Discrete-log attacks and factorization Part I

Tanja Lange
Technische Universiteit Eindhoven

11 & 13 June 2019

with some slides by
Daniel J. Bernstein

Main goal of this course:
We are the attackers.
We want to break ECC and RSA.

First need to understand ECC.

Main motivation for ECC:
Avoid index-calculus attacks
that plague finite-field DL.

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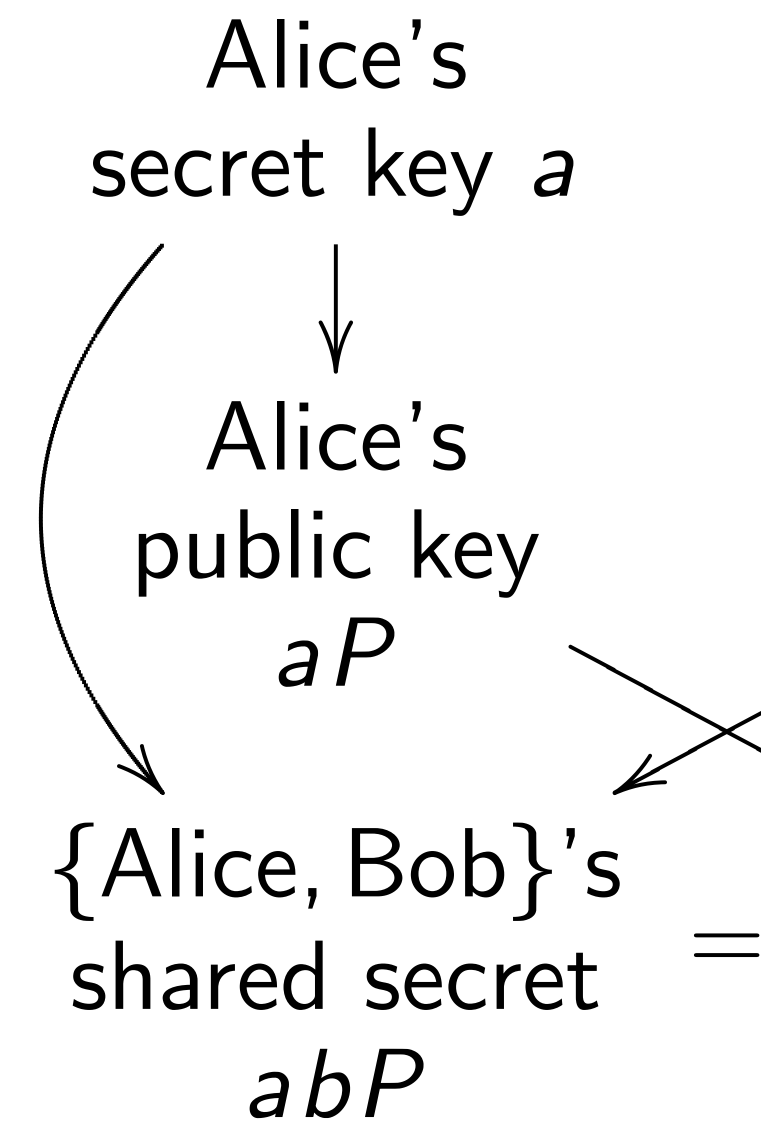

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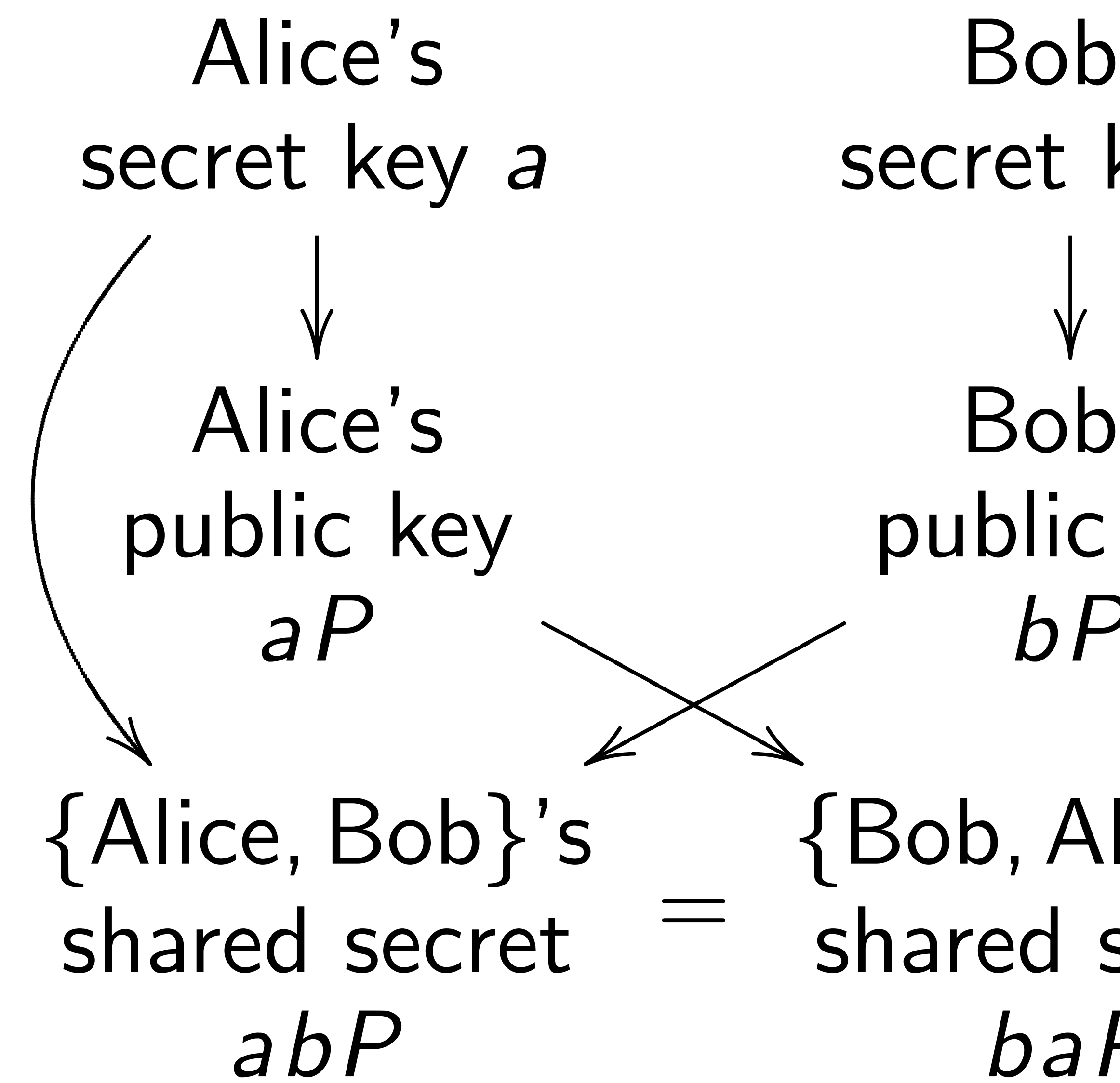

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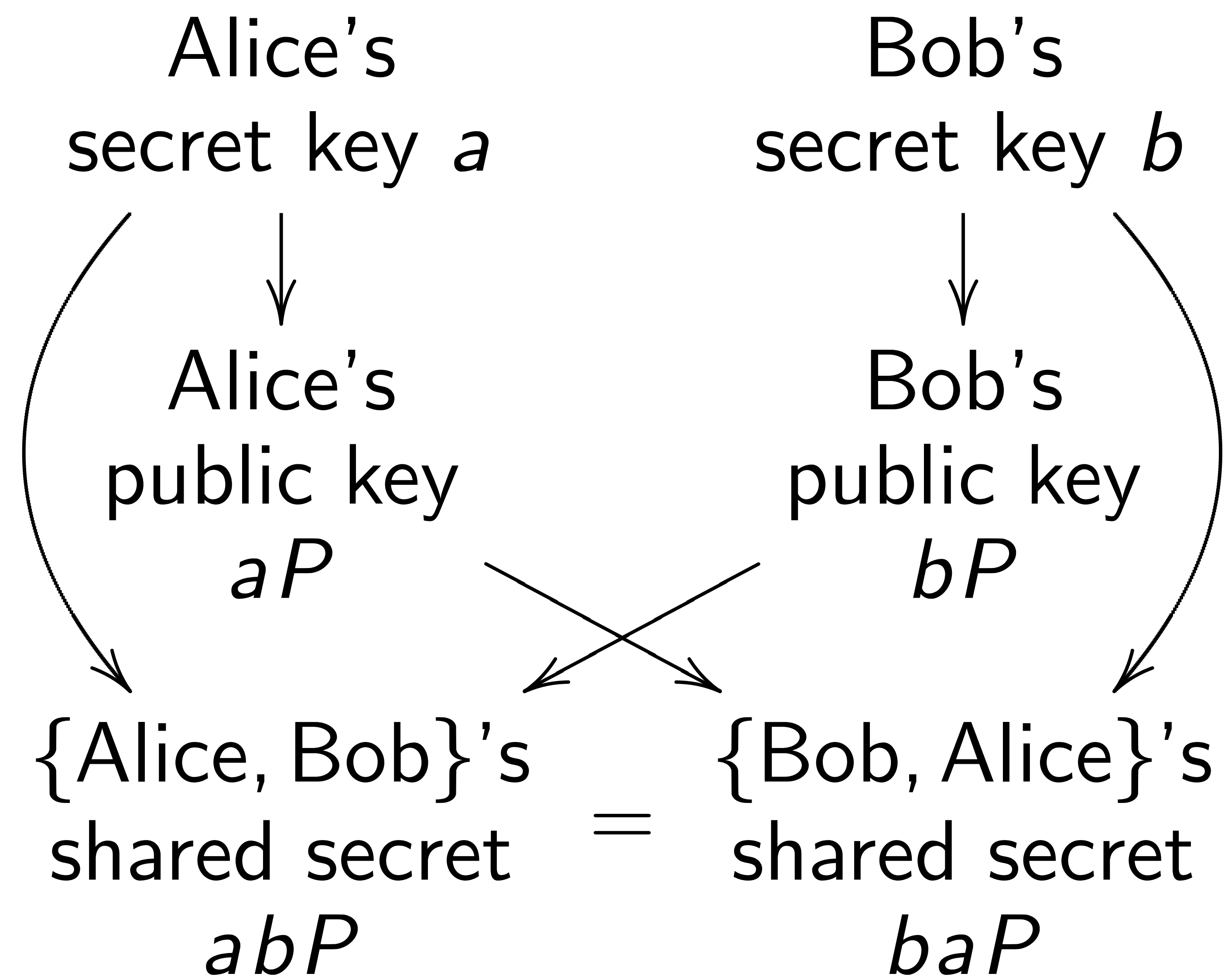

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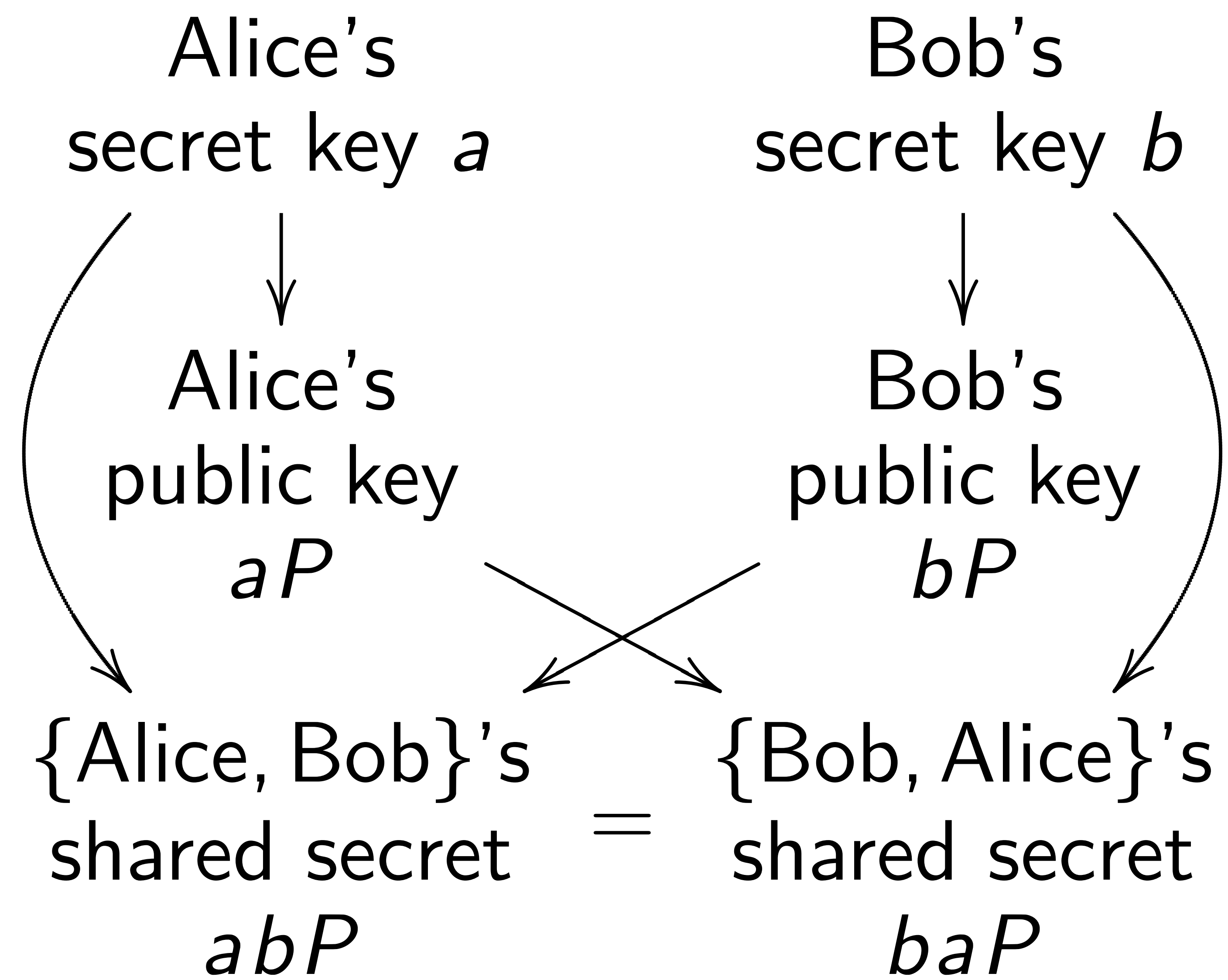

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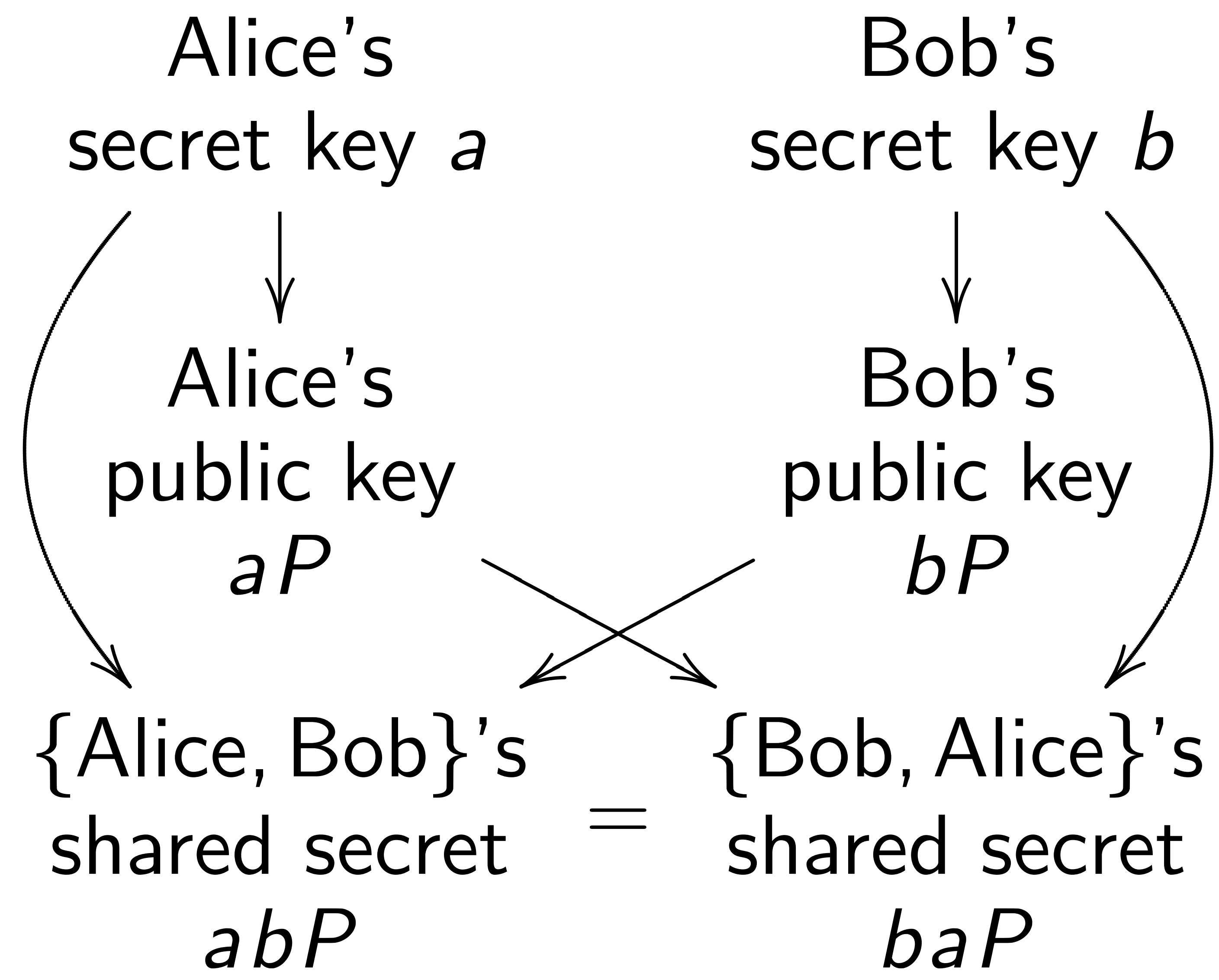

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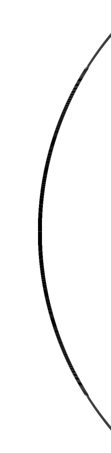

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Diffie-Hellman key exchange

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Alice's secret key a → Alice's public key aP

Bob's secret key b → Bob's public key bP

{Alice, Bob}'s shared secret abP = {Bob, Alice}'s shared secret baP

What does P look like &
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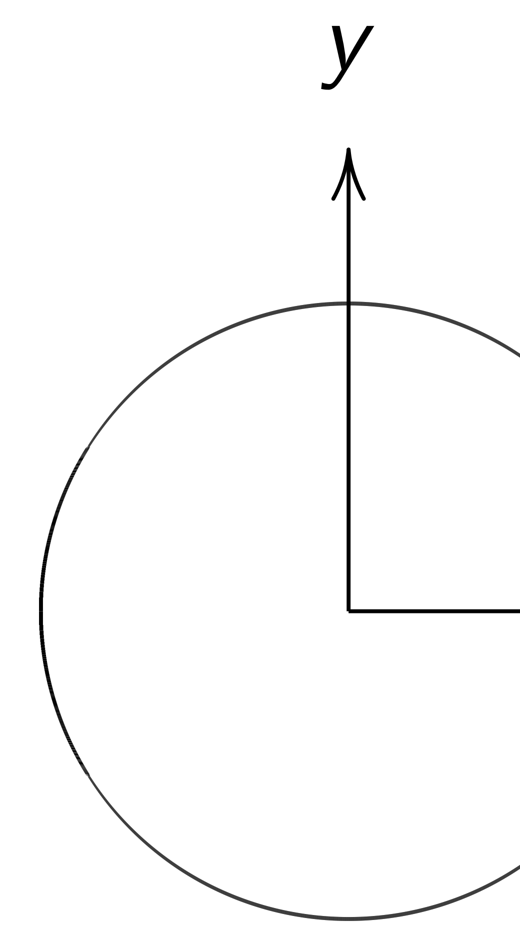

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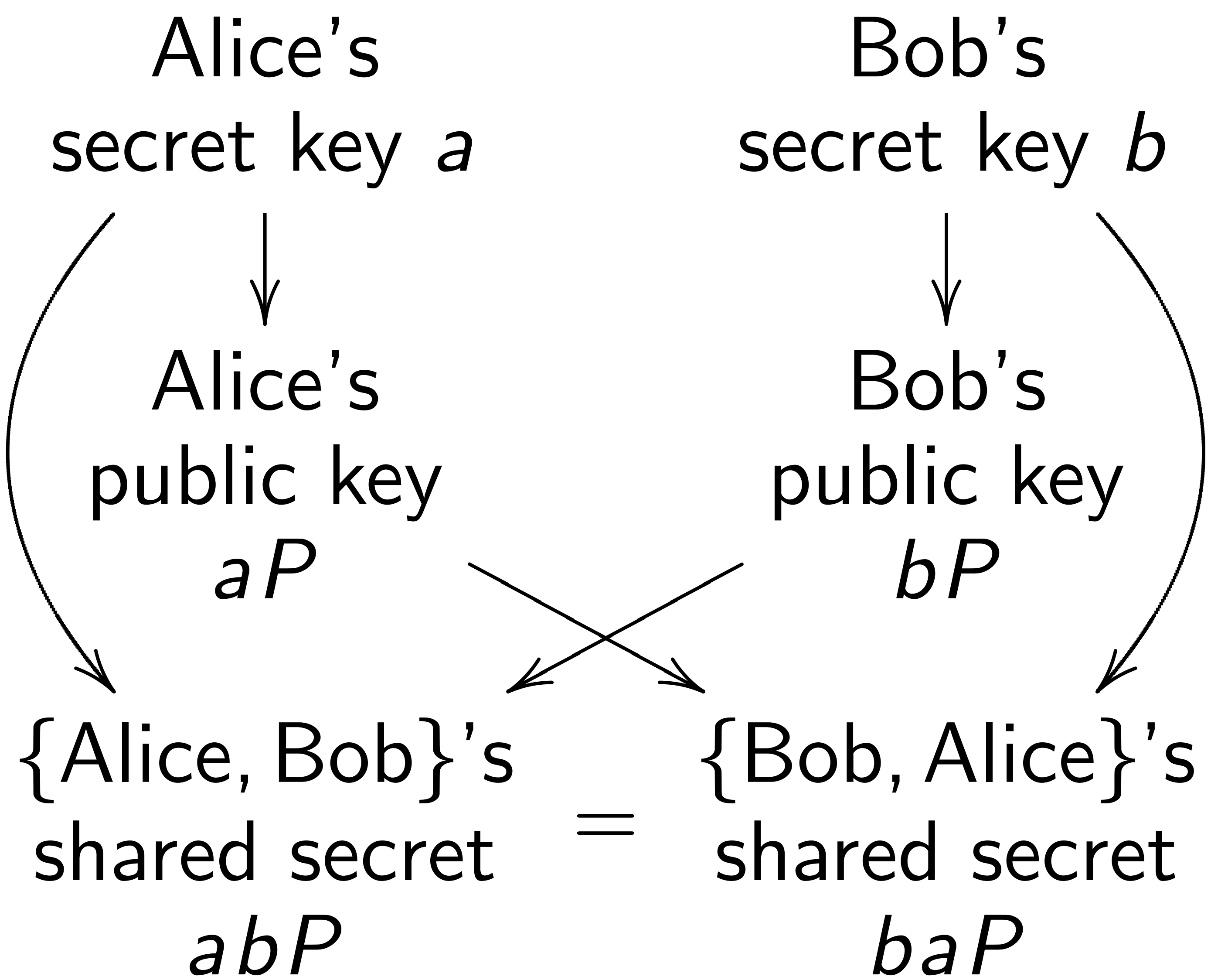

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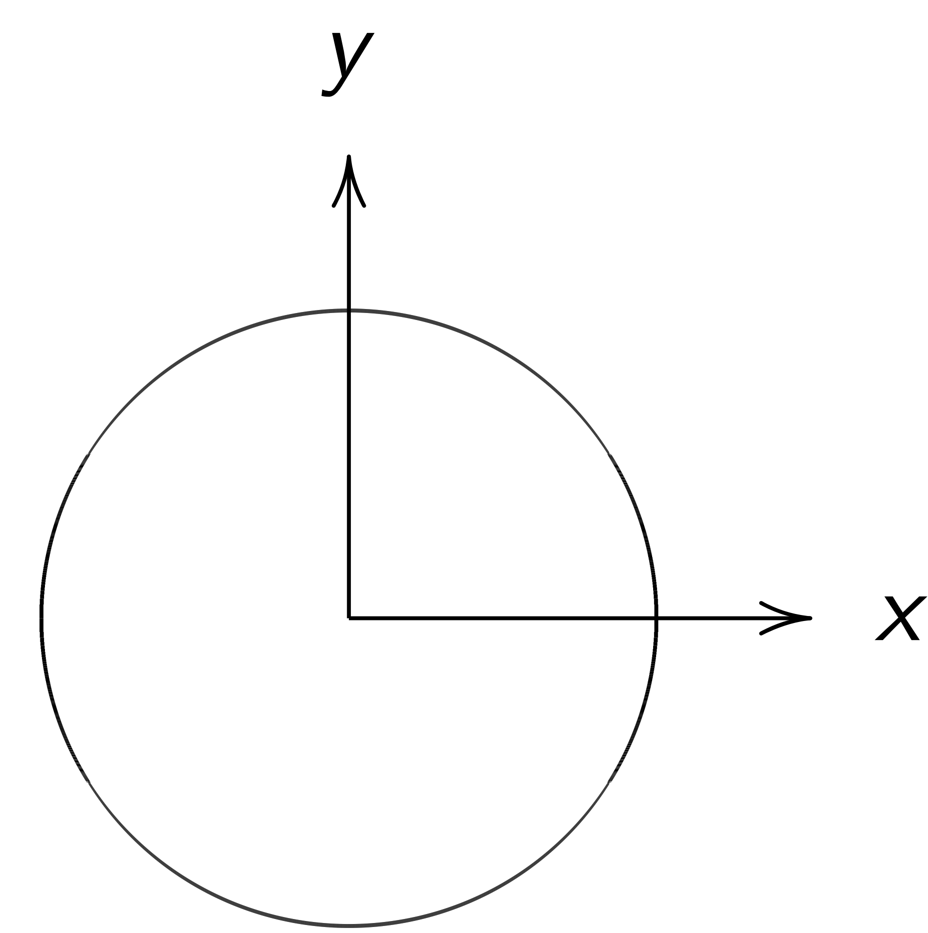

