup.

Weil descent:

Maps DLP in E over $\mathbf{F}_{p^{m\,n}}$

to DLP on variety J over $\mathbf{F}_{p^n}$.

J has larger dimension; elements represented as polynomials of low degree. $\Rightarrow$ index calculus.

This is efficient if dimension of J is not too big.

Particularly nice to compute with J if it is the Jacobian of a hyperelliptic curve C.

For genus g get complexity $\tilde{O}(p^{2-\frac{2}{g+1}})$ with the factor base described before, since polynomials have degree $<= g$.