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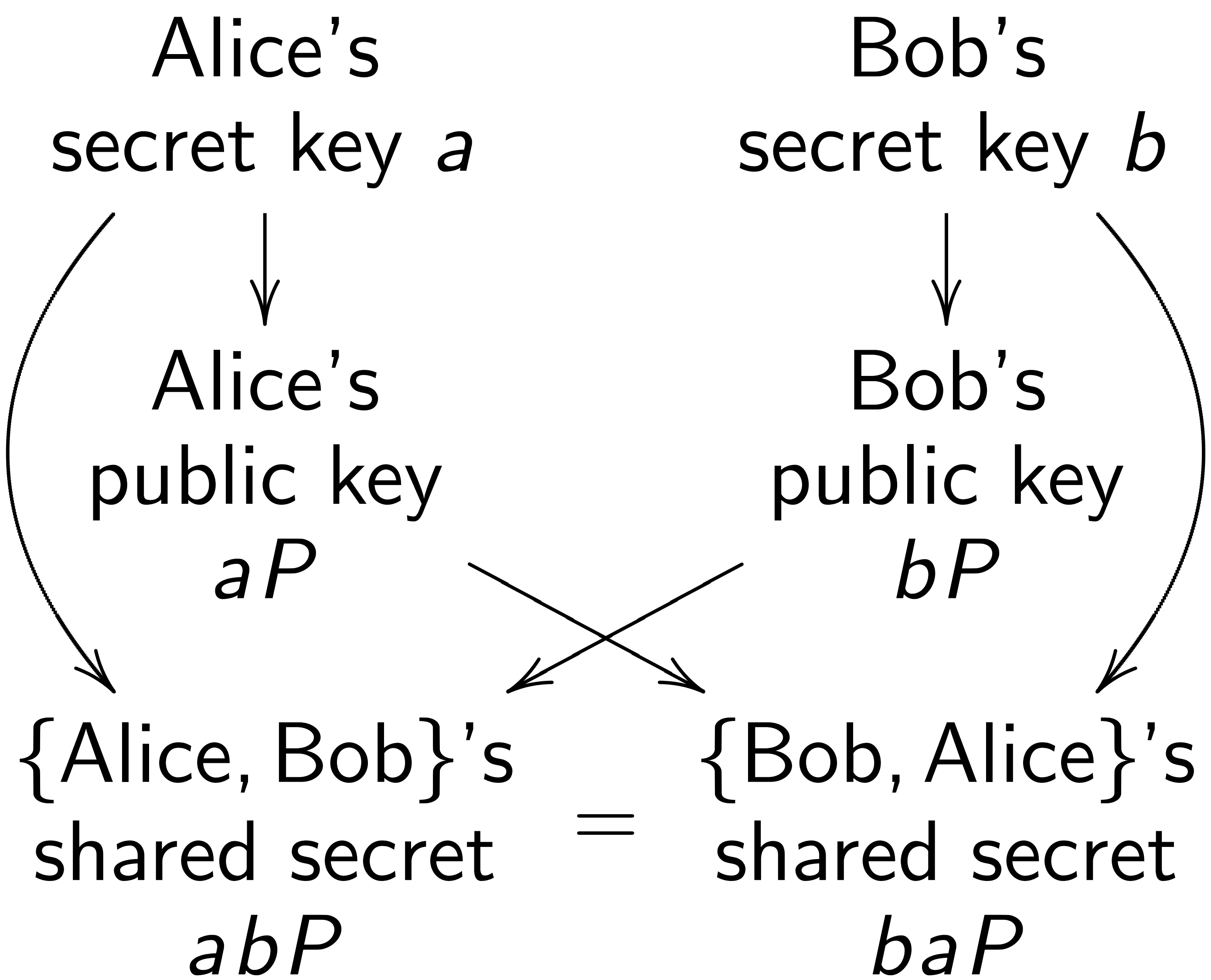

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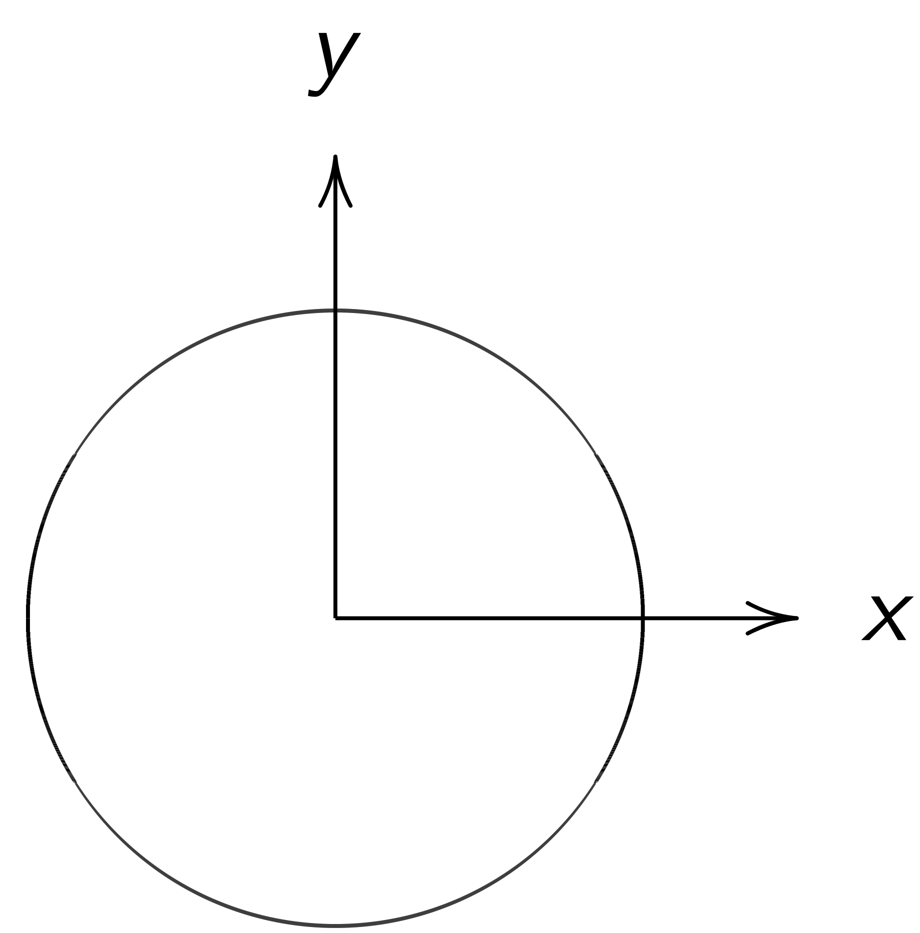

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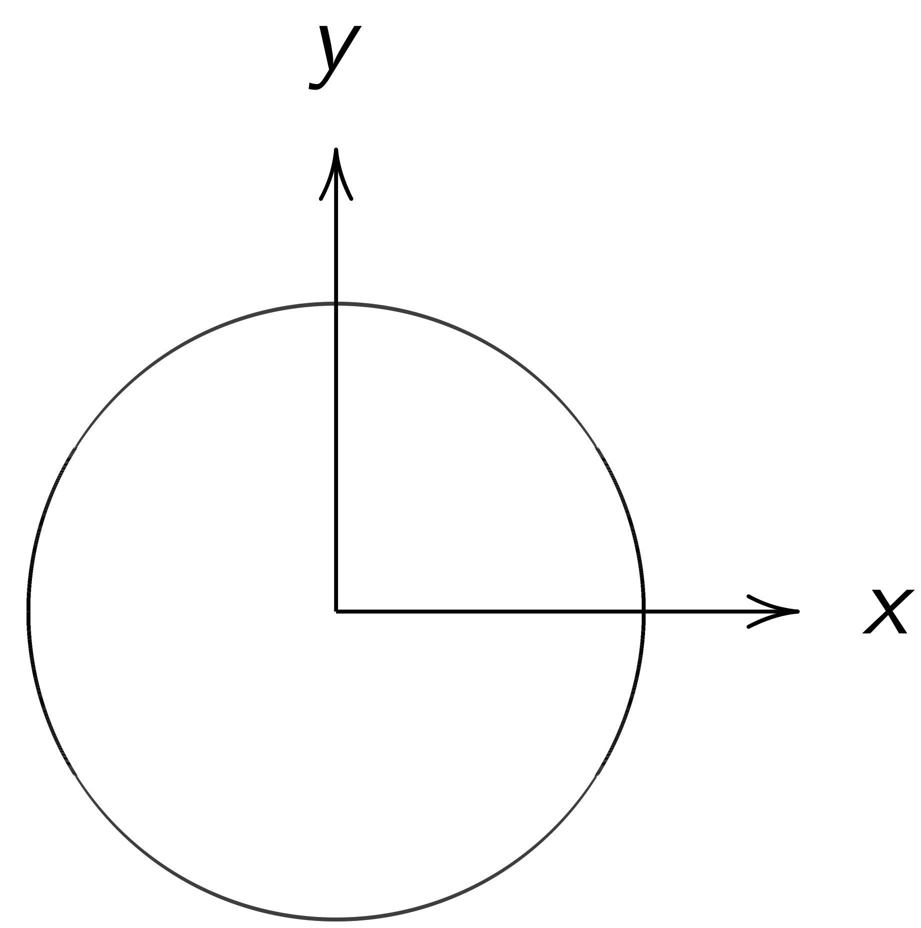

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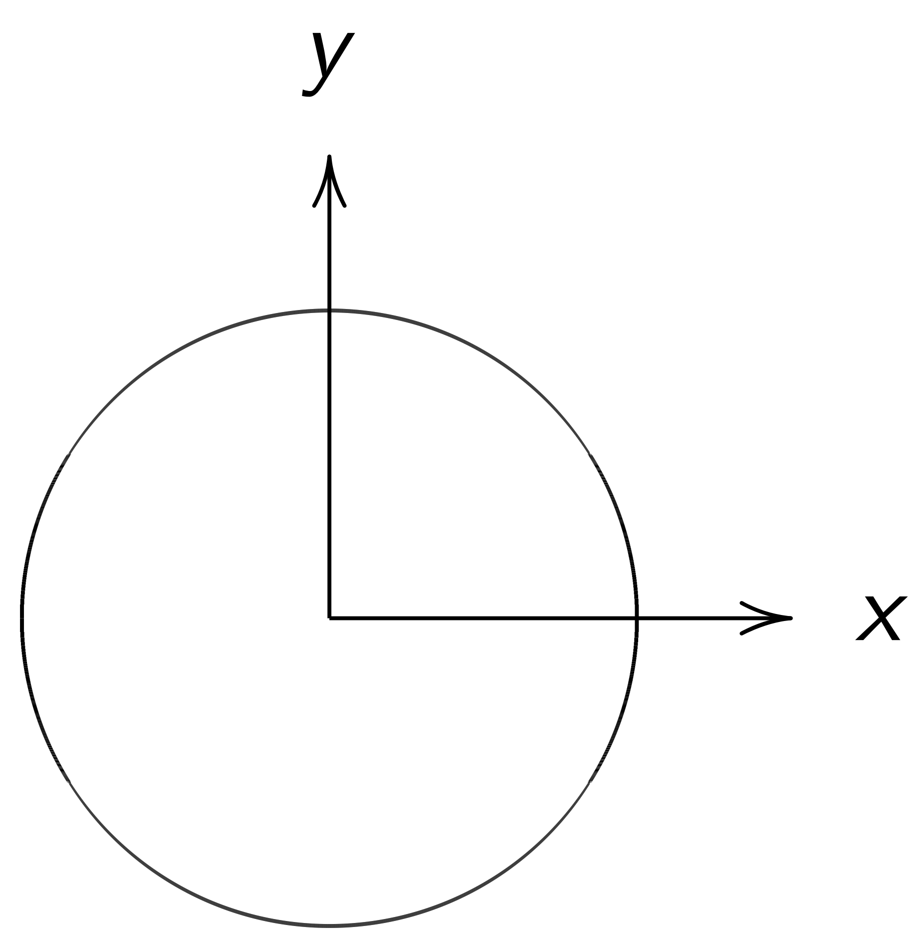

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Examples of point

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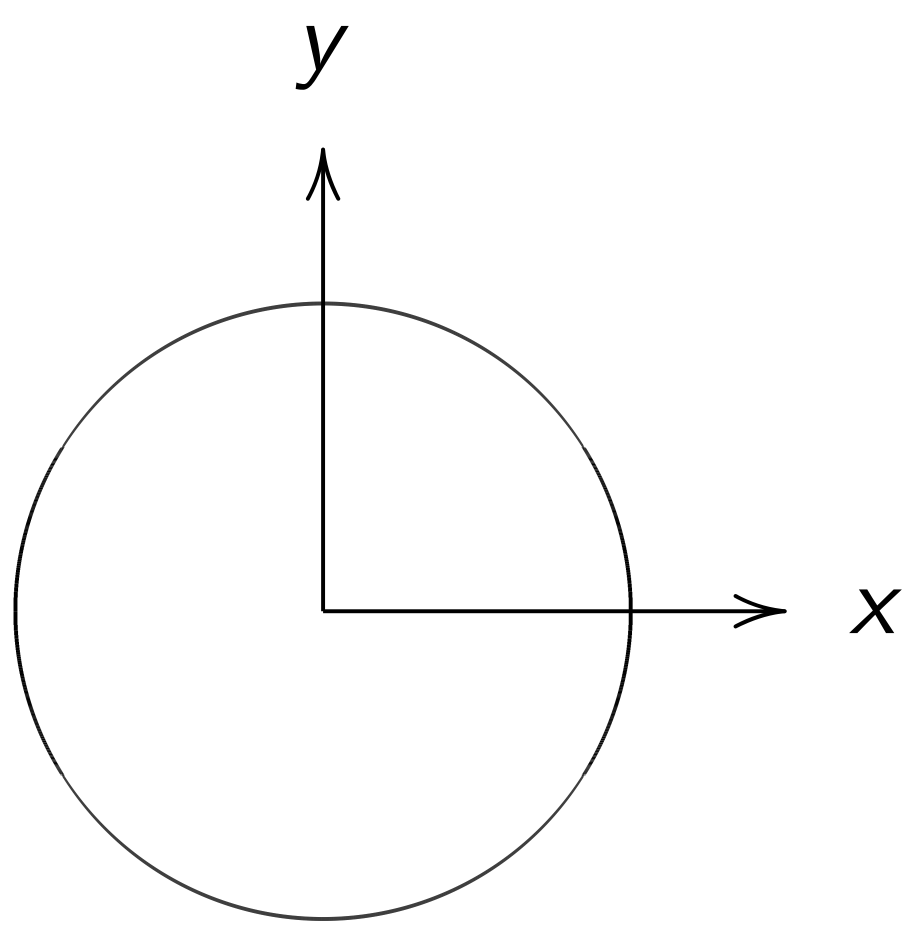

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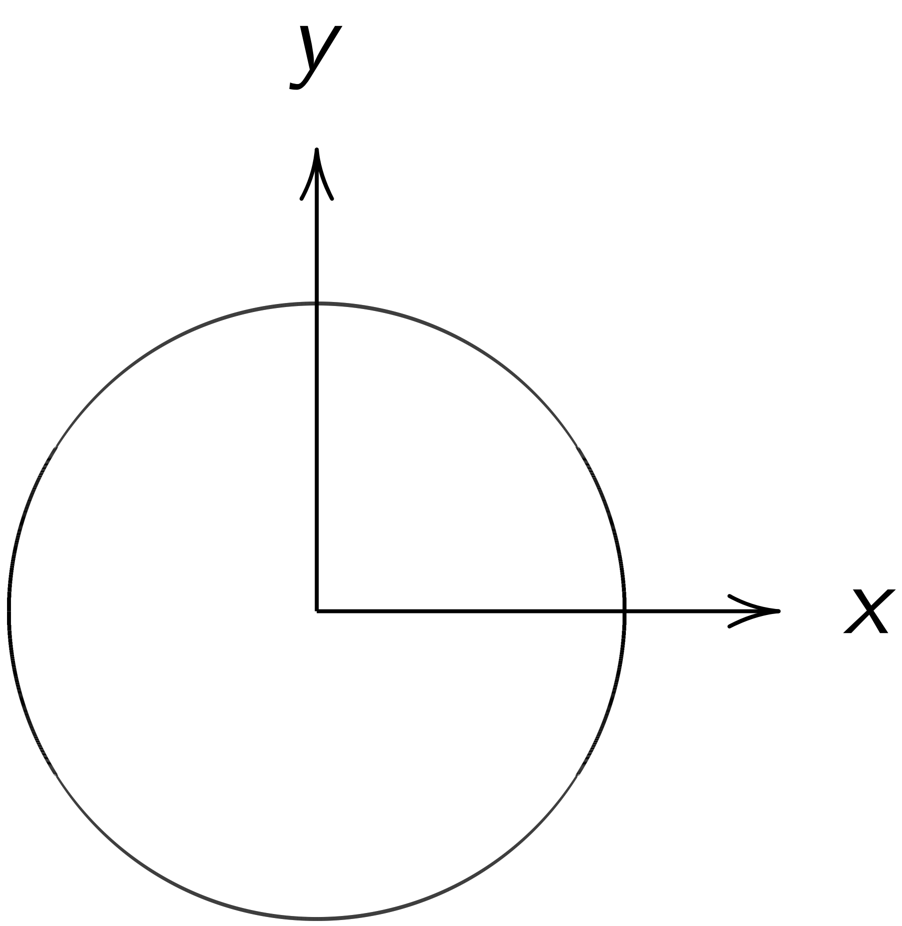

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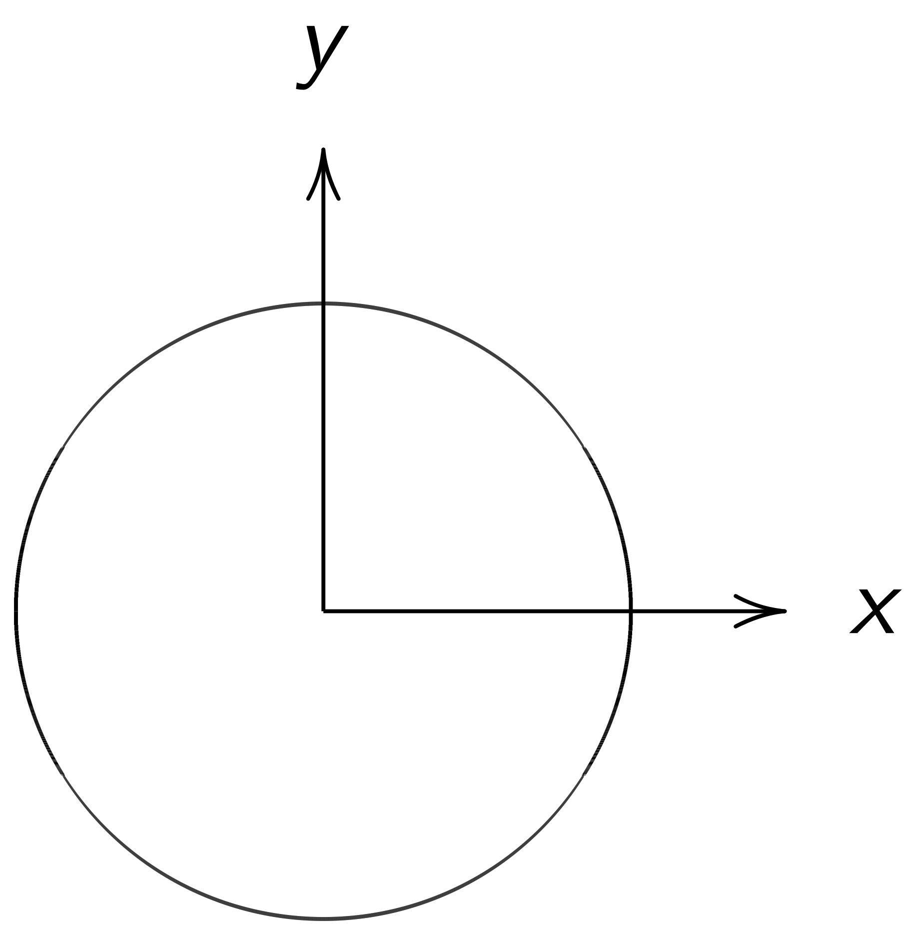

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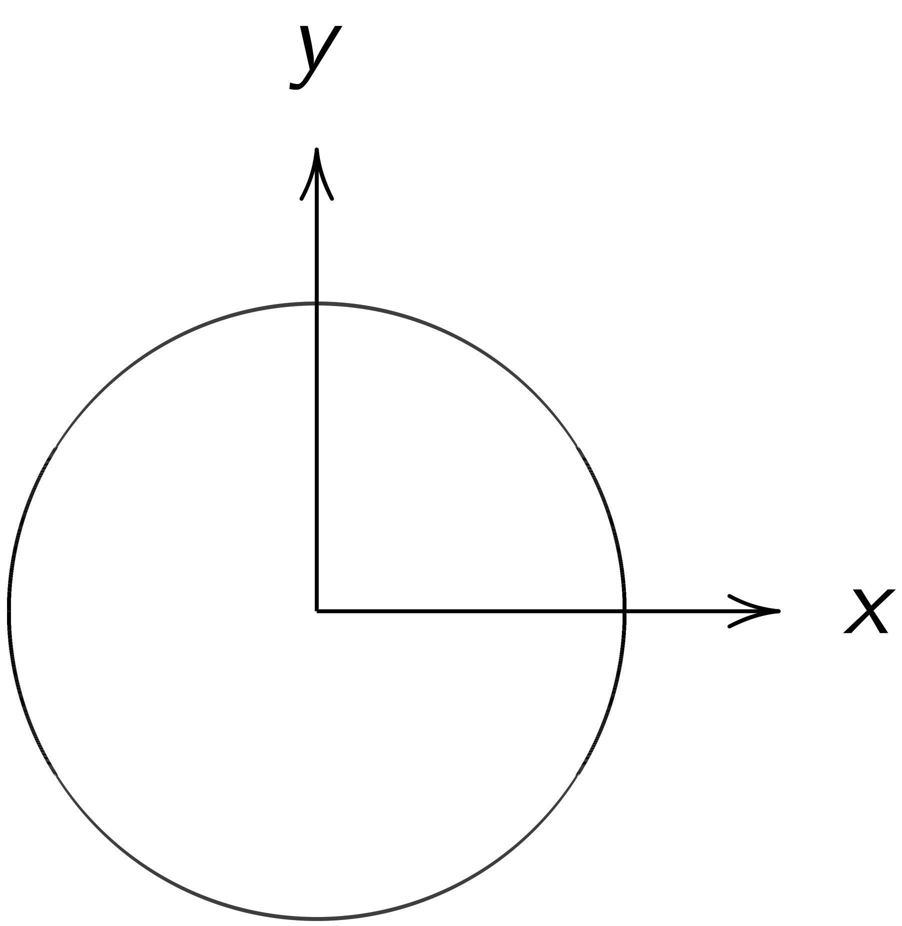

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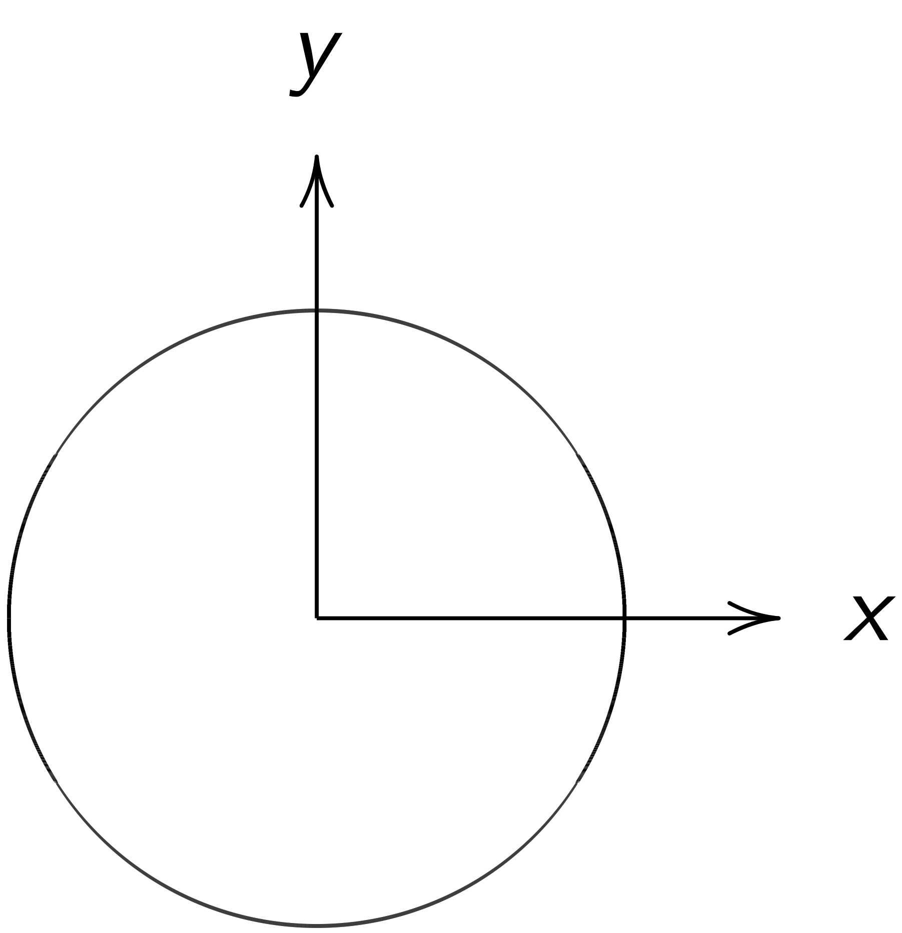

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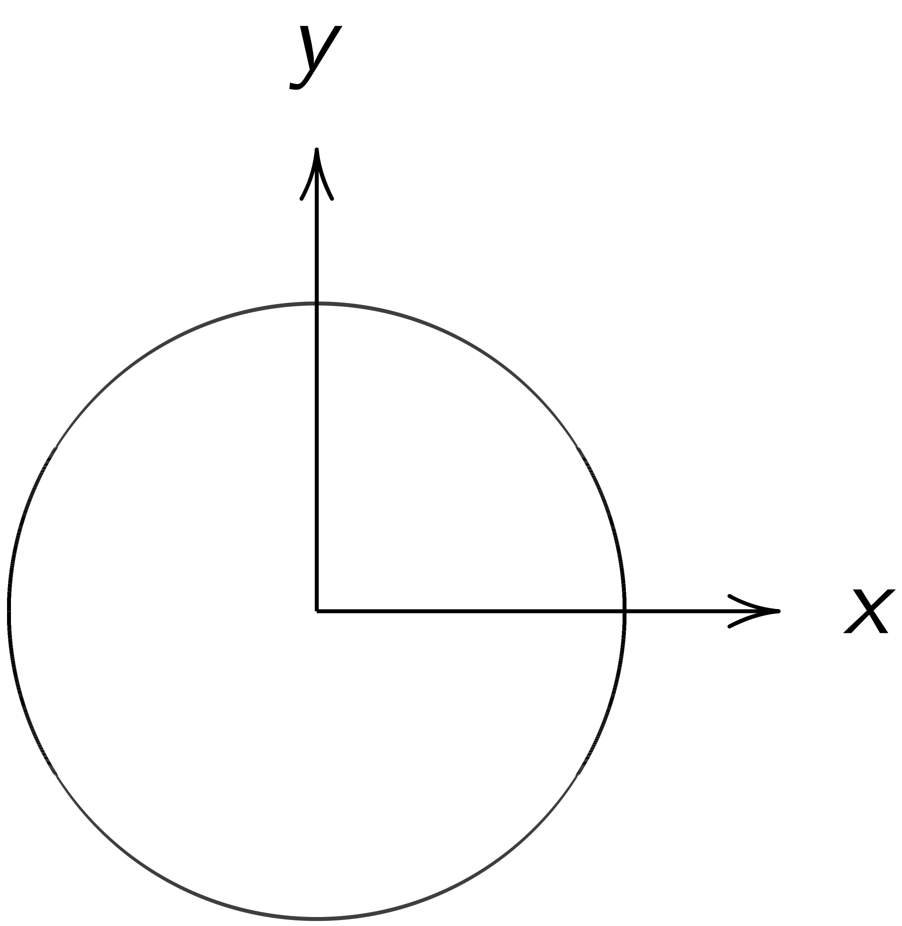

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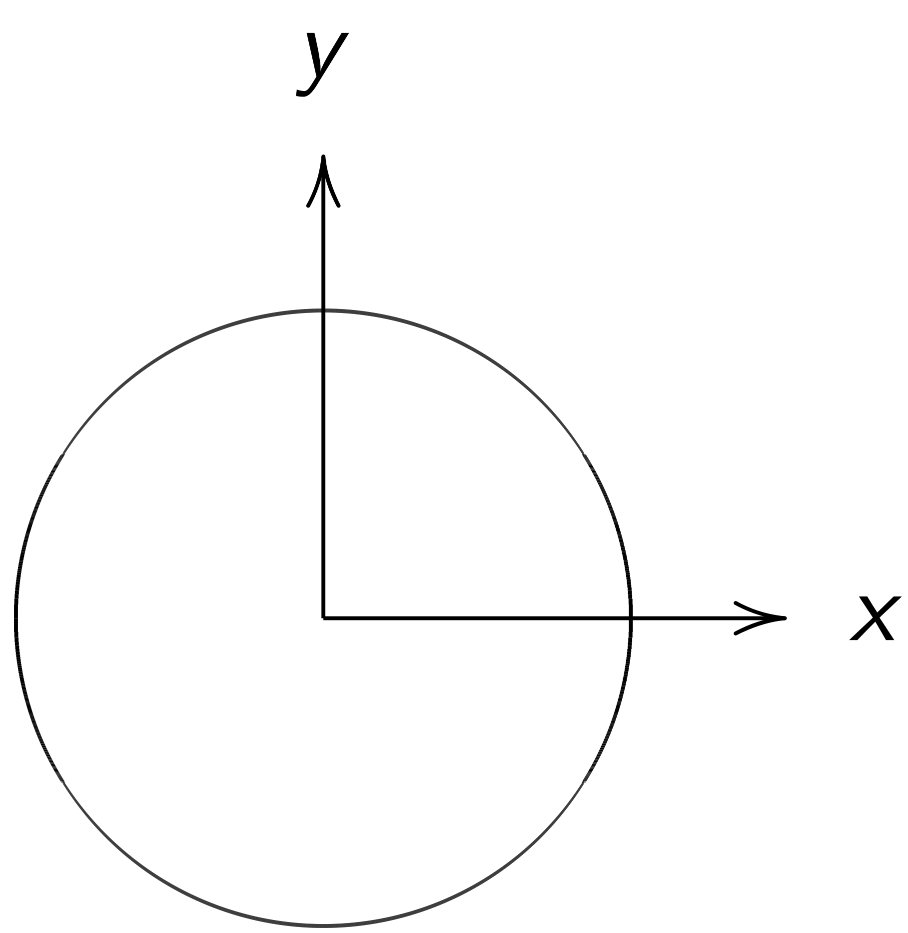

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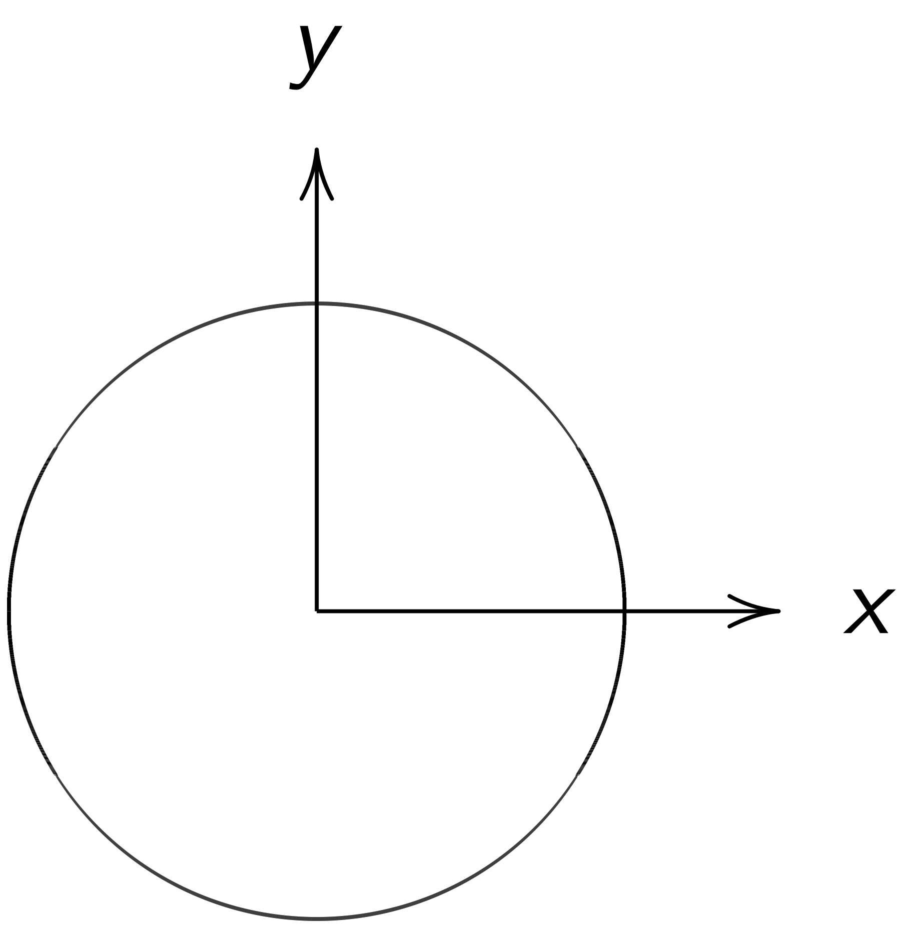

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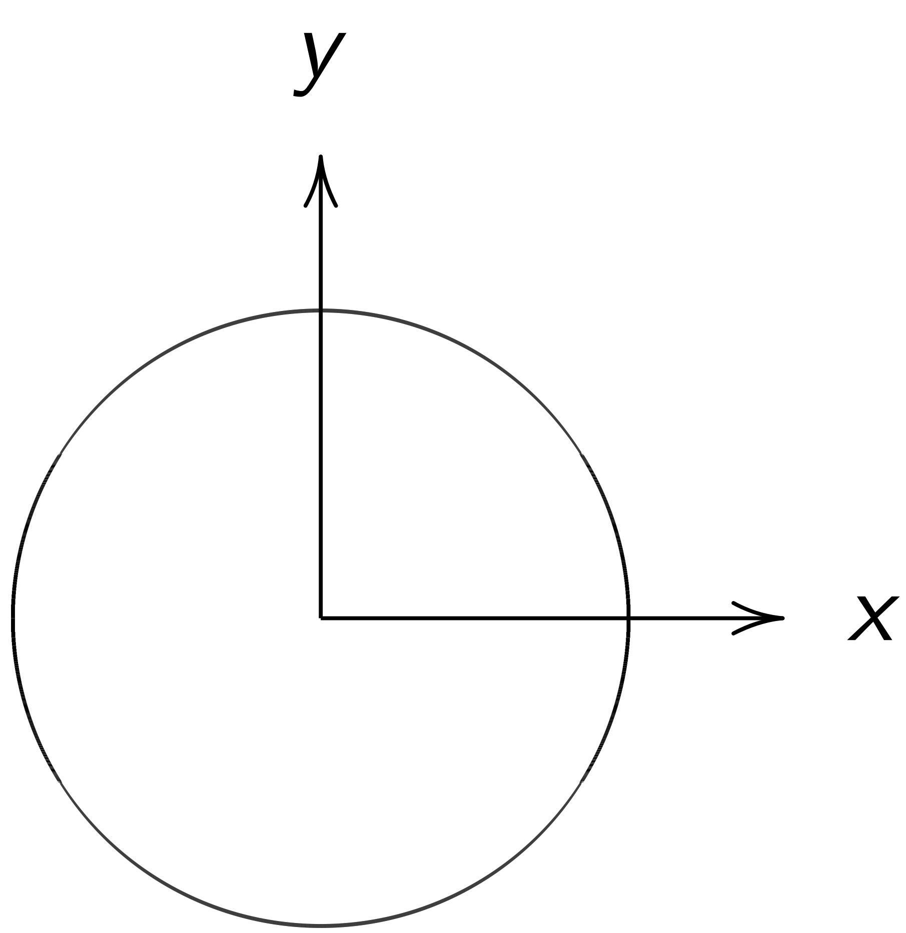

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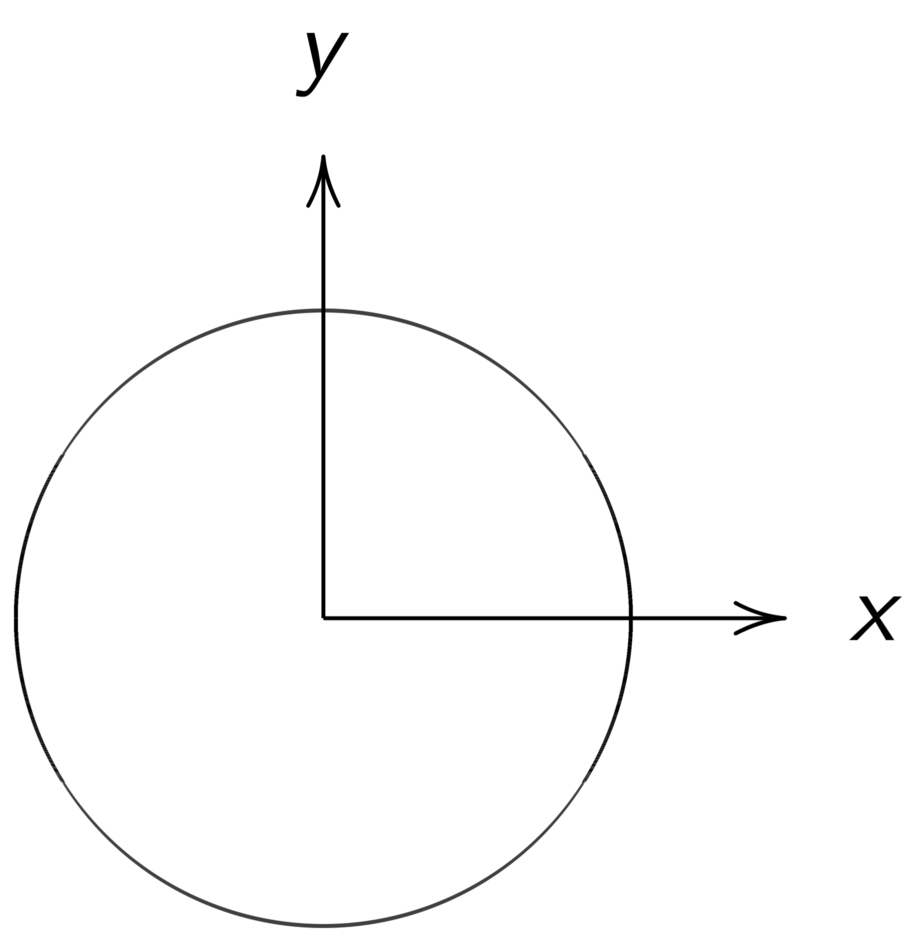

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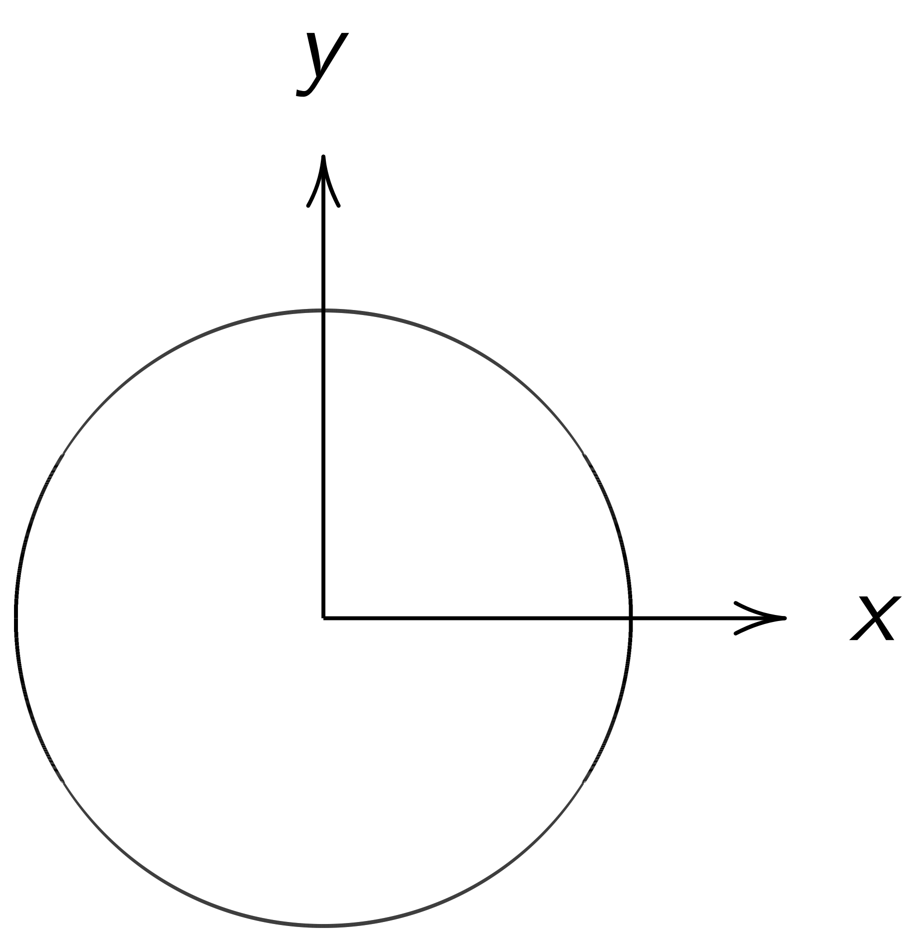

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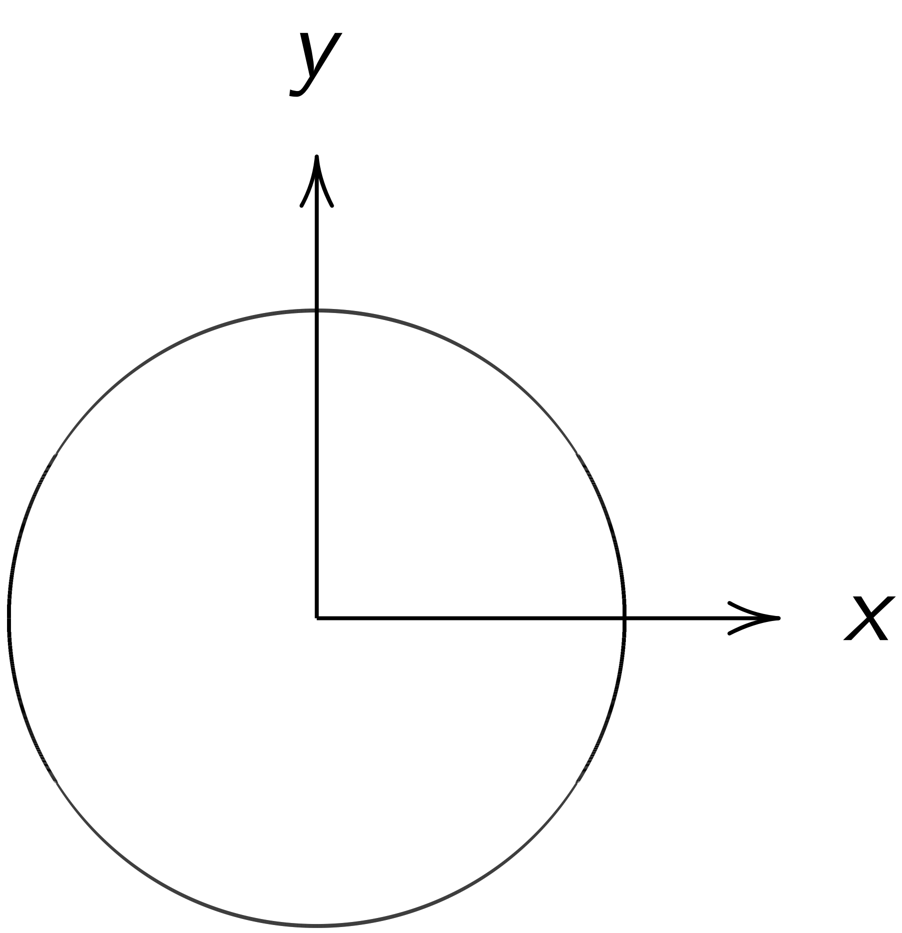

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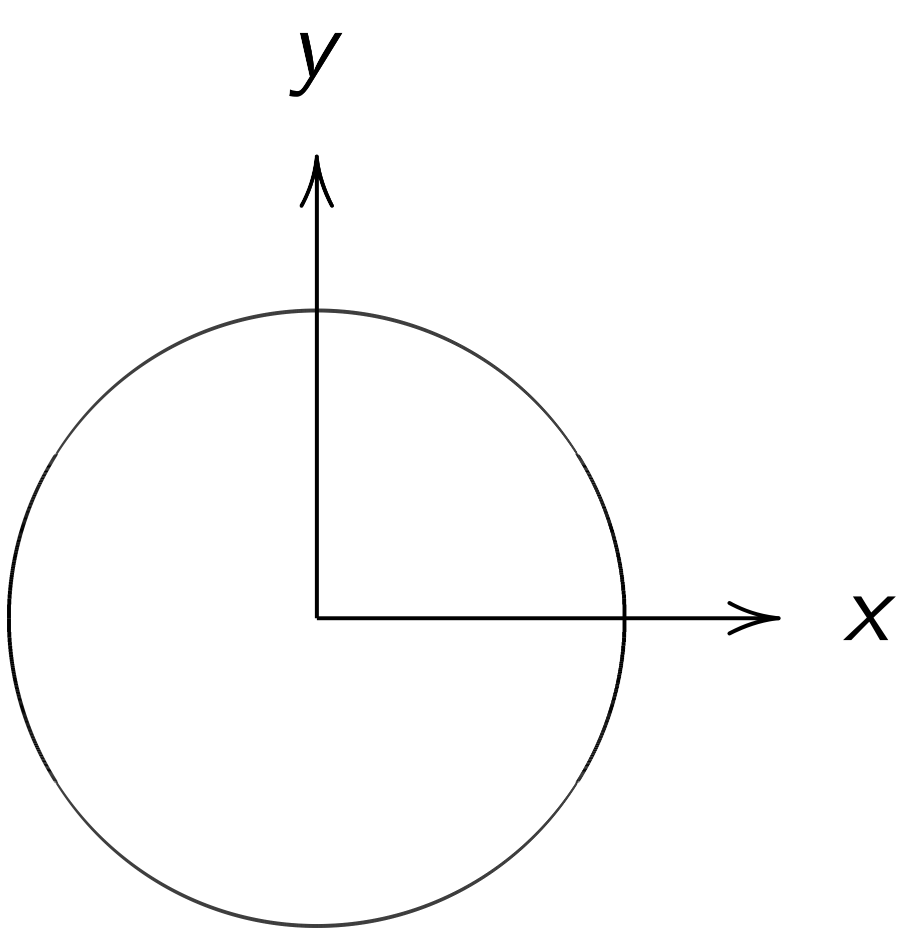

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Many more.

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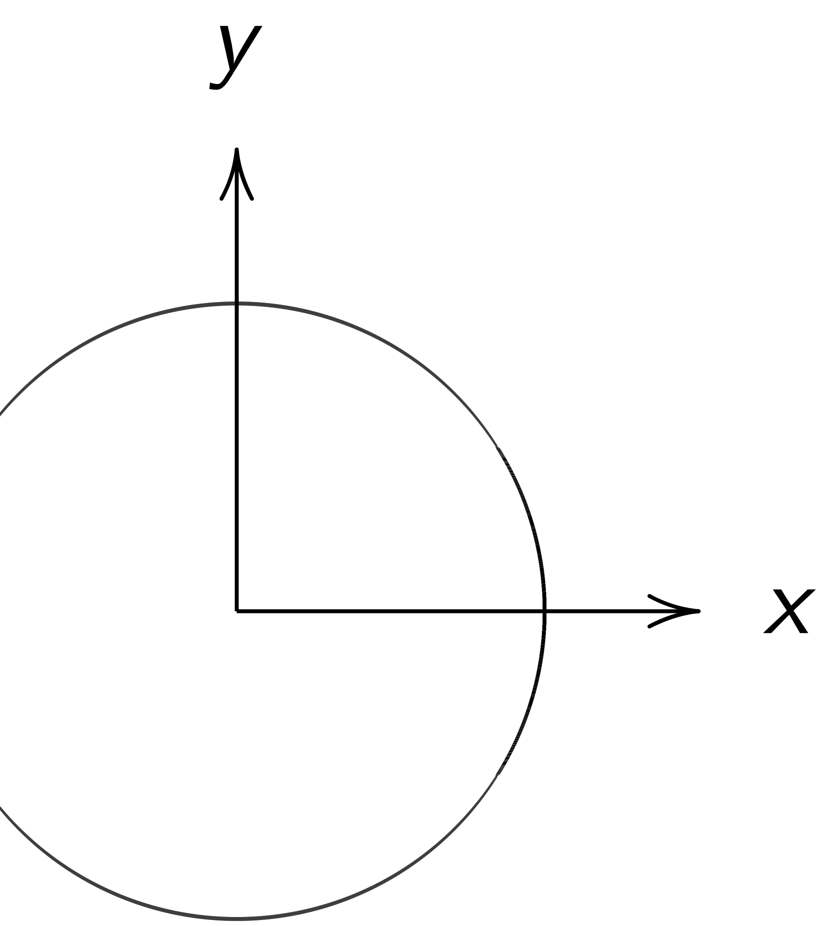

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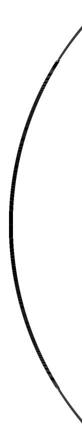
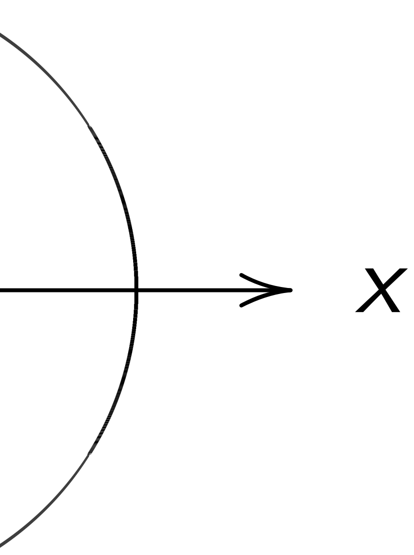


$x^2 + y^2$

$x = \sin$



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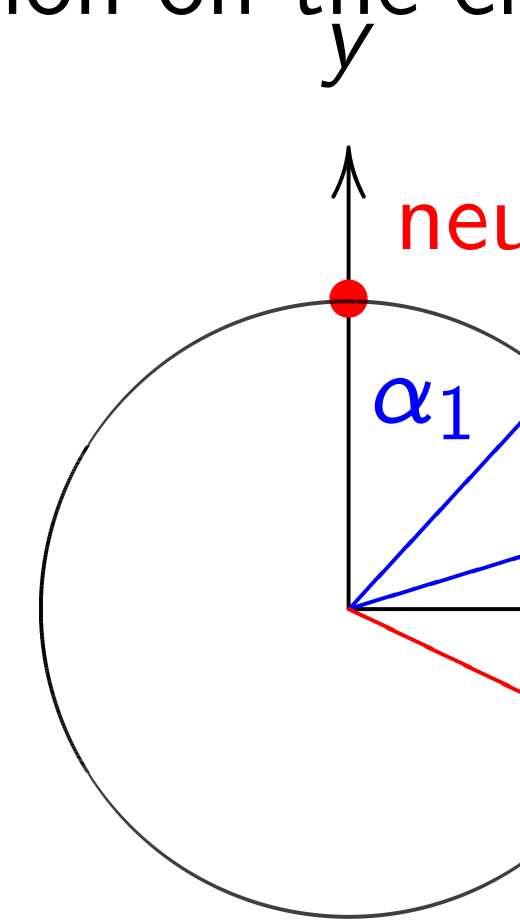

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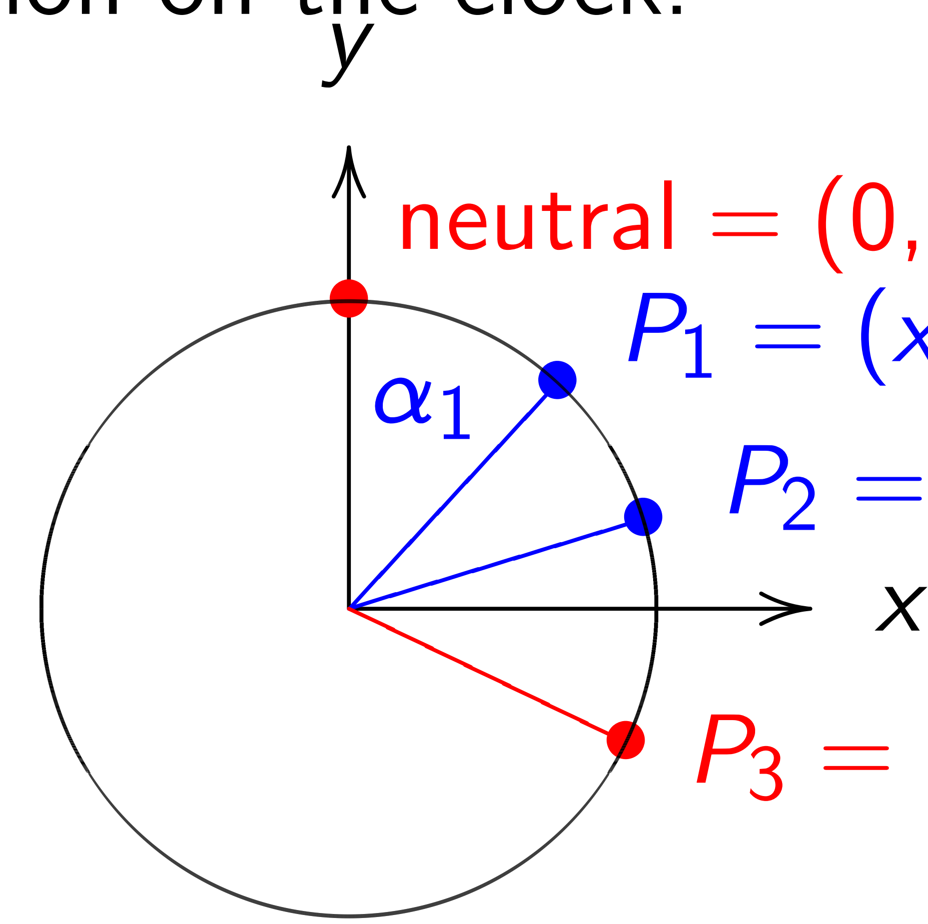

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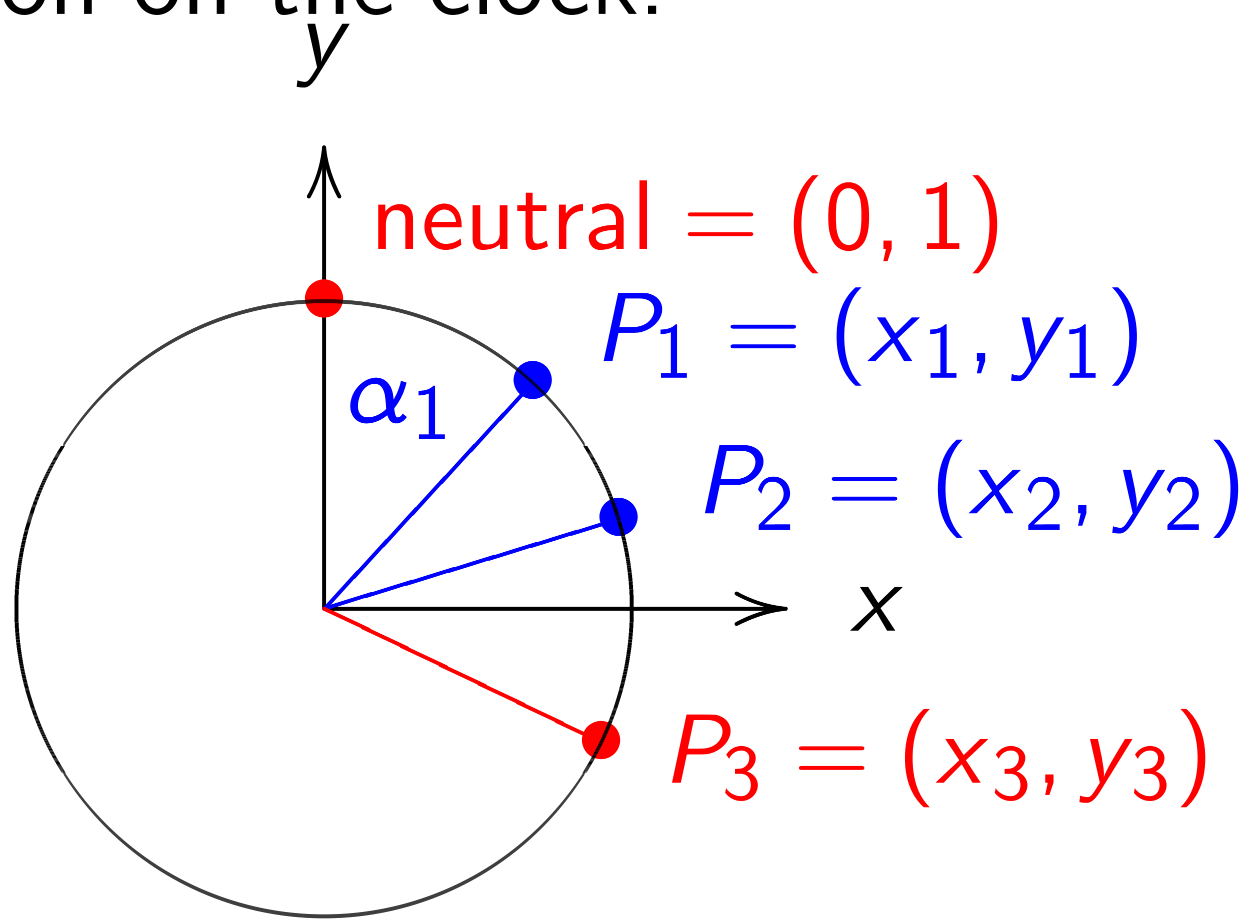

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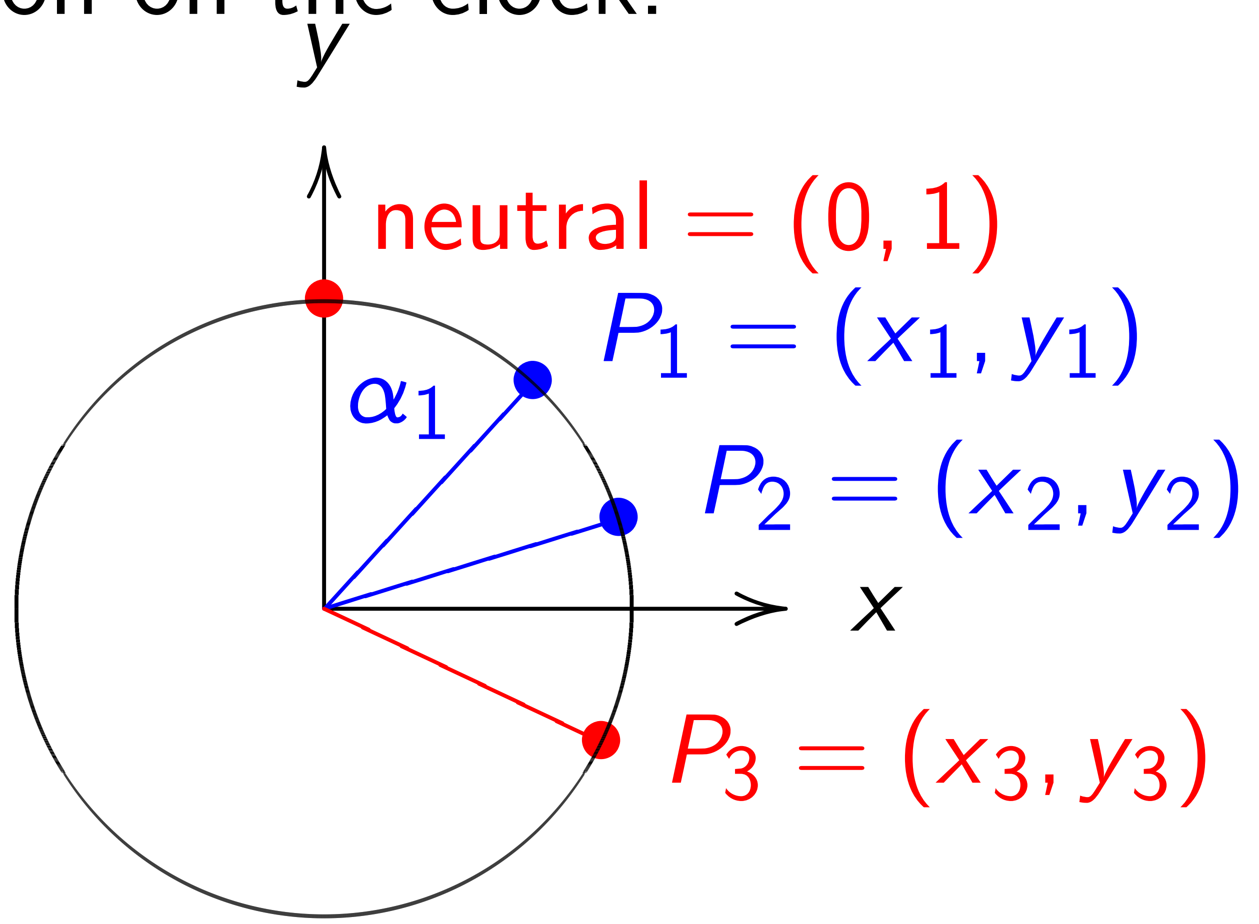

$x^2 + y^2 = 1$, parametrized by

$x = \sin\alpha$, $\quad y = \cos\alpha$. Recall

$(\sin(\alpha_1 + \alpha_2), \cos(\alpha_1 + \alpha_2)) =$

Examples of points on this curve:

$(0, 1) =$ "12:00".

$(0, -1) =$ "6:00".

$(1, 0) =$ "3:00".

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$(\sqrt{3/4}, 1/2) =$ "2:00".

$(1/2, -\sqrt{3/4}) =$ "5:00".

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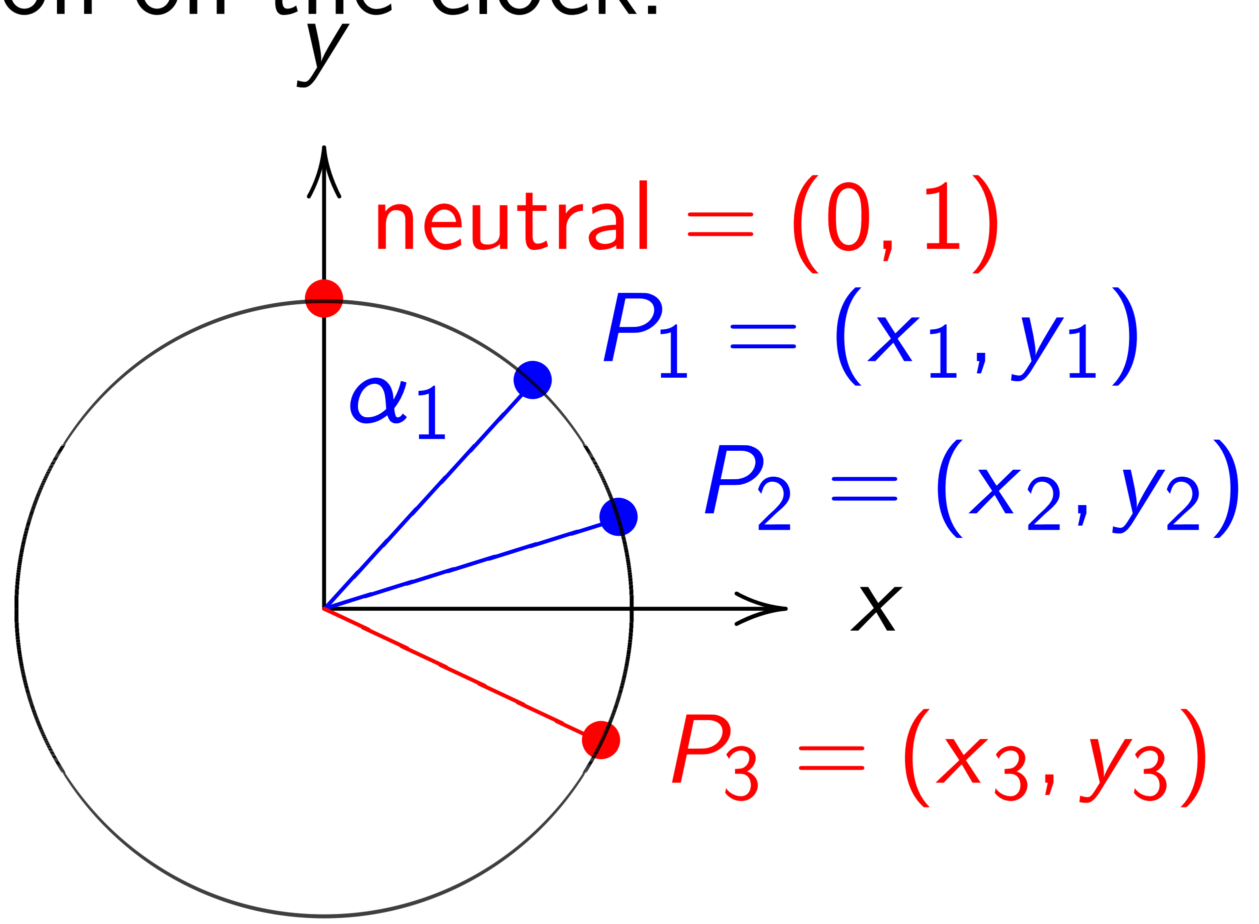

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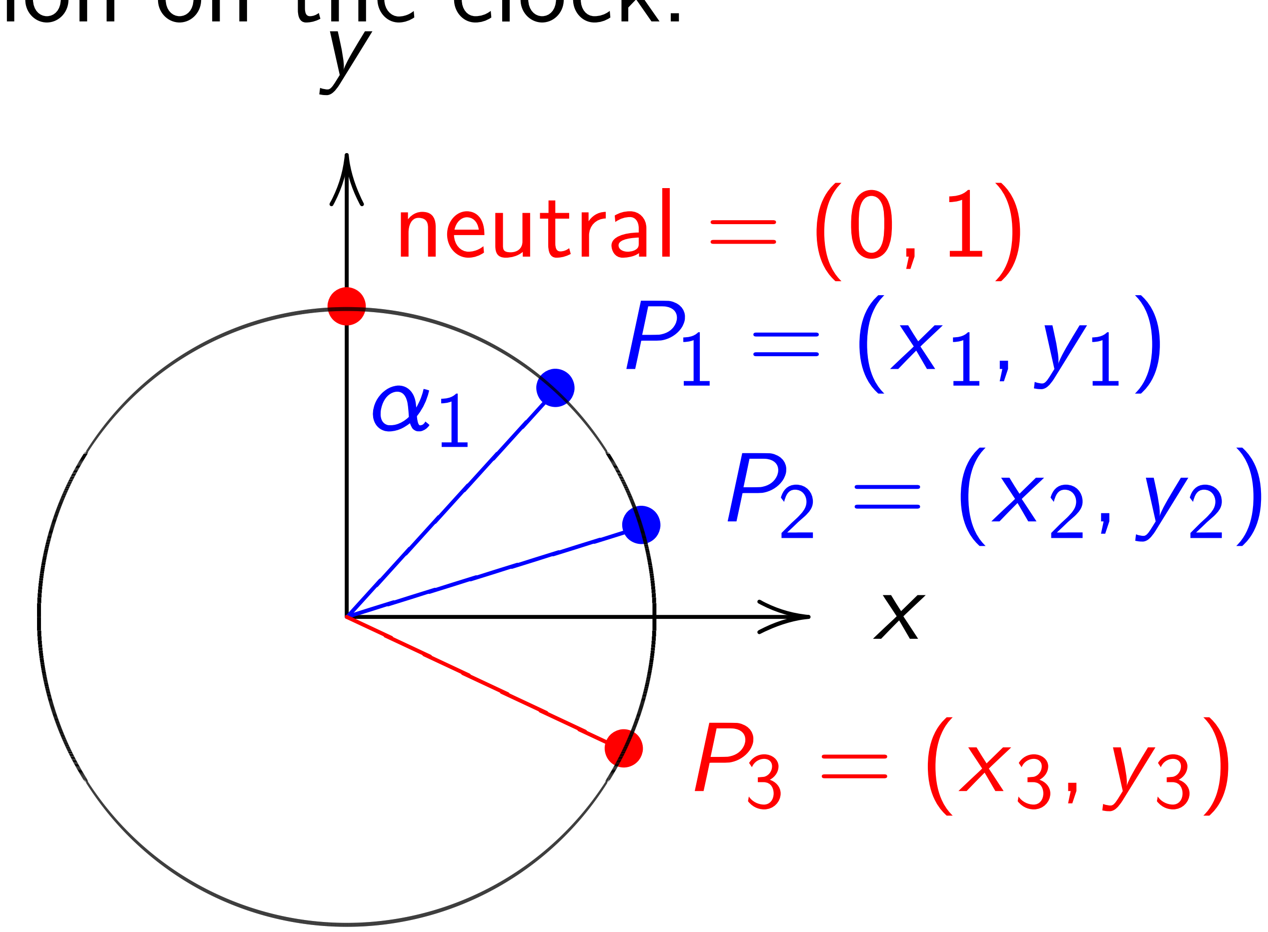

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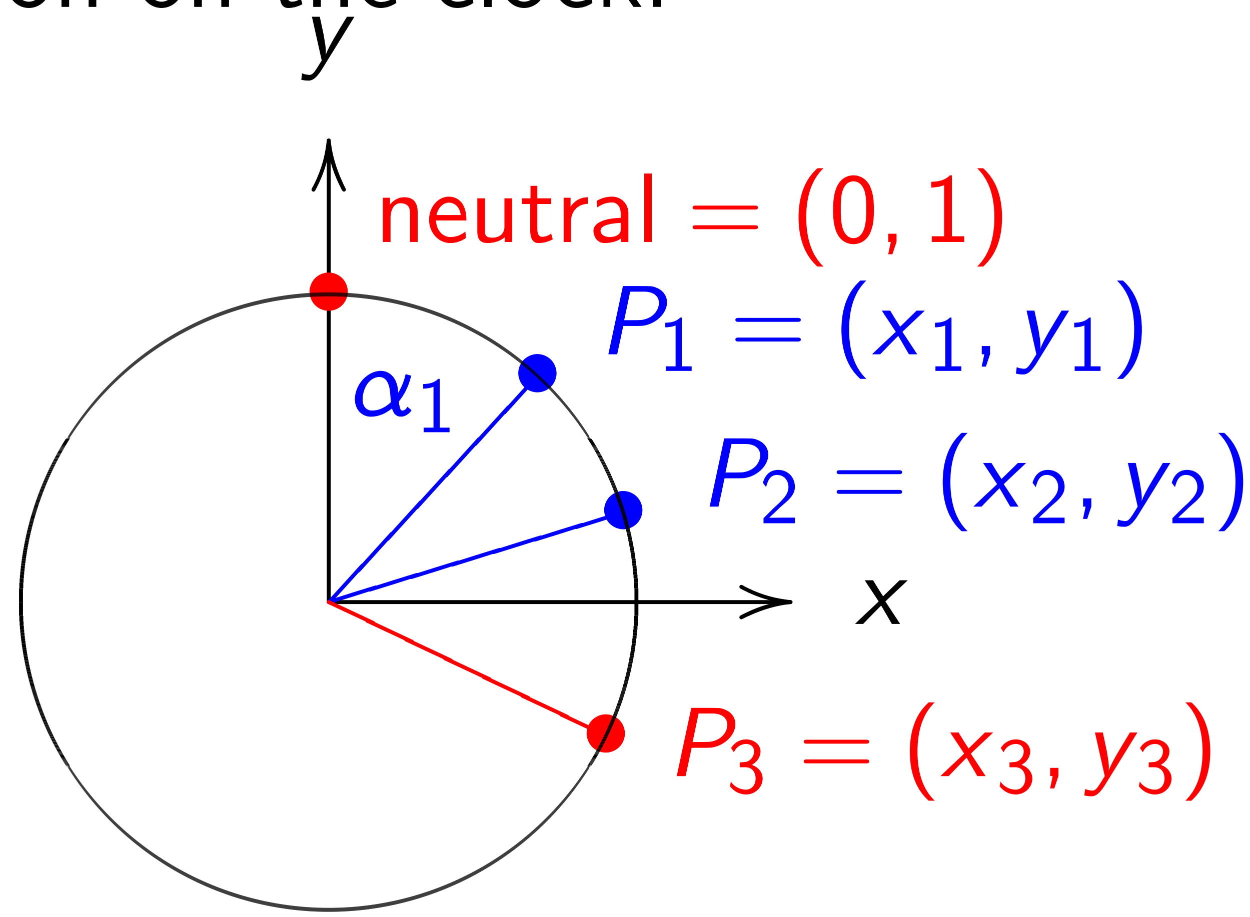

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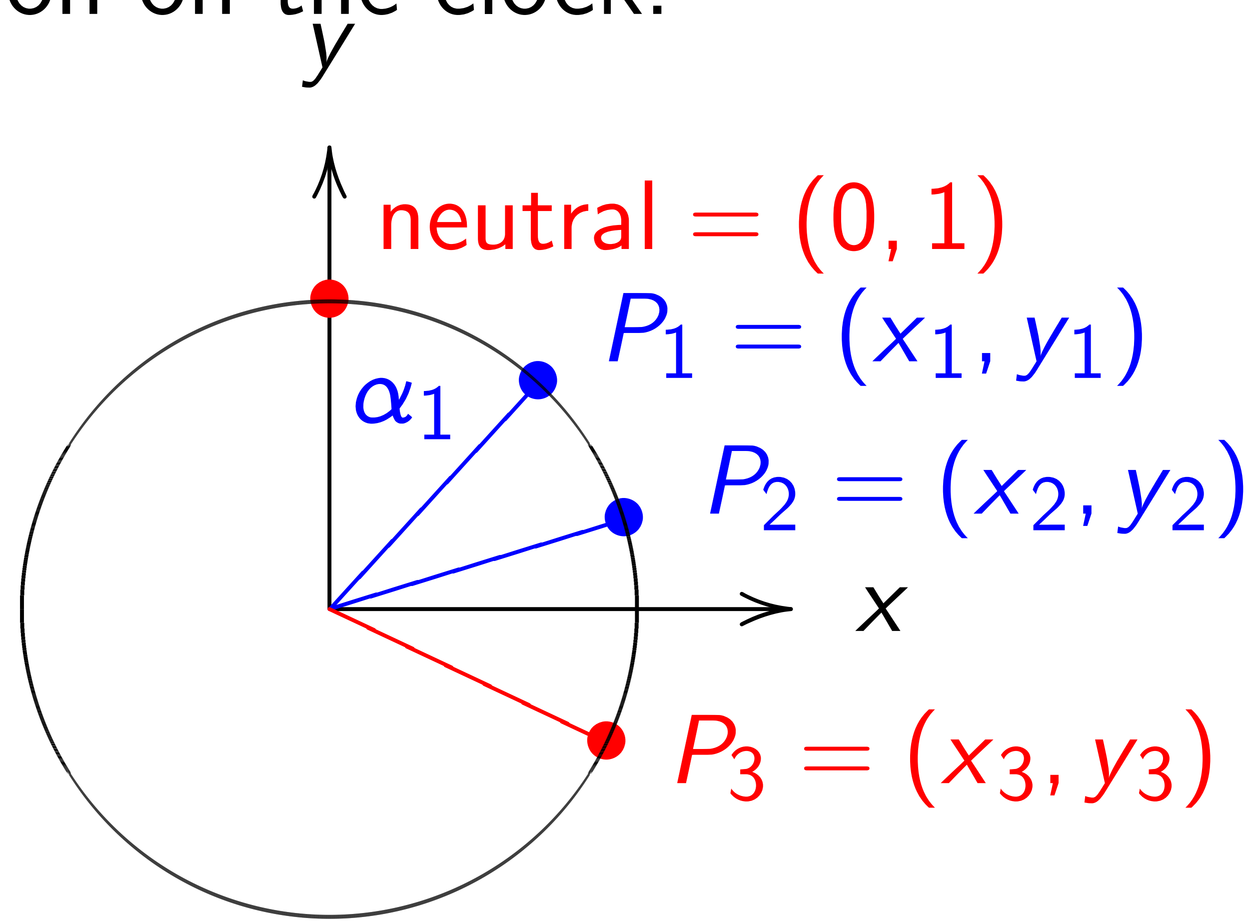

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Adding two points

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Neutral element: a

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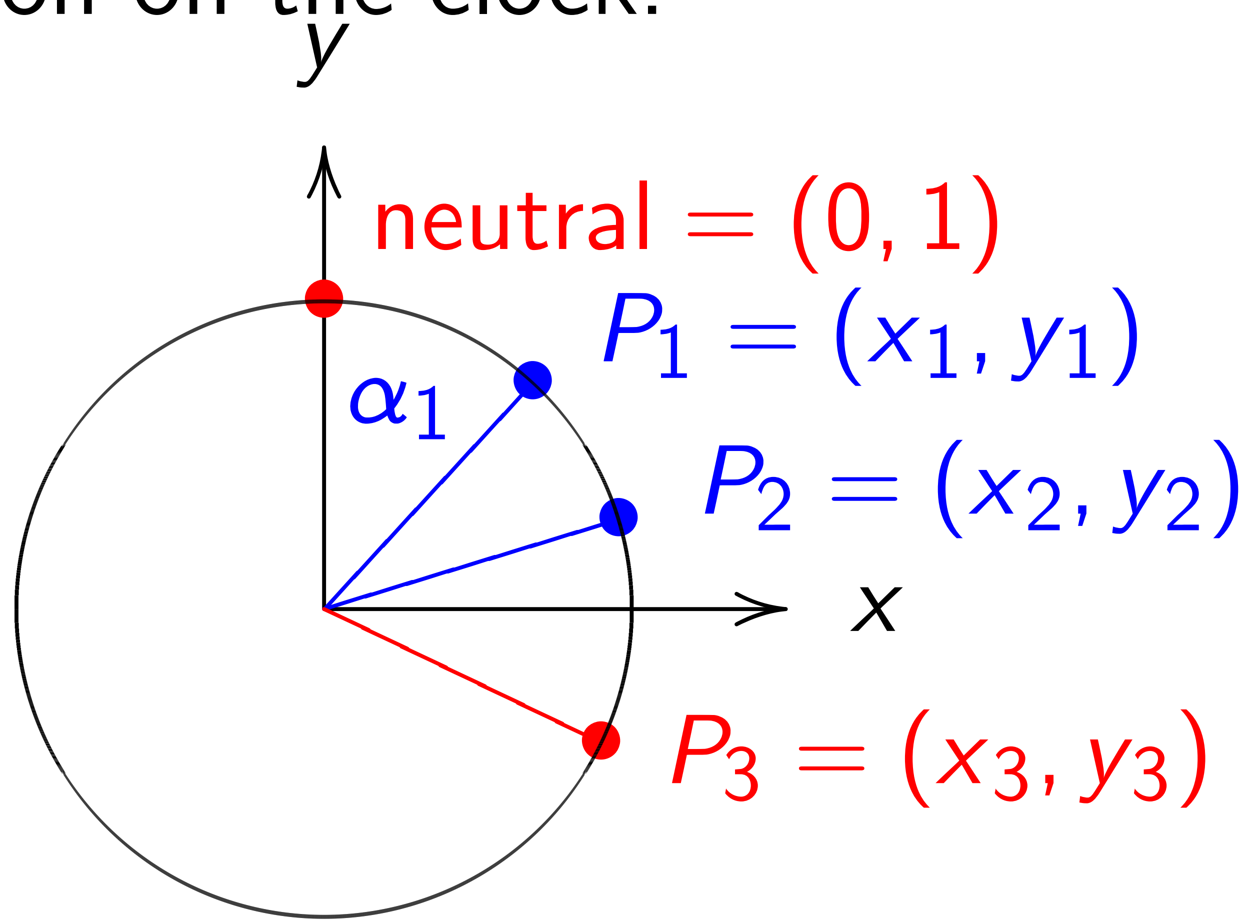
Addition on the clock:



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Adding two points correspon
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Angles modulo 360° are a g
so points on clock are a gro

Neutral element: angle α =
point (0, 1); "12:00".
The point with $\alpha = 180°$
has order 2 and equals 6:00.
3:00 and 9:00 have order 4.
Inverse of point with α
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There are many more points
where angle α is not "nice."

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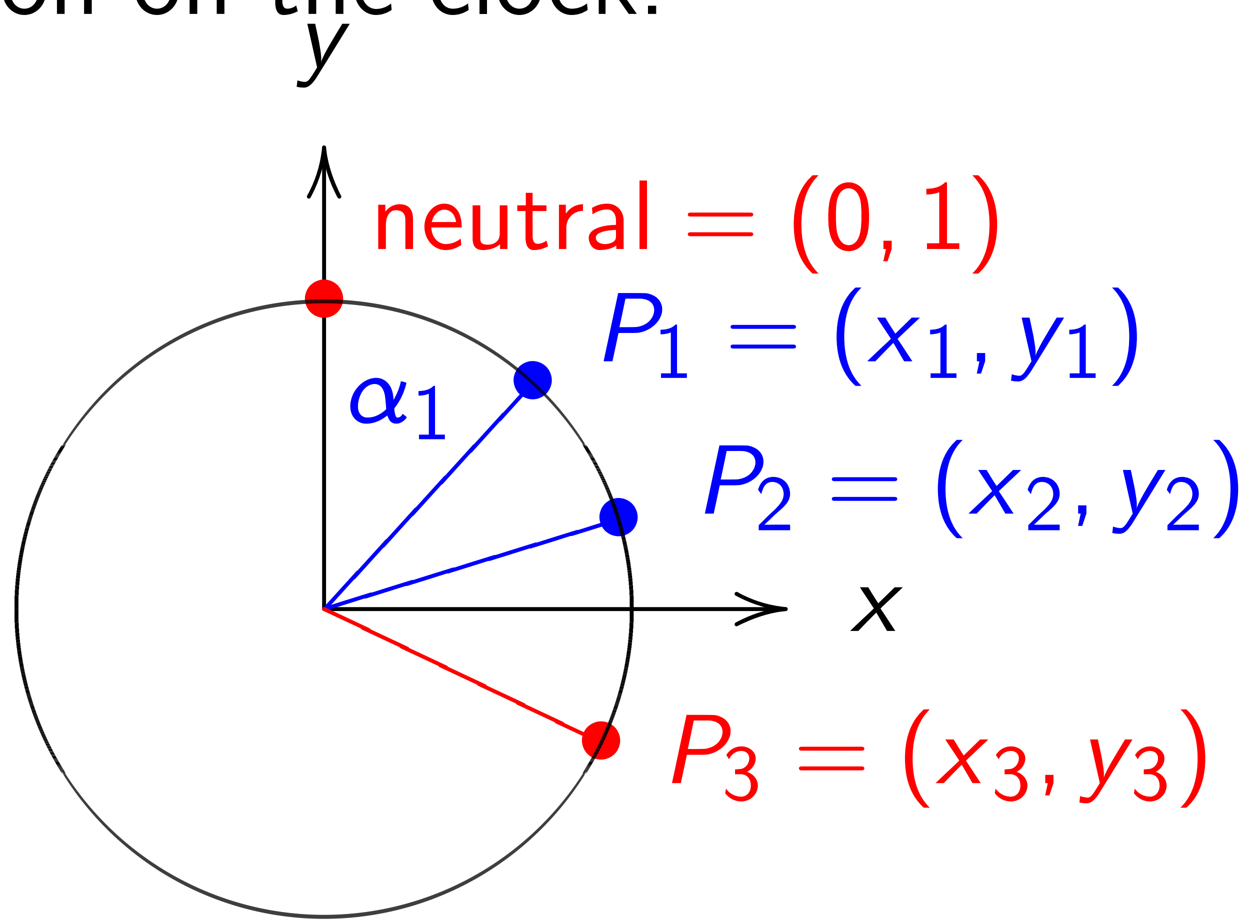

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Adding two points corresponds to adding the angles α_1 and α_2. Angles modulo 360° are a group, so points on clock are a group.

Neutral element: angle $\alpha = 0$; point $(0, 1)$; "12:00".
The point with $\alpha = 180^\circ$ has order 2 and equals 6:00.
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Inverse of point with α is point with $-\alpha$ since $\alpha + (-\alpha) = 0$.
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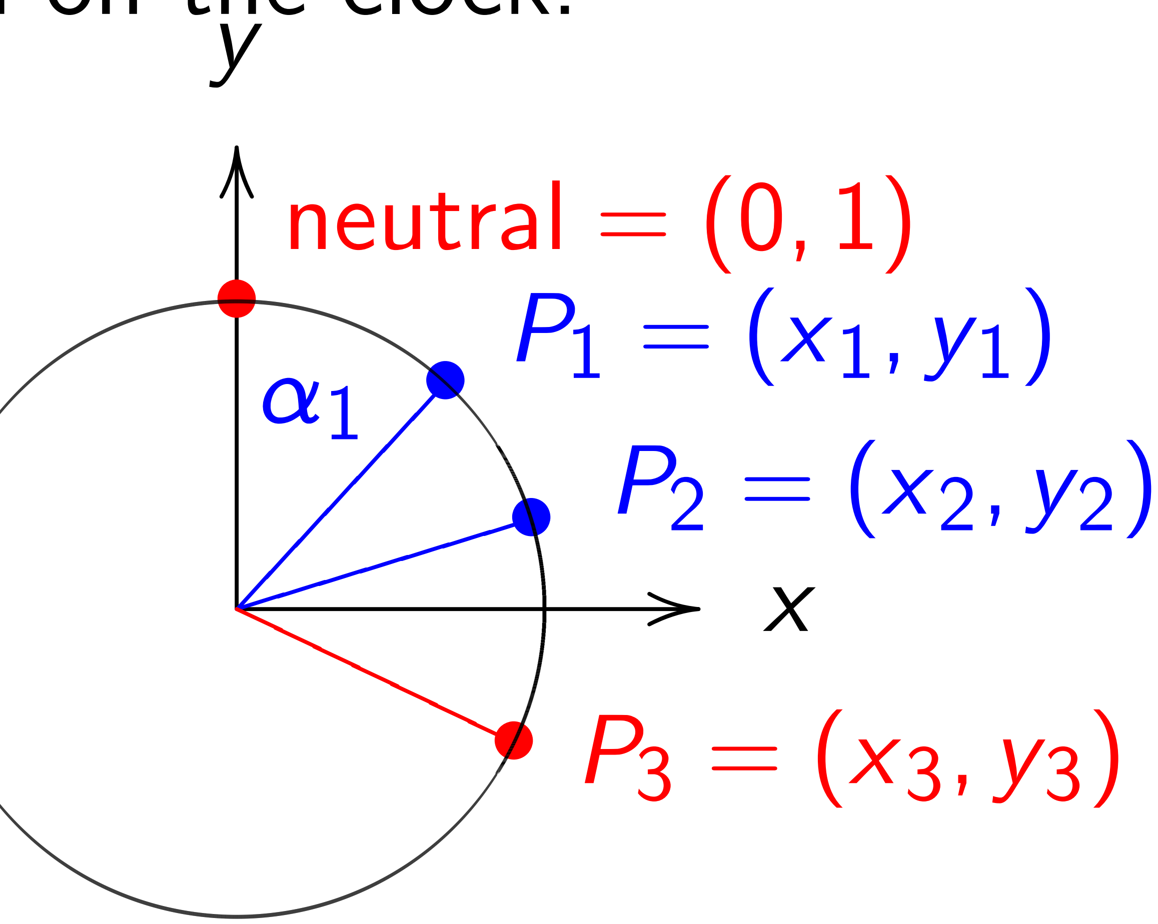

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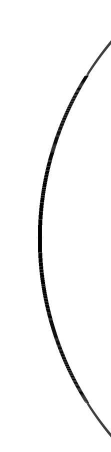

$x^2 + y^2$

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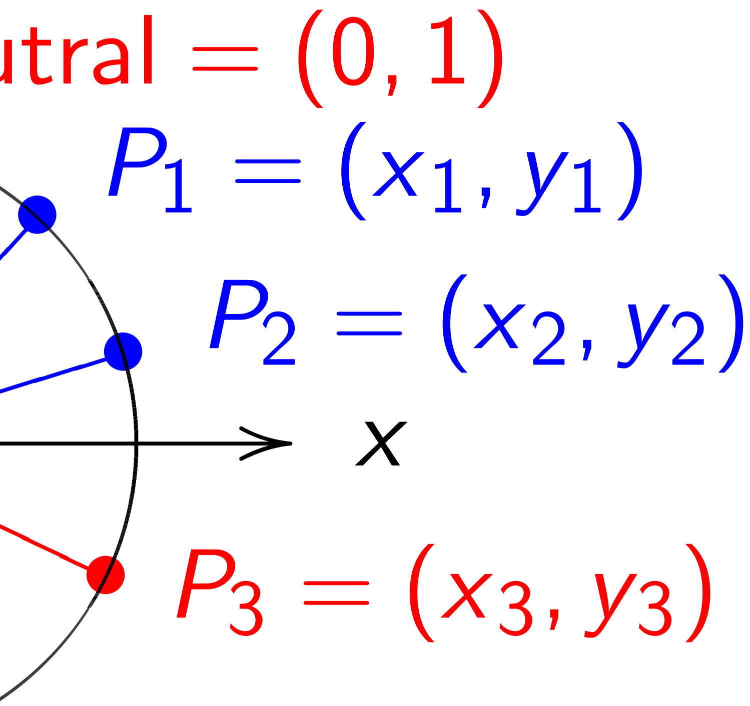

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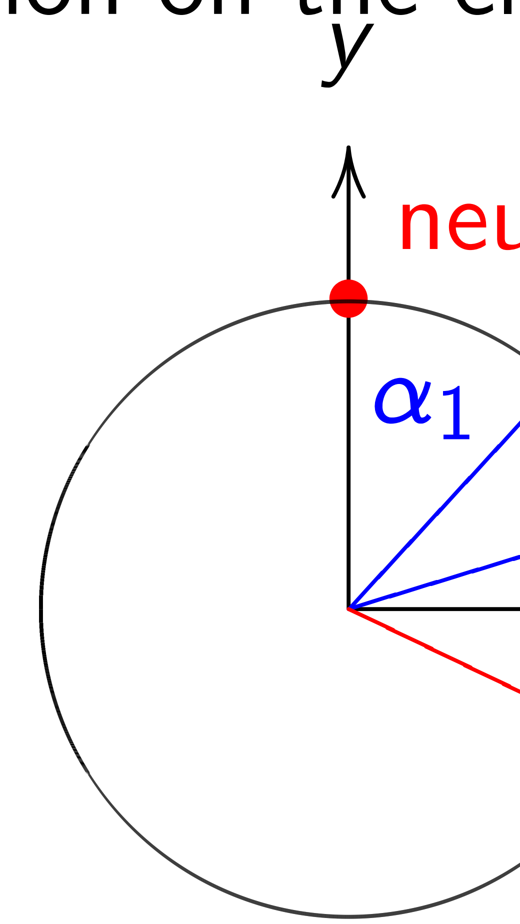

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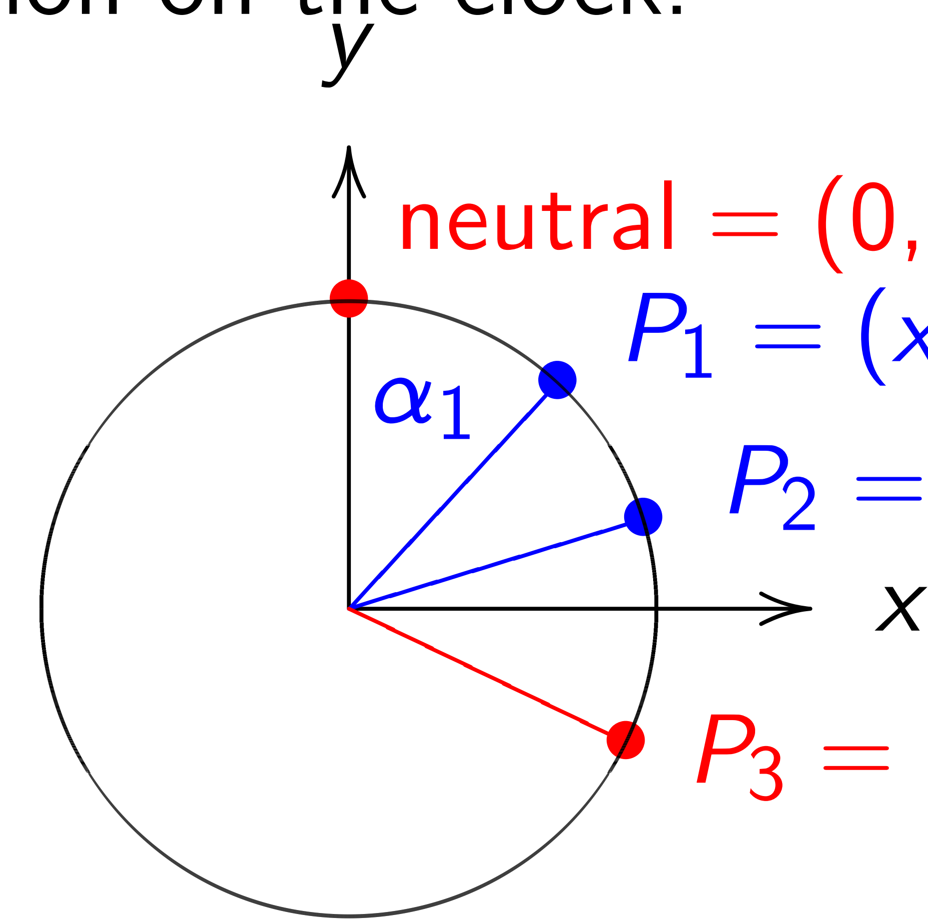

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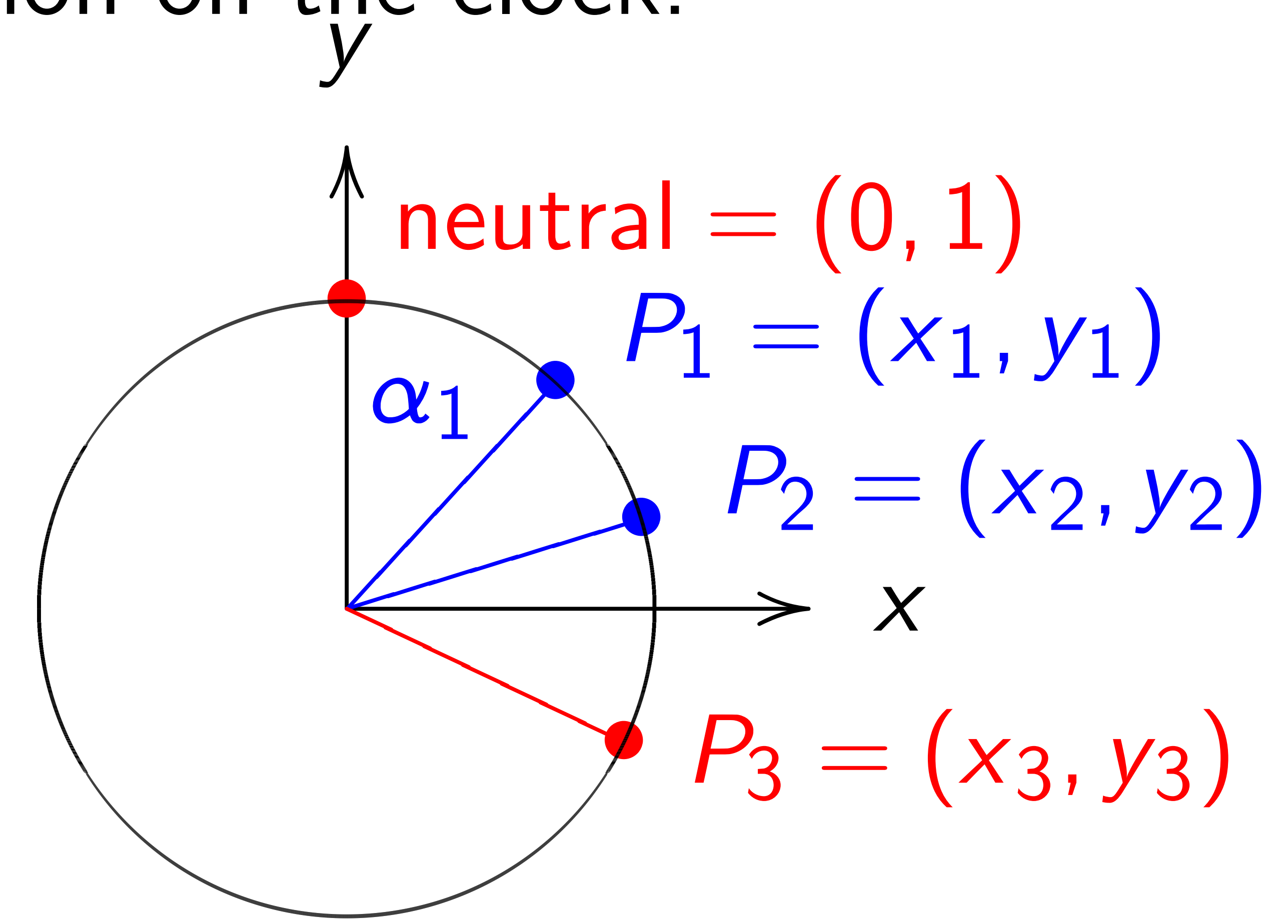

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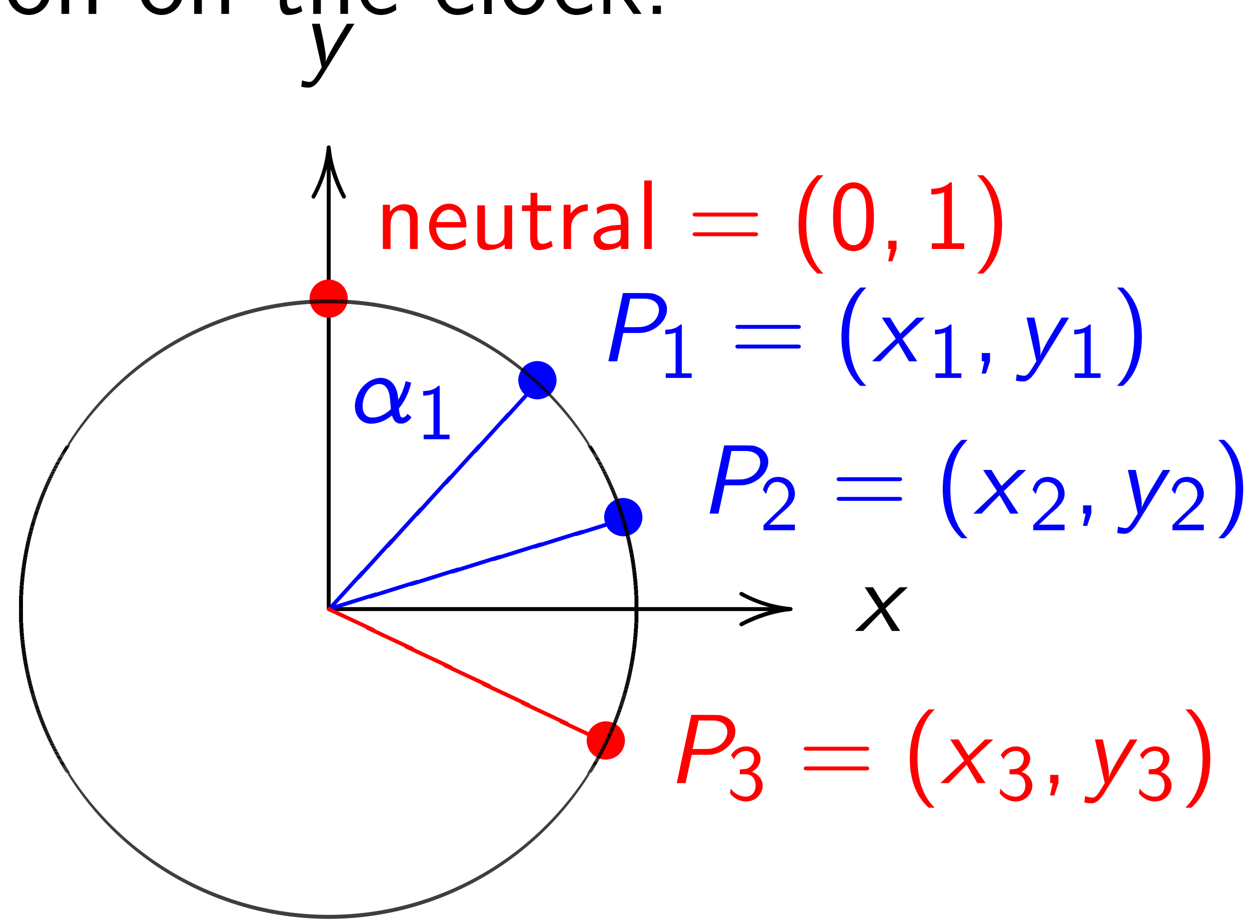

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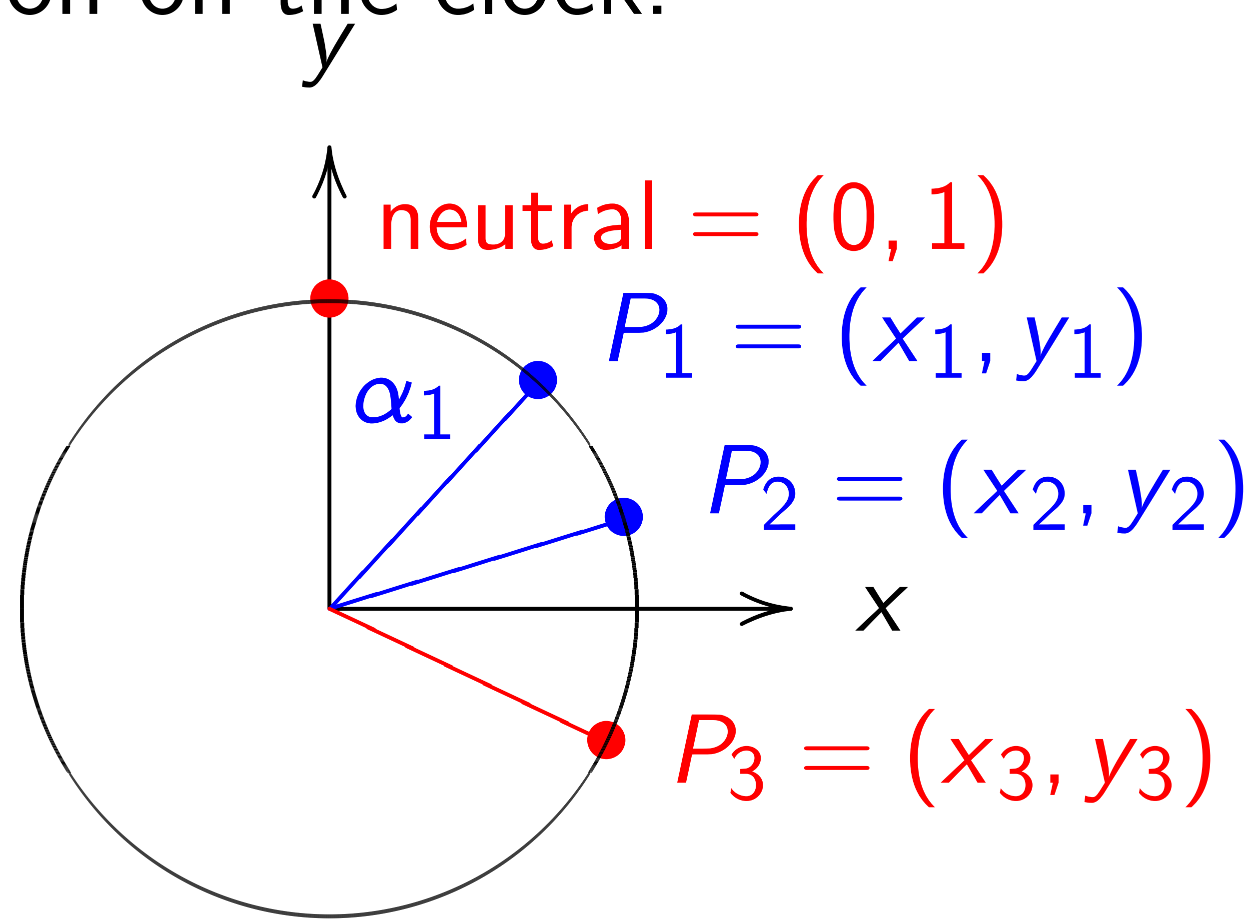

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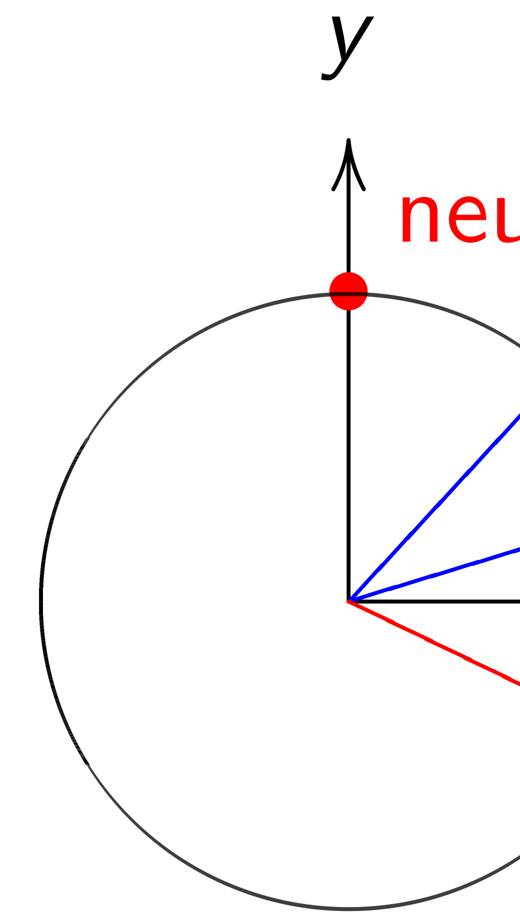

Use Cartesian coor
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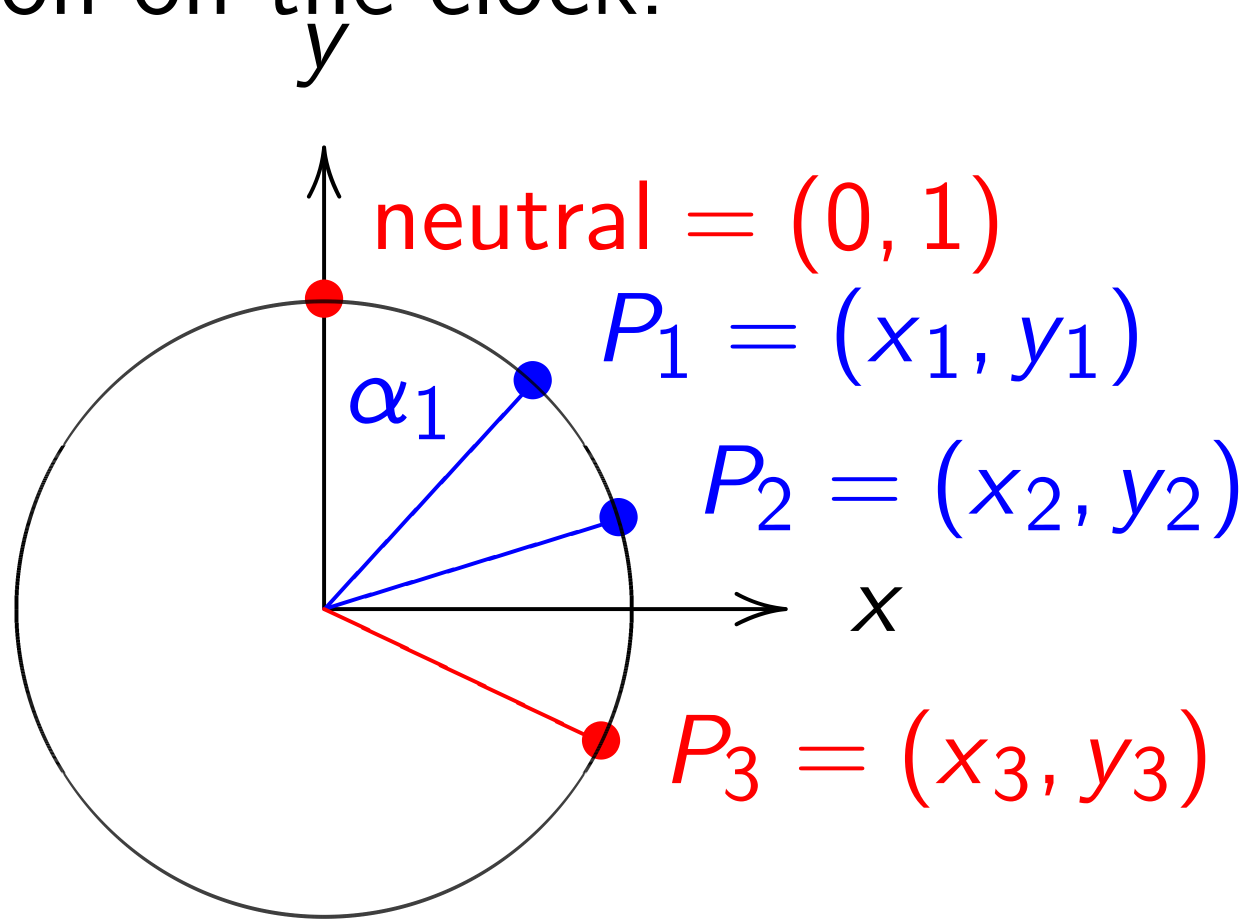

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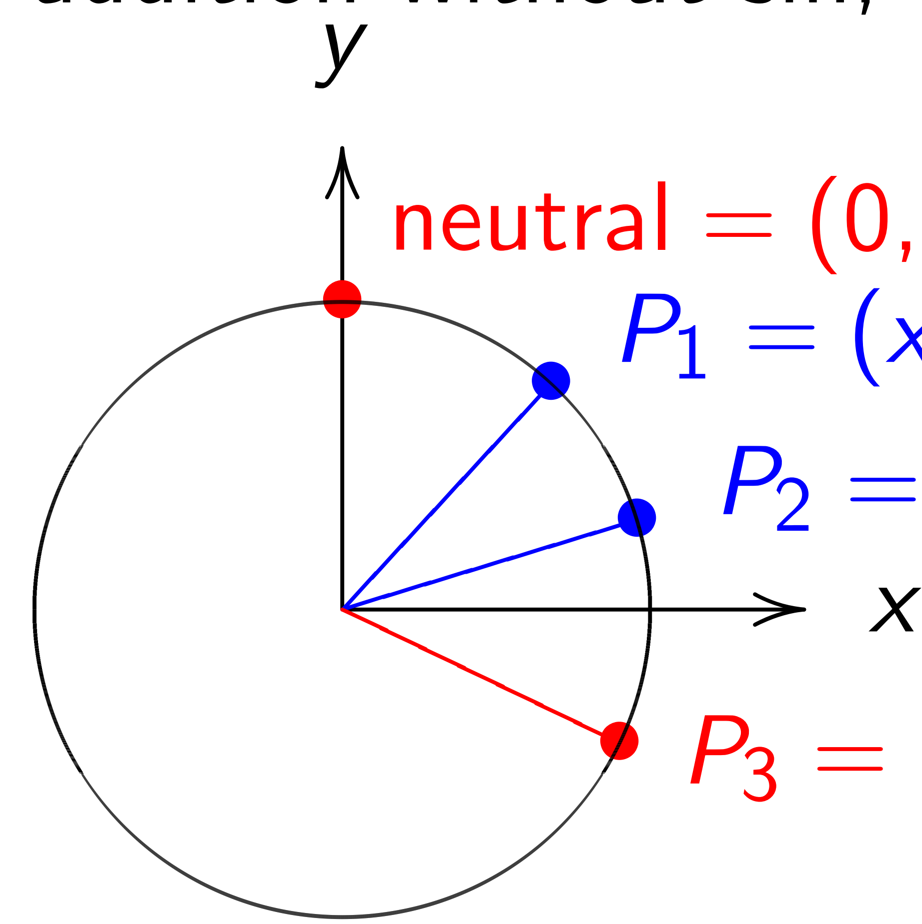

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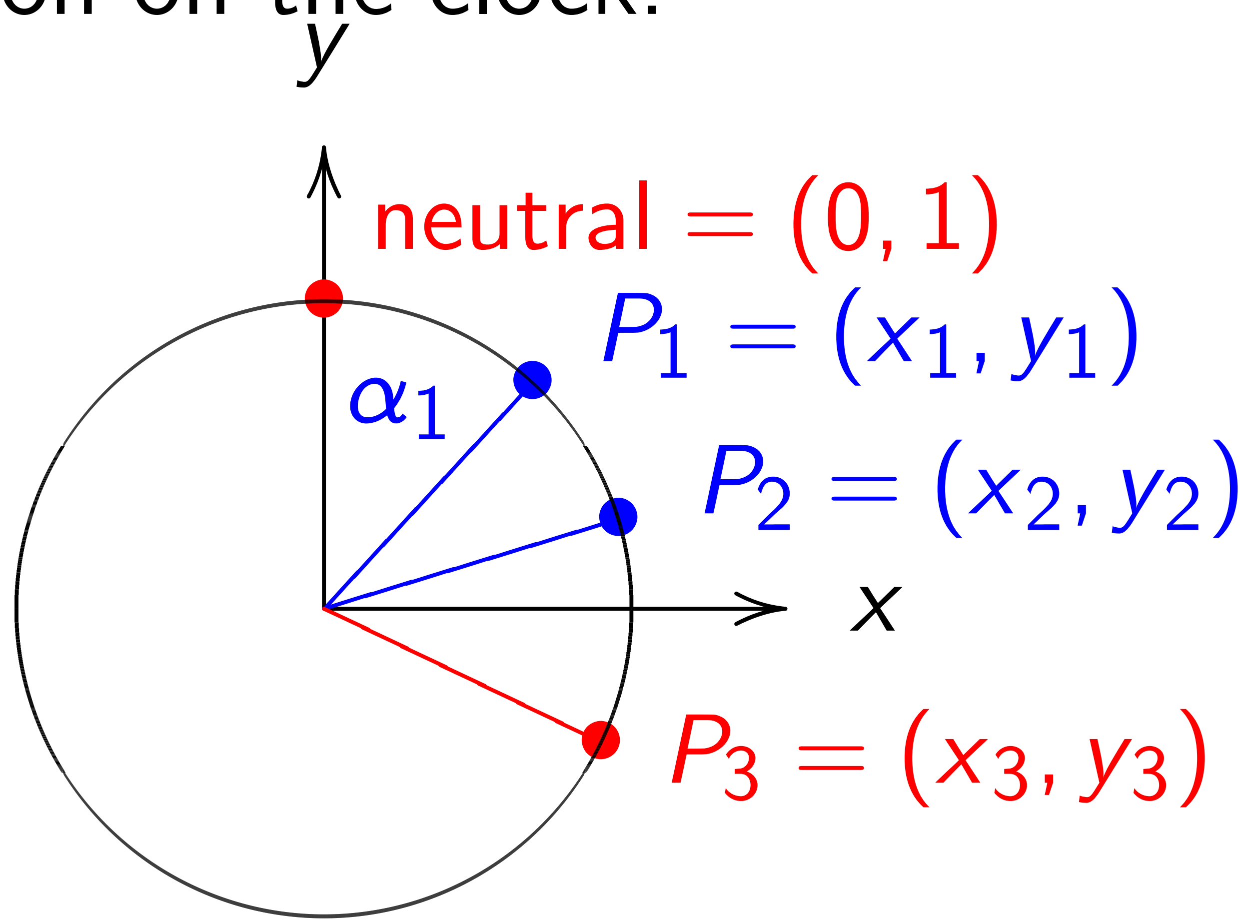

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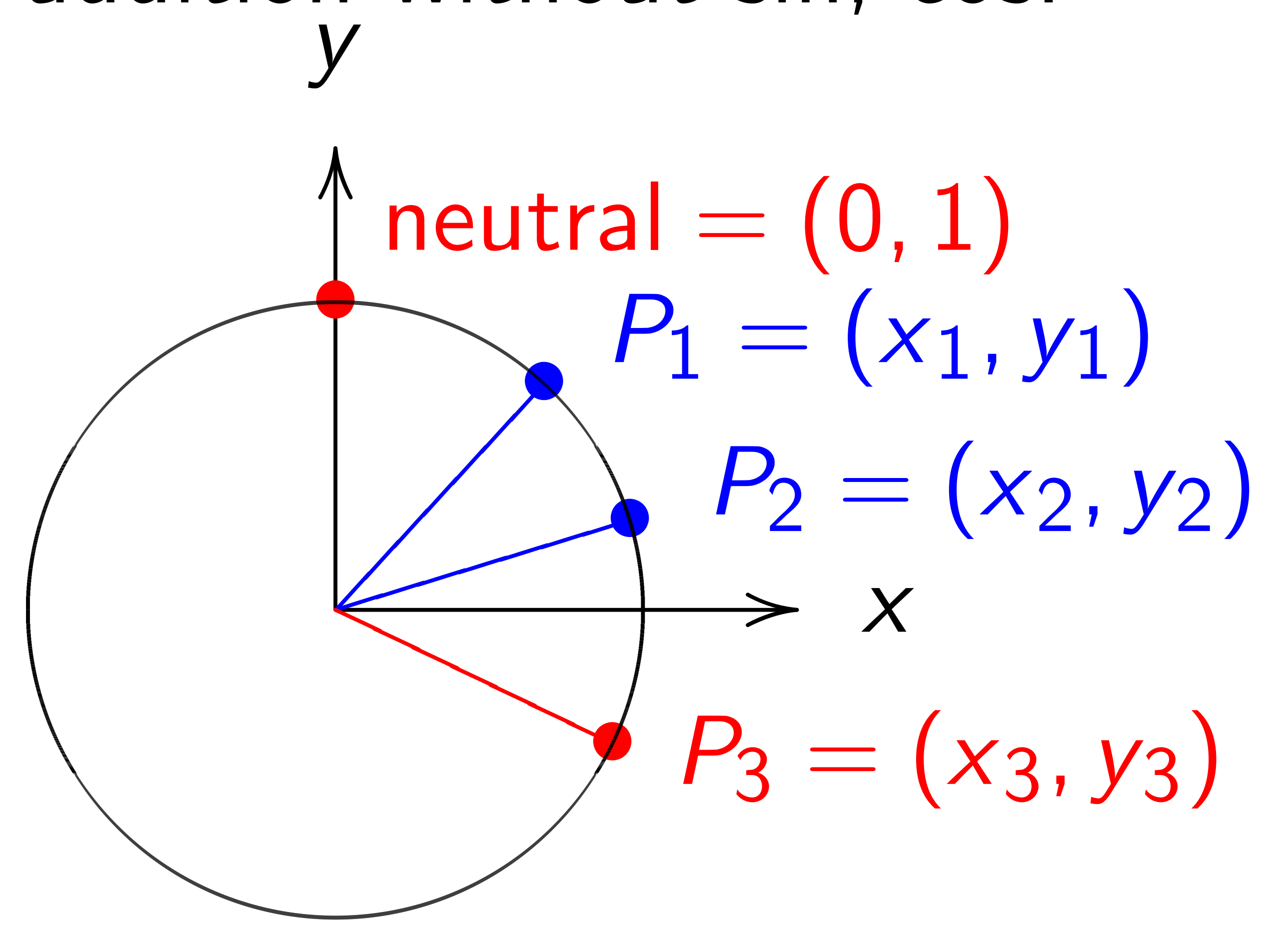
Clock addition without sin, cos:



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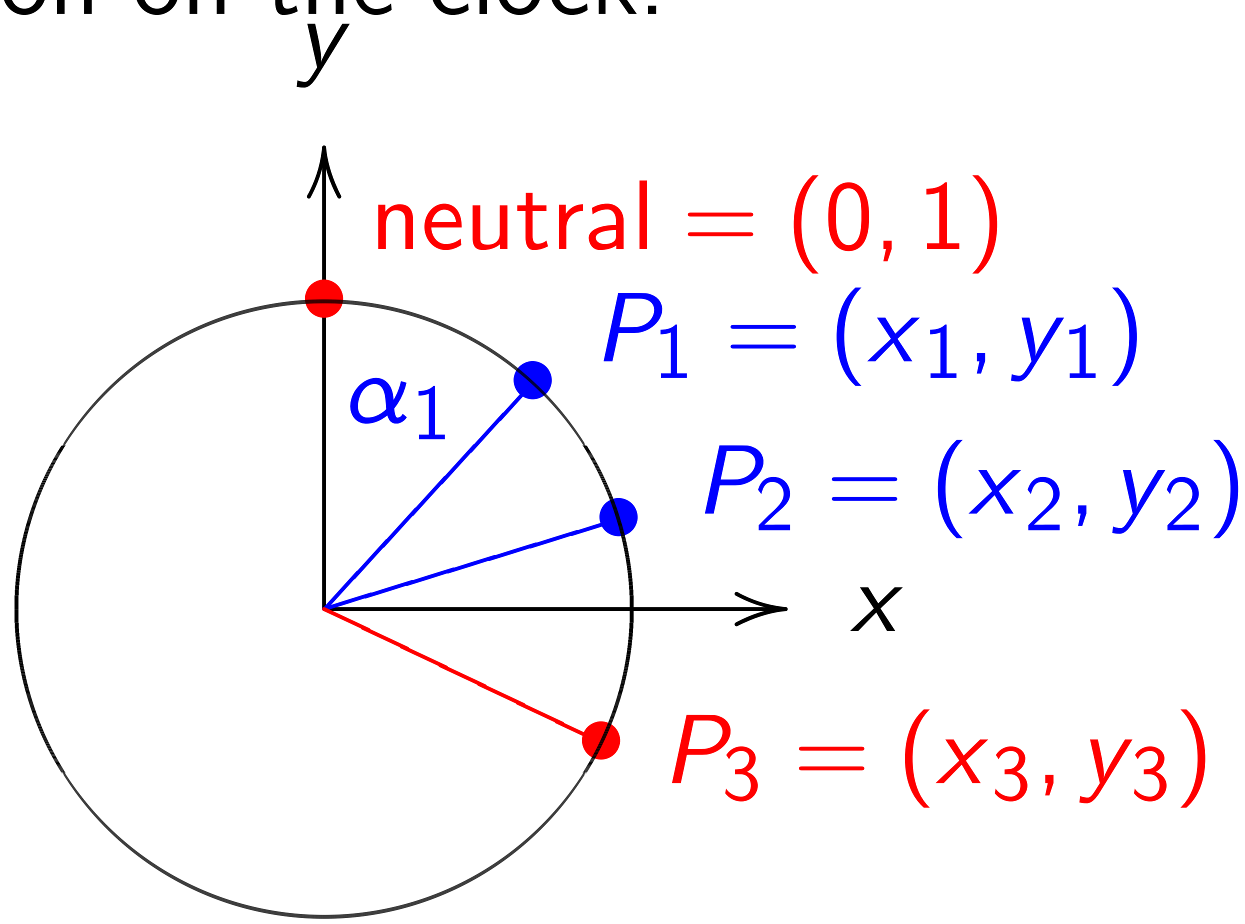

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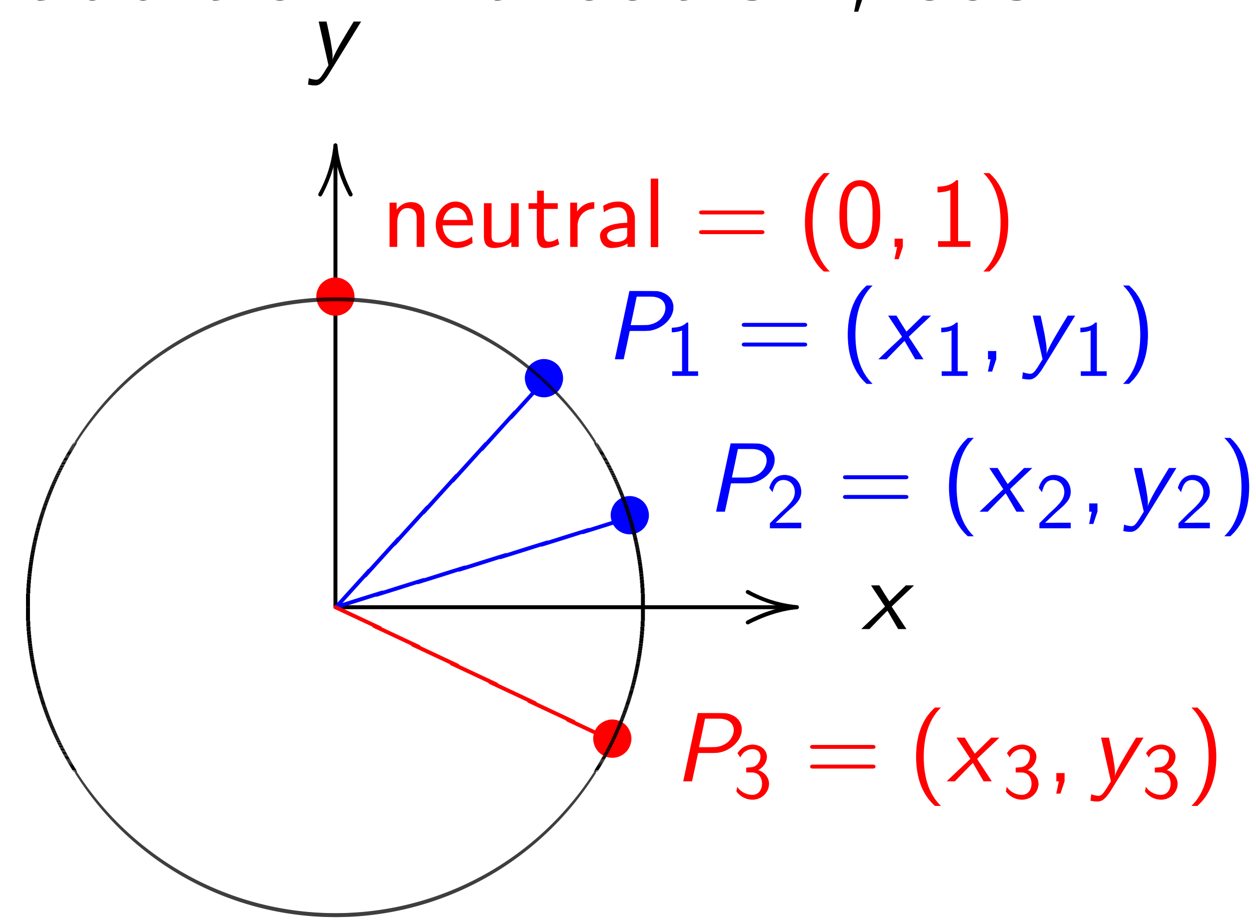

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Note $(x_1, y_1) + (-x_1, y_1) = (0, 1)$.
$kP = \underbrace{P + P + \cdots + P}_{k \text{ copies}}$ for $k \geq 0$.

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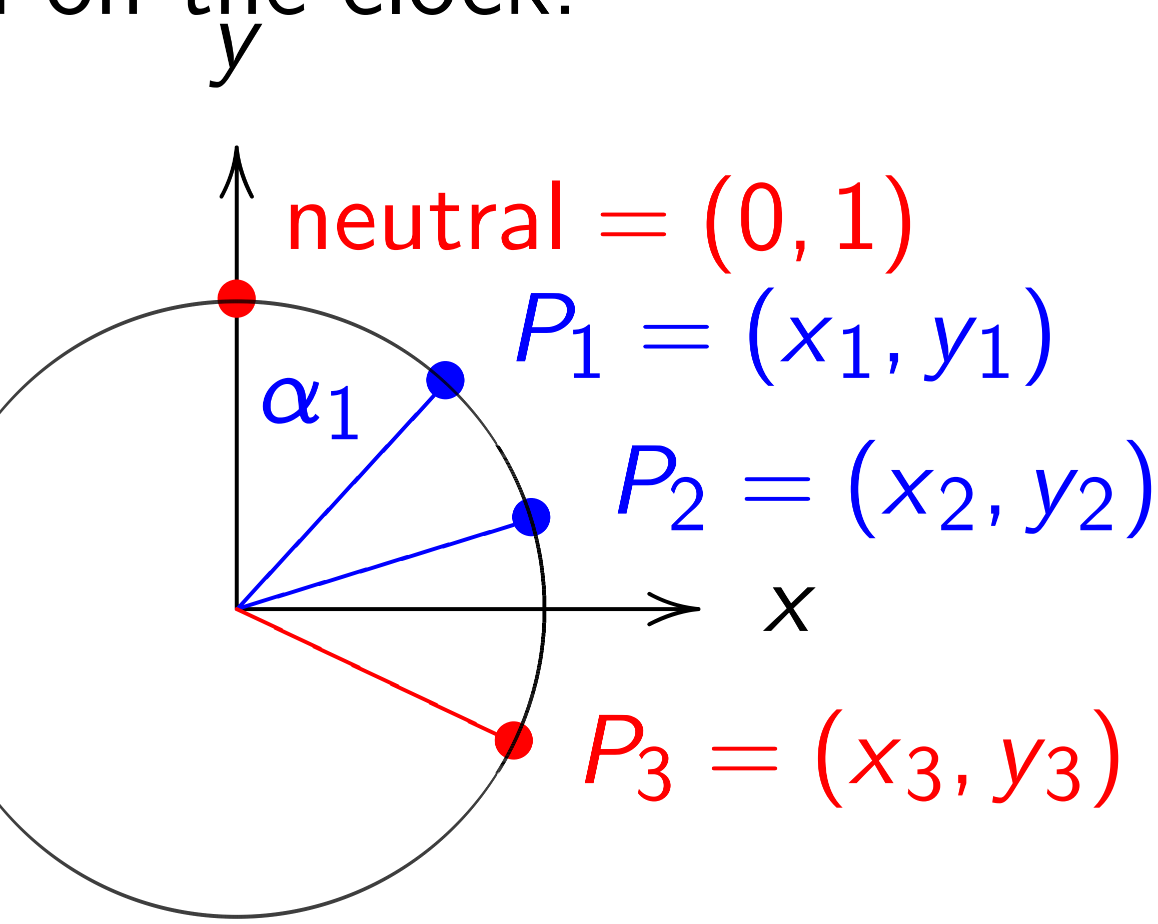

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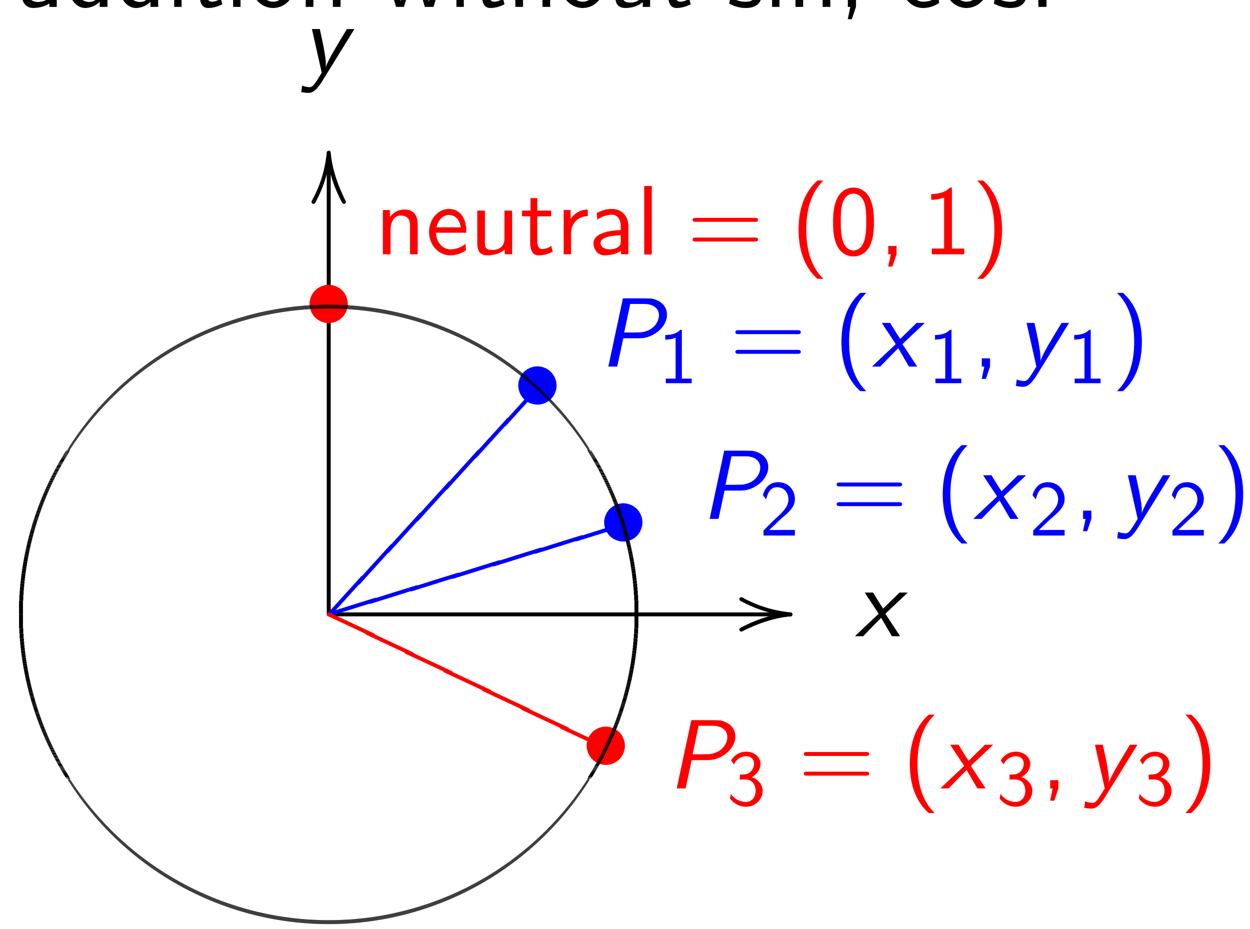

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Example

"2:00" +

$= (\sqrt{3}/$

$= (-1/2$

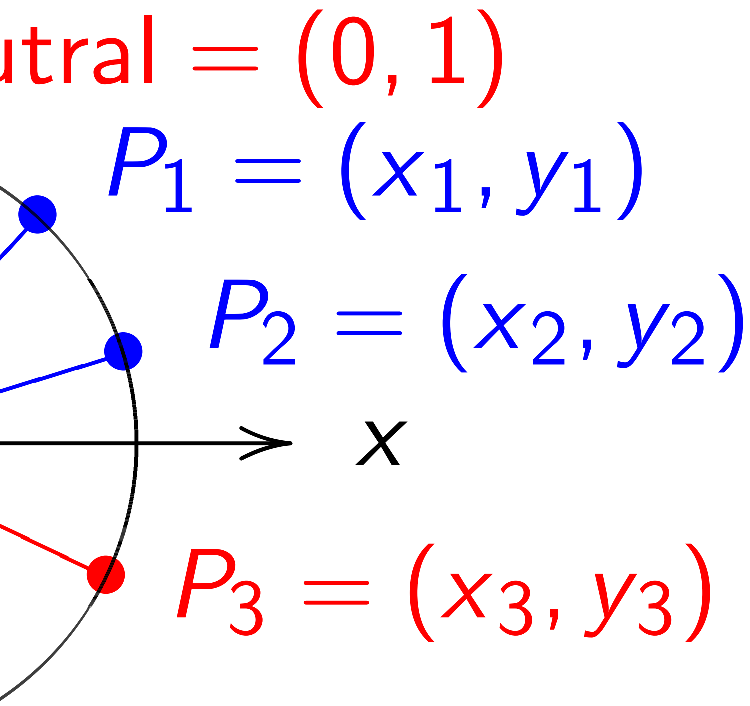
"5:00" +

$= (1/2,$

$= (\sqrt{3}/$

$2\left(\frac{3}{5}, \frac{4}{5}\right)$

ock:



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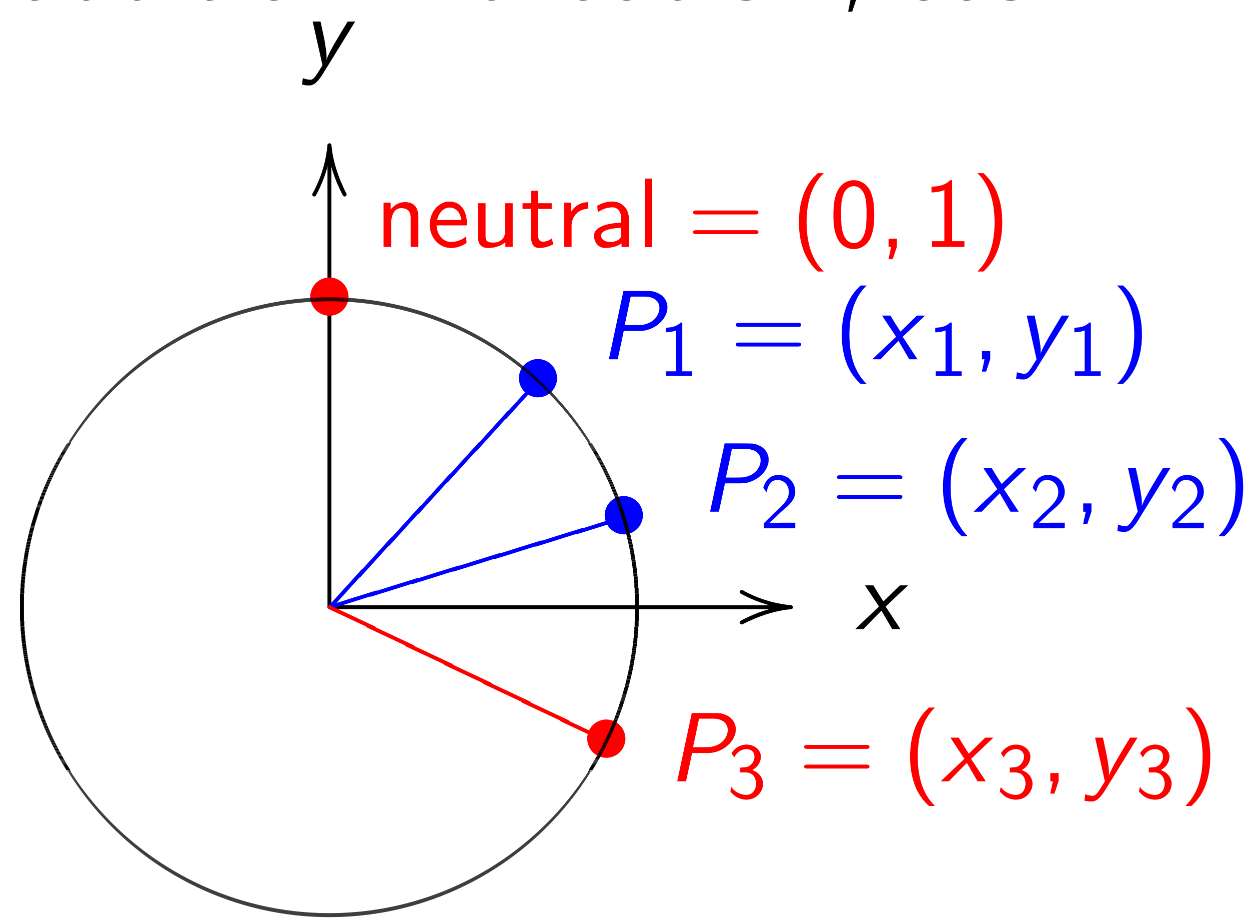

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Examples of clock

"2:00" + "5:00"

$= (\sqrt{3/4}, 1/2) +$

$= (-1/2, -\sqrt{3/4}$

"5:00" + "9:00"

$= (1/2, -\sqrt{3/4})$

$= (\sqrt{3/4}, 1/2) =$

$2\left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{24}{25},\right.$

$1)$

$x_1, y_1)$

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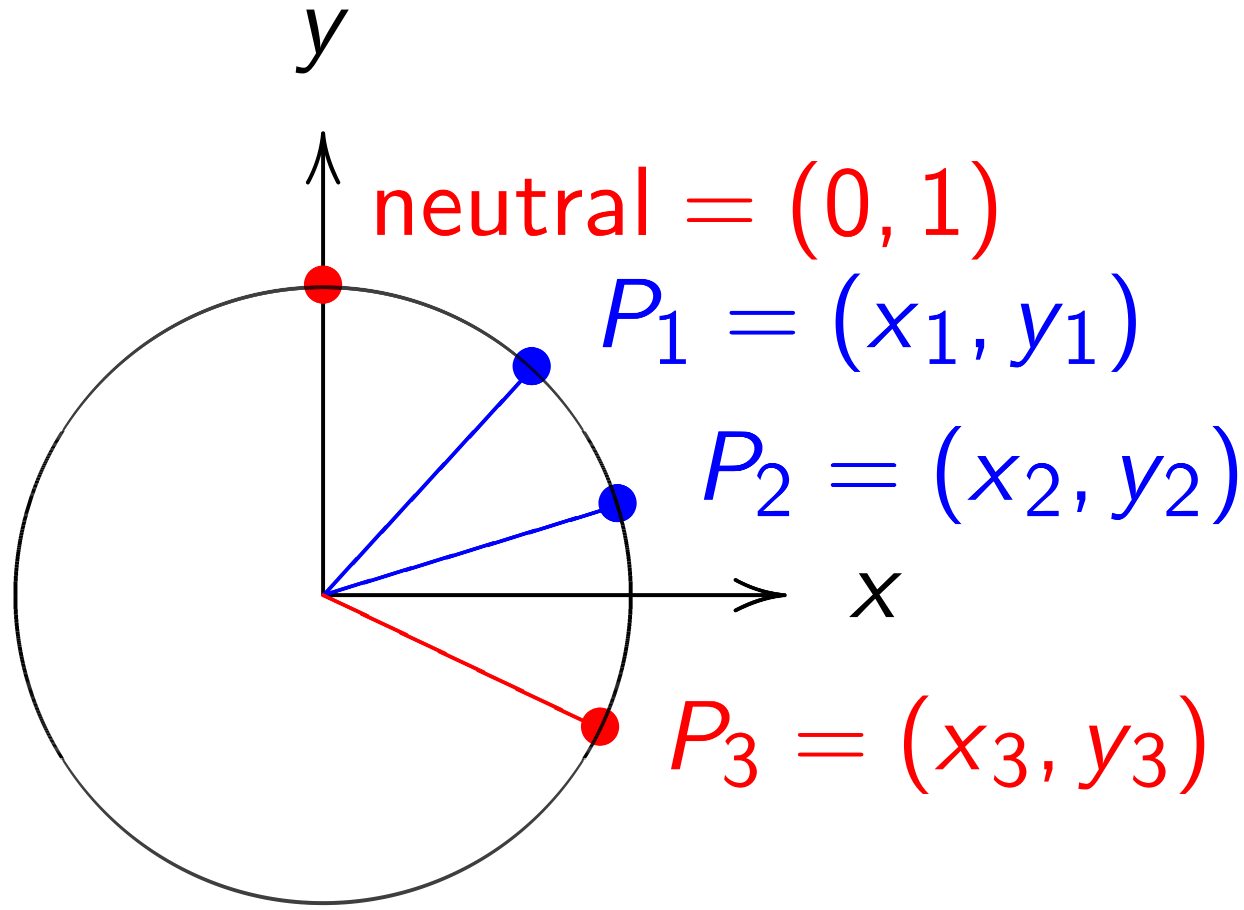
Clock addition without sin, cos:



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"2:00" + "5:00"

$= (\sqrt{3/4}, 1/2) + (1/2, -\sqrt{3}$

$= (-1/2, -\sqrt{3/4}) =$ "7:00"

"5:00" + "9:00"

$= (1/2, -\sqrt{3/4}) + (-1, 0)$

$= (\sqrt{3/4}, 1/2) =$ "2:00".

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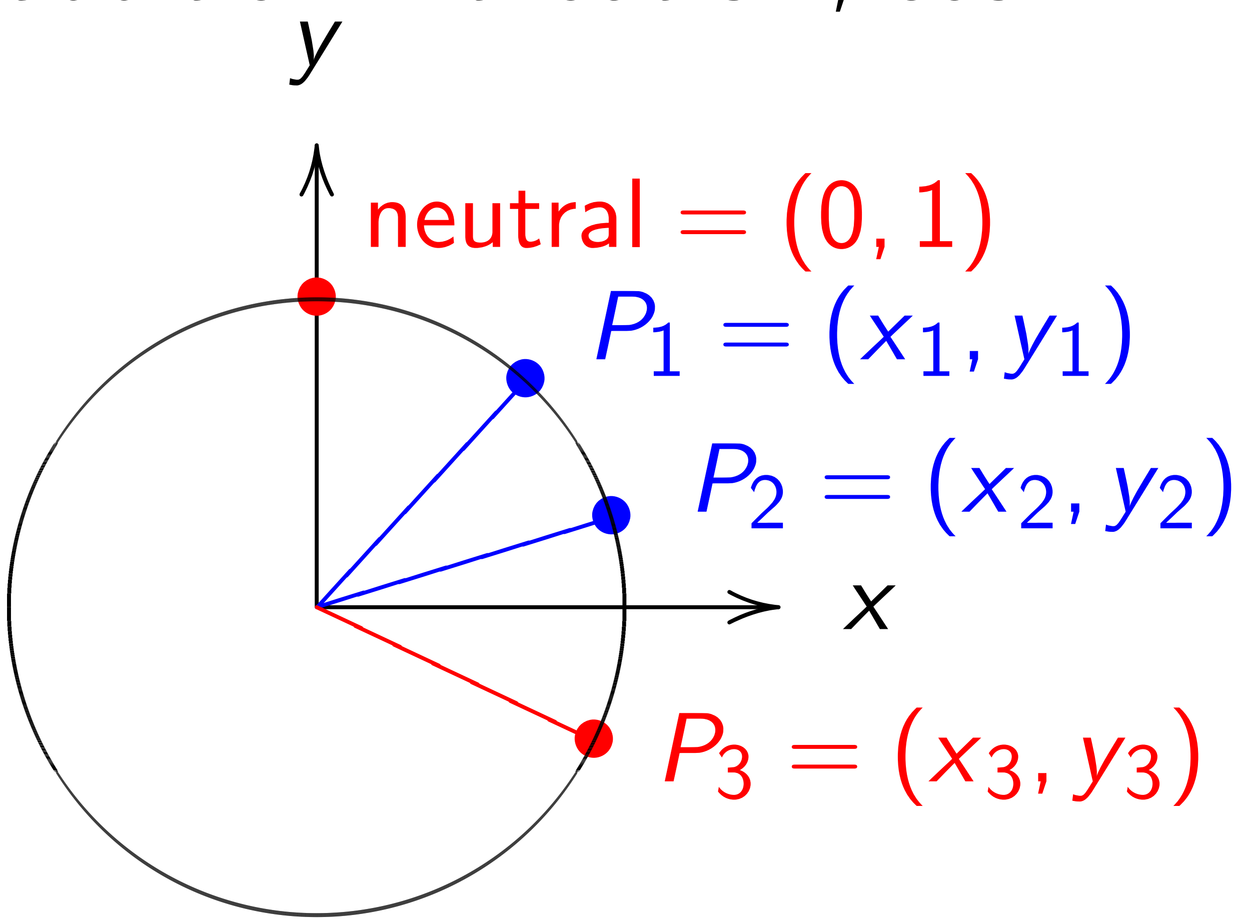

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Examples of clock addition:

"2:00" + "5:00"

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$= (-1/2, -\sqrt{3/4}) =$ "7:00".

"5:00" + "9:00"

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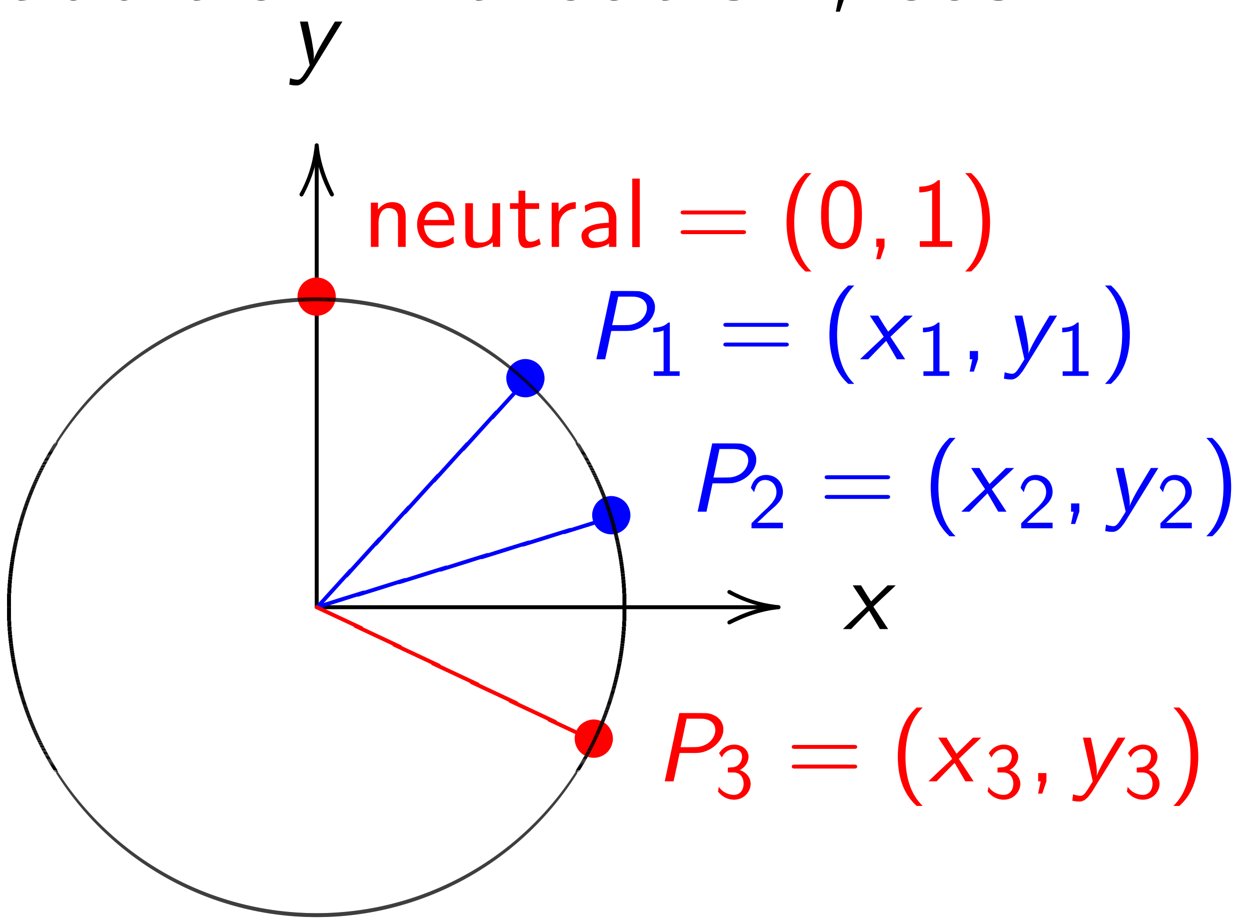

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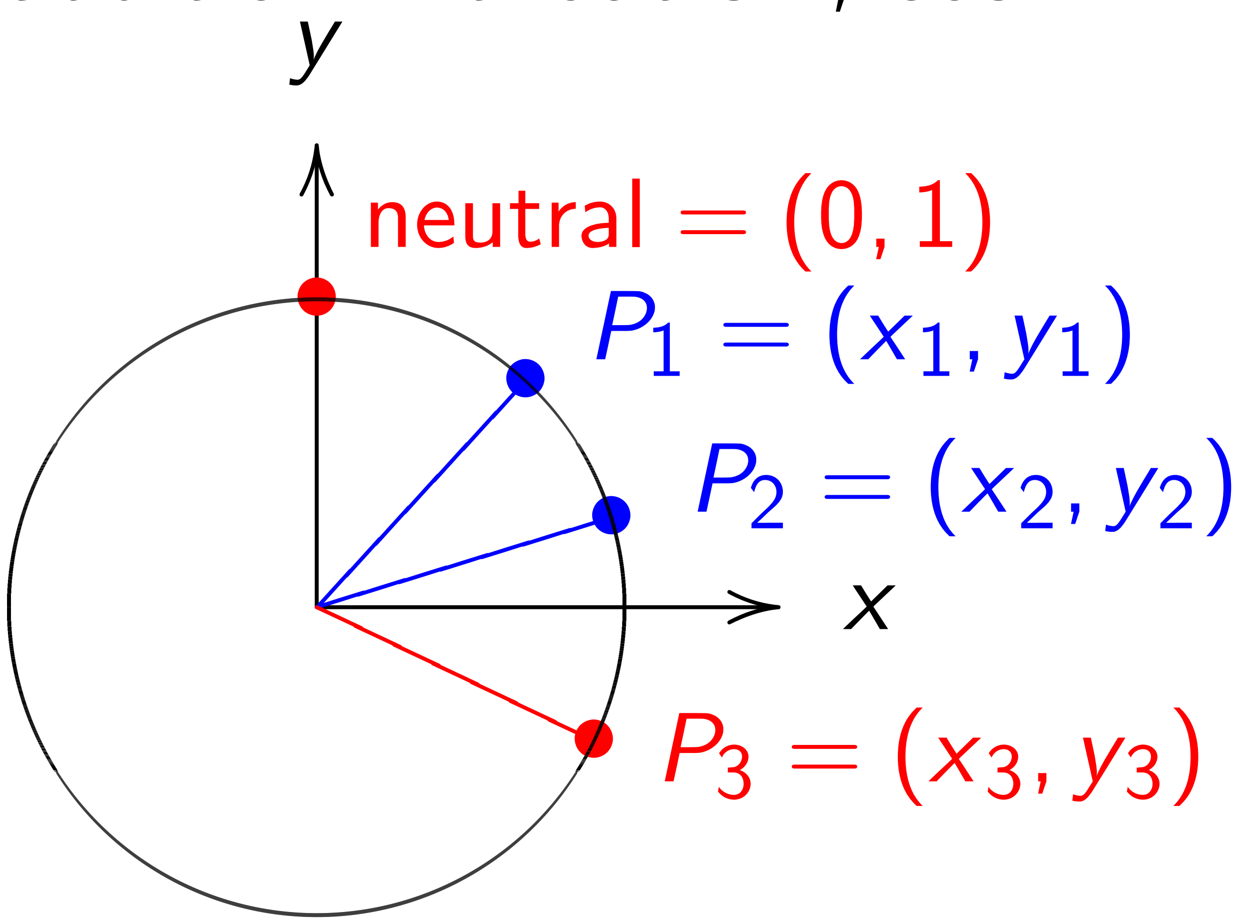

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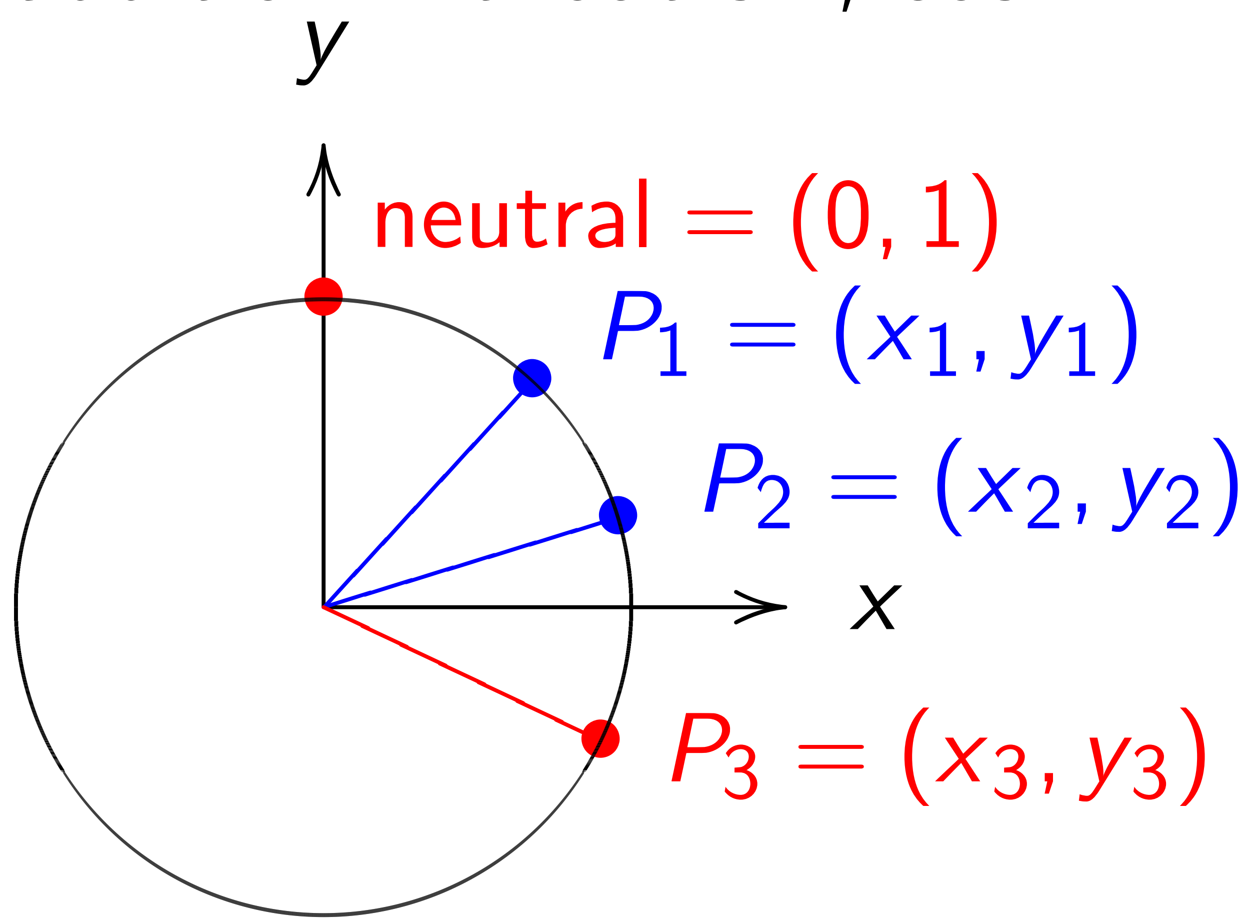

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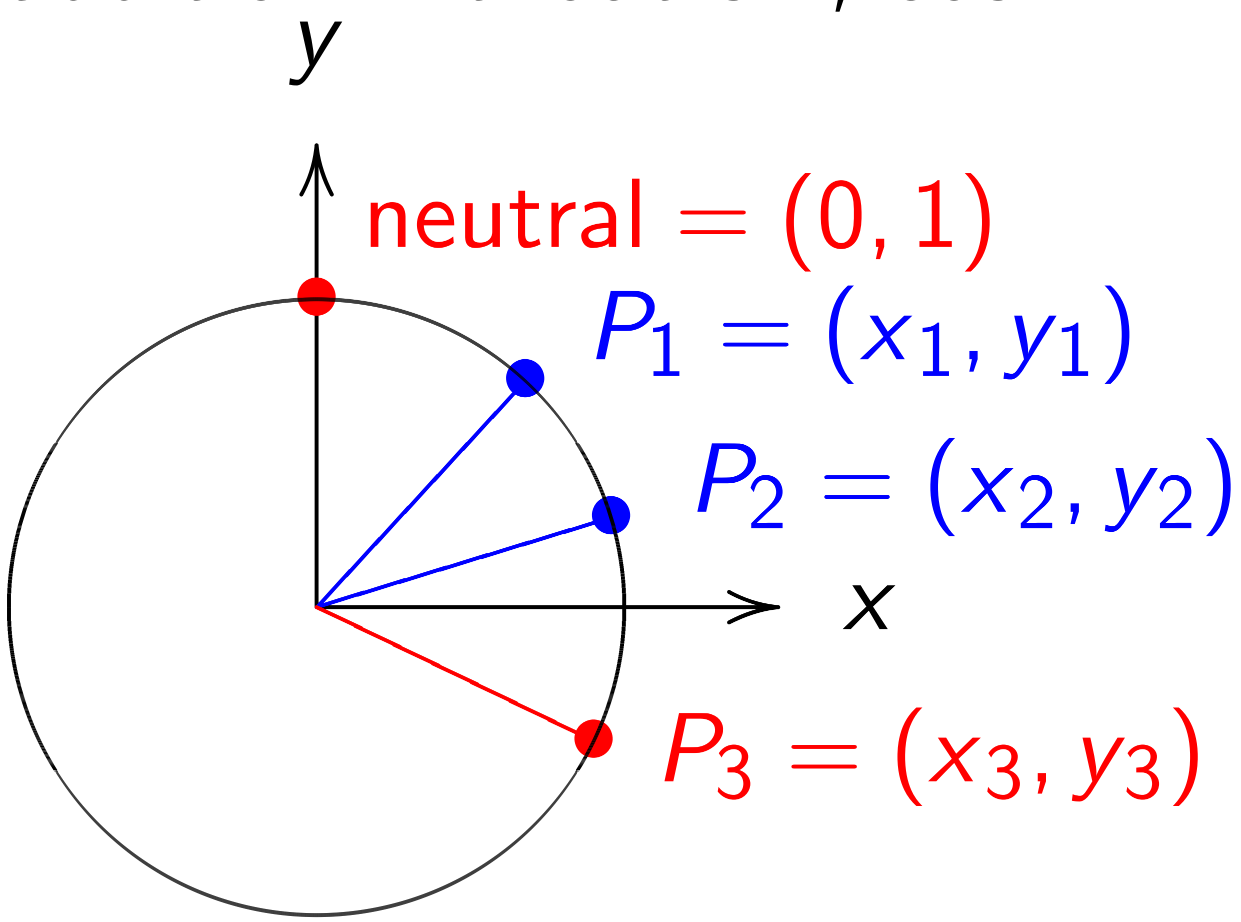

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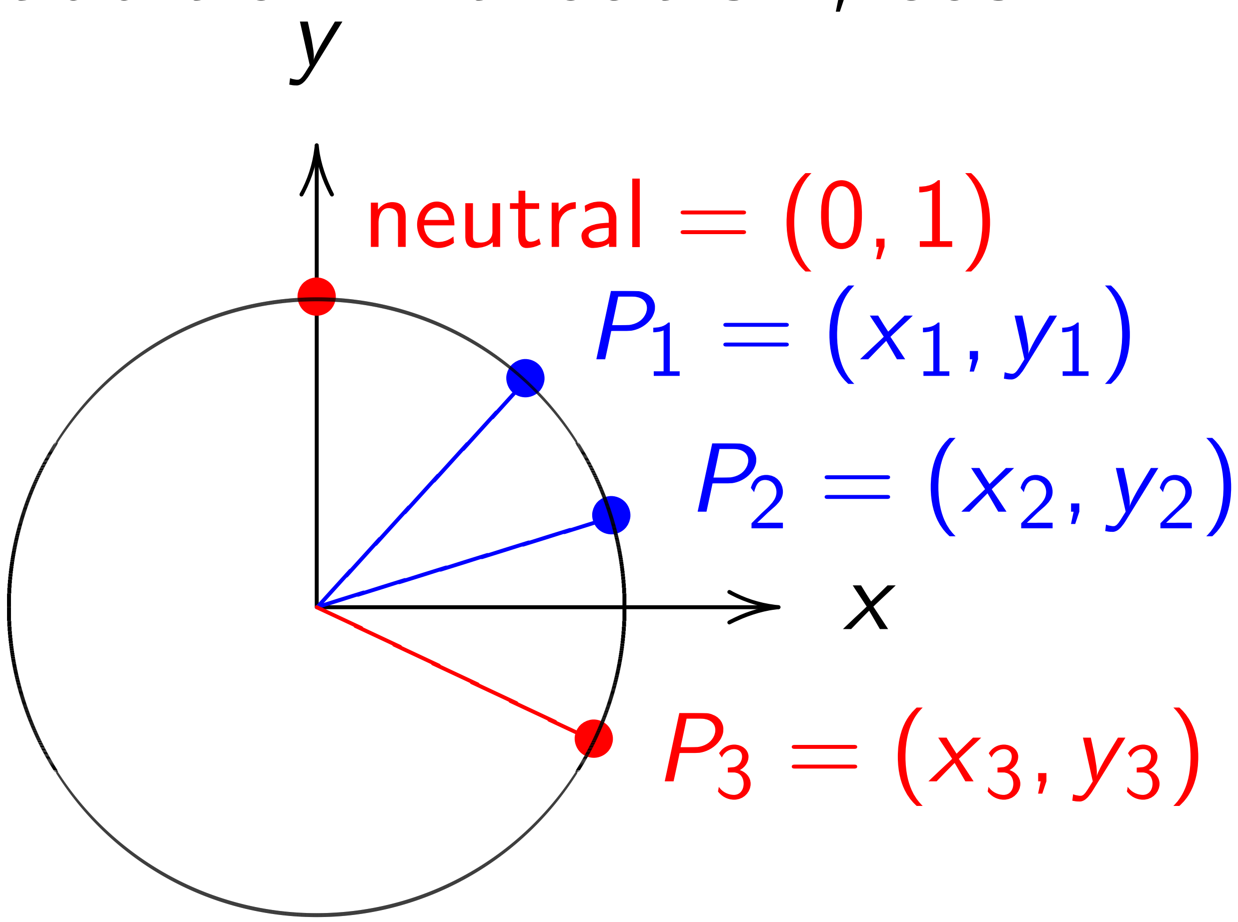

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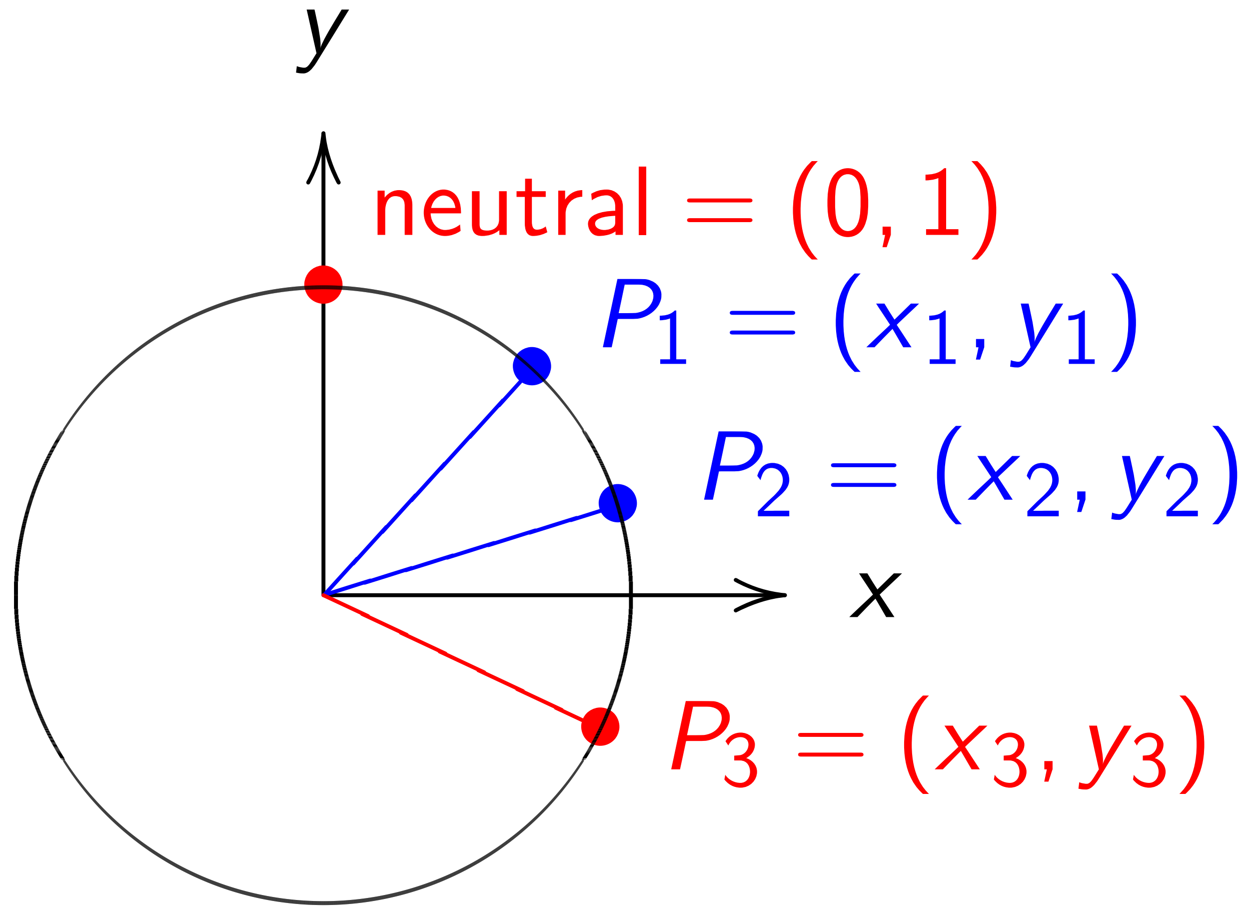

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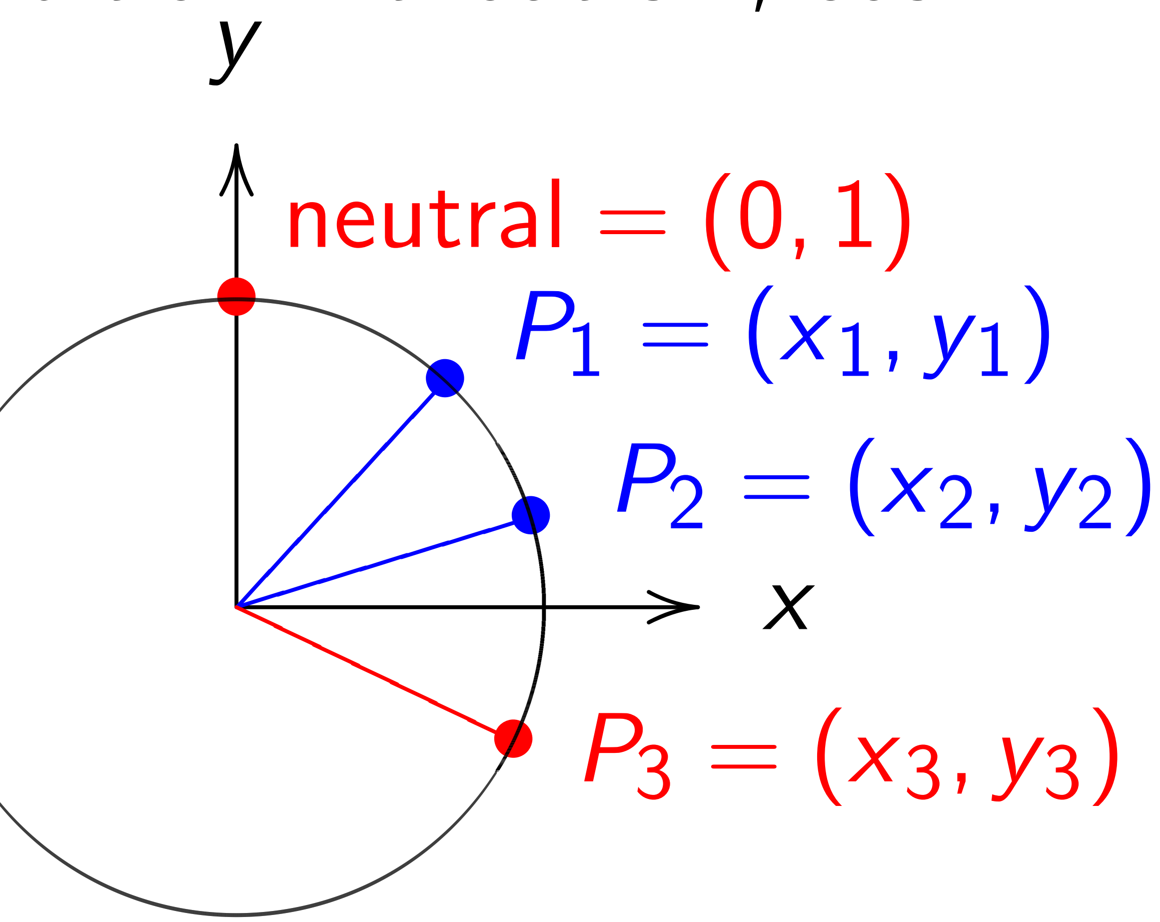

tesian coordinates for

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$, y_1) + (x_2, y_2) = (x_3, y_3)$

$+ y_1 x_2, y_1 y_2 - x_1 x_2)$.

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k copies

Examples of clock addition:

$$\text{“2:00”} + \text{“5:00”}$$
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Clocks o

Clock($\mathbf{F}$

$\{(x, y) \in$

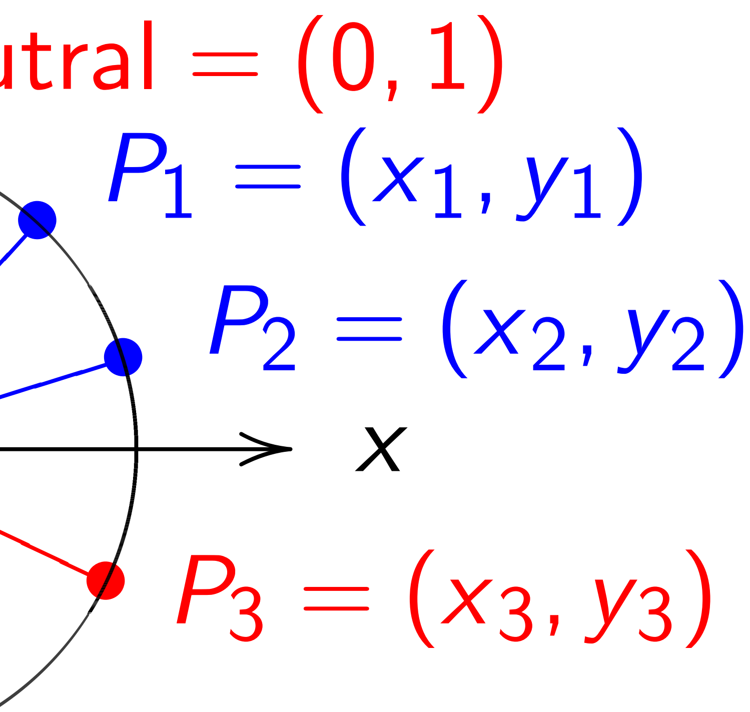
Here $\mathbf{F}_7$

$= \{0, 1,$

with $+,$

E.g. $2 \cdot$

thout sin, cos:



rdinates for

n formula

$y^2 = 1$:

$_2, y_2) = (x_3, y_3)$

$_1 y_2 - x_1 x_2)$.

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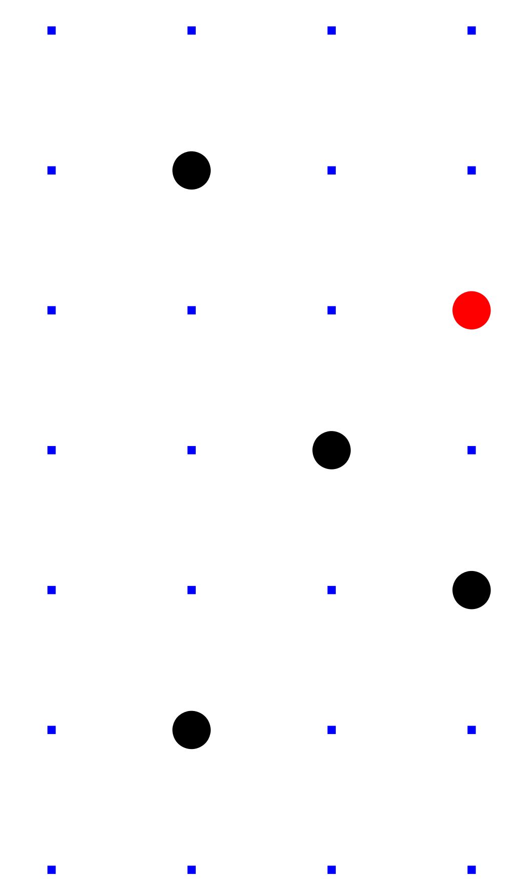
Clocks over finite



$\text{Clock}(\mathbf{F}_7) =$

$\{(x, y) \in \mathbf{F}_7 \times \mathbf{F}_7$

Here $\mathbf{F}_7 = \{0, 1, 2$

$= \{0, 1, 2, 3, -3, -$

with $+, -, \times$ mod

E.g. $2 \cdot 5 = 3$ and

cos:

$1)$

$x_1, y_1)$

(x_2, y_2)

(x_3, y_3)

or

$x_3, y_3)$

$y_2)$.

$(0, 1)$.

$k \geq 0$.

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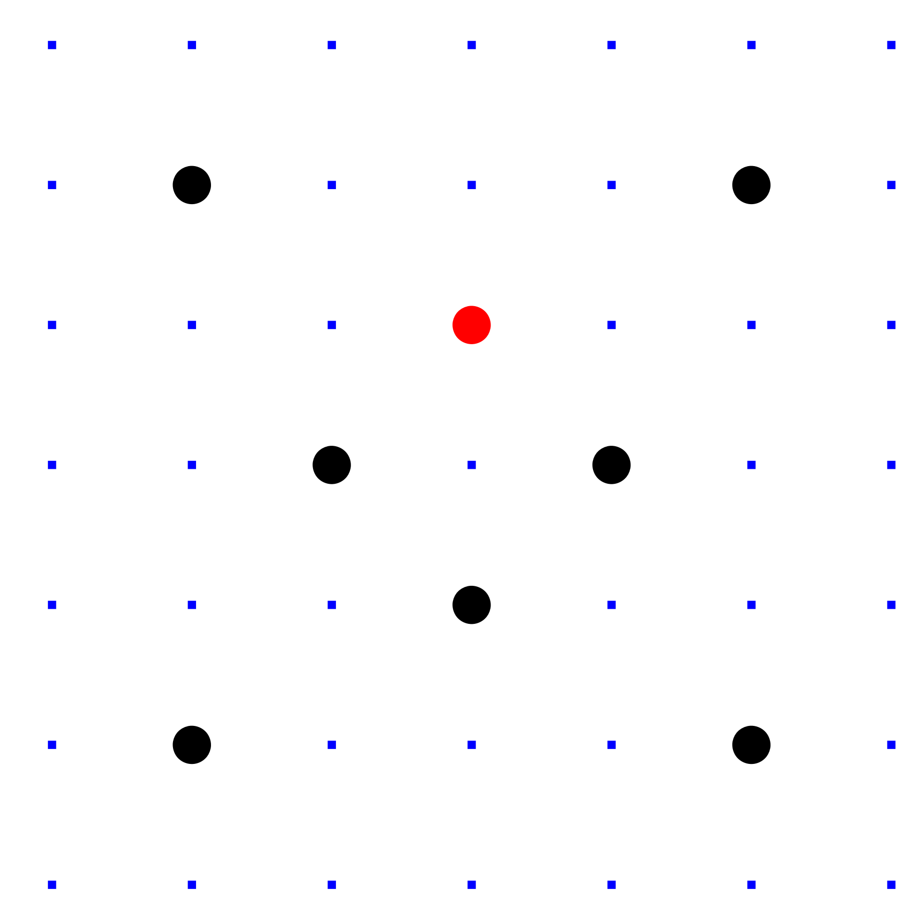
Clocks over finite fields



$\text{Clock}(\mathbf{F}_7) =$

$\{(x, y) \in \mathbf{F}_7 \times \mathbf{F}_7 : x^2 + y^2$

Here $\mathbf{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$

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with $+, -, \times$ modulo 7.

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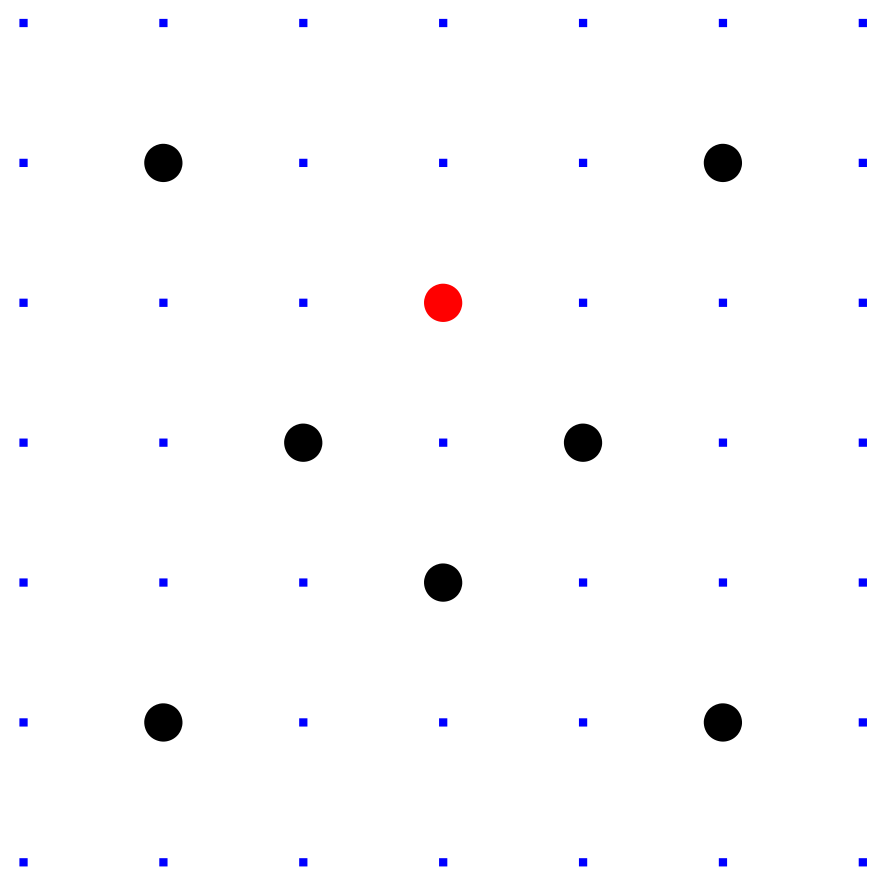

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es of clock addition:

– “5:00”

$\overline{4}, 1/2) + (1/2, -\sqrt{3/4})$

$2, -\sqrt{3/4}) =$ “7:00”.

– “9:00”

$-\sqrt{3/4}) + (-1, 0)$

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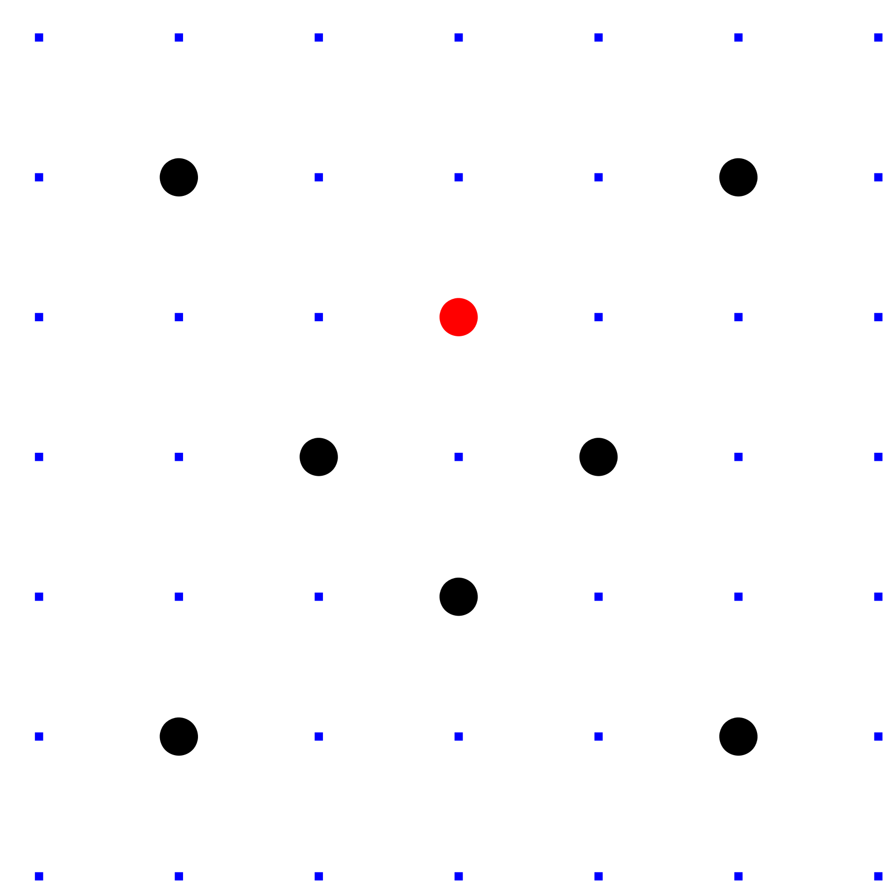

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```
>>> for i
...   for
...
...
...
(0, 1)
(0, 6)
(1, 0)
(2, 2)
(2, 5)
(5, 2)
(5, 5)
(6, 0)
>>>
```

addition:

$(1/2, -\sqrt{3/4})$

$) =$ “7:00”.

$+ (-1, 0)$

“2:00”.

$\frac{7}{25}\Big)$.

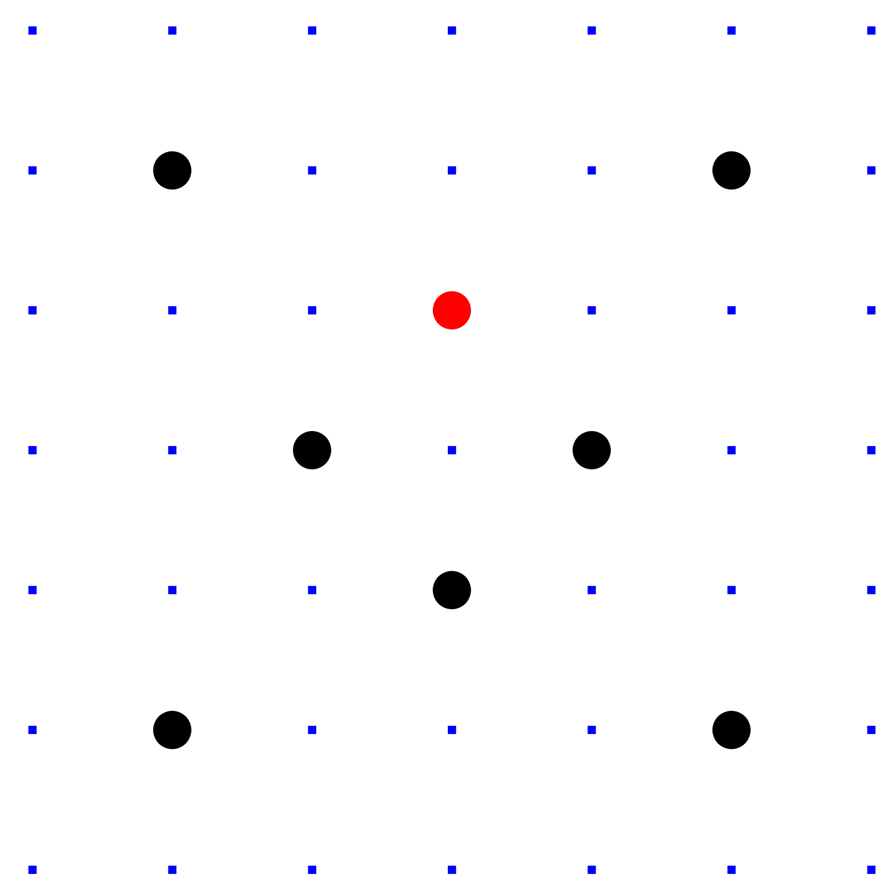
$, \frac{-44}{125}\Big)$.

$, \frac{-527}{625}\Big)$.

$= (x_1, y_1)$.

$_1) = (0, 1)$.

Clocks over finite fields



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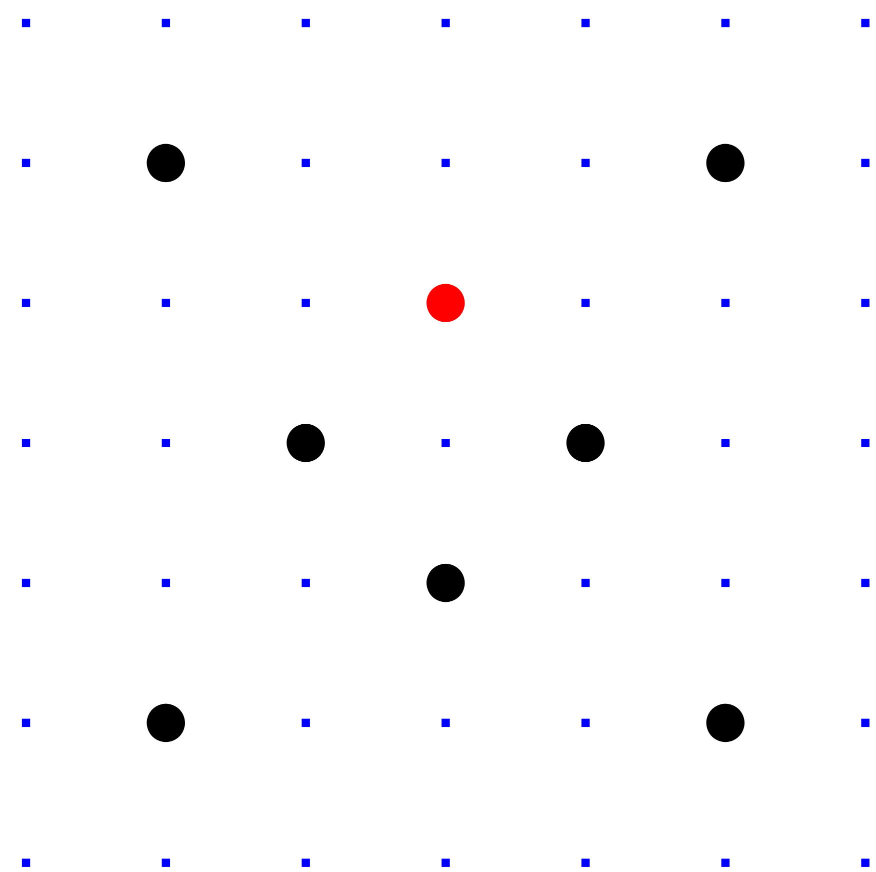
```
>>> for x in range
...   for y in ran
...     if (x*x+y*
...       print (x
...
(0, 1)
(0, 6)
(1, 0)
(2, 2)
(2, 5)
(5, 2)
(5, 5)
(6, 0)
>>>
```

3/4)

.

).

Clocks over finite fields



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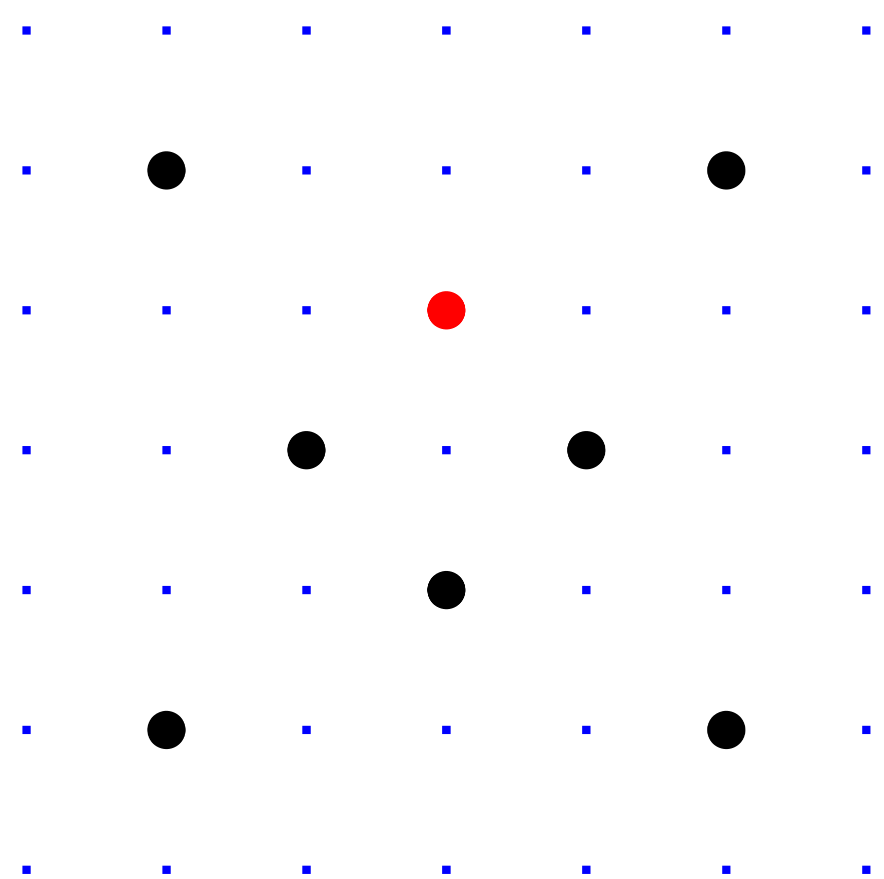
with $+, -, \times$ modulo 7.
E.g. $2 \cdot 5 = 3$ and $3/2 = 5$ in $\mathbf{F}_7$.

```
>>> for x in range(7):
...   for y in range(7):
...     if (x*x+y*y) % 7 ==
...       print (x,y)
...
(0, 1)
(0, 6)
(1, 0)
(2, 2)
(2, 5)
(5, 2)
(5, 5)
(6, 0)
>>>
```

Clocks over finite fields



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(2, 5)
(5, 2)
(5, 5)
(6, 0)
>>>
```

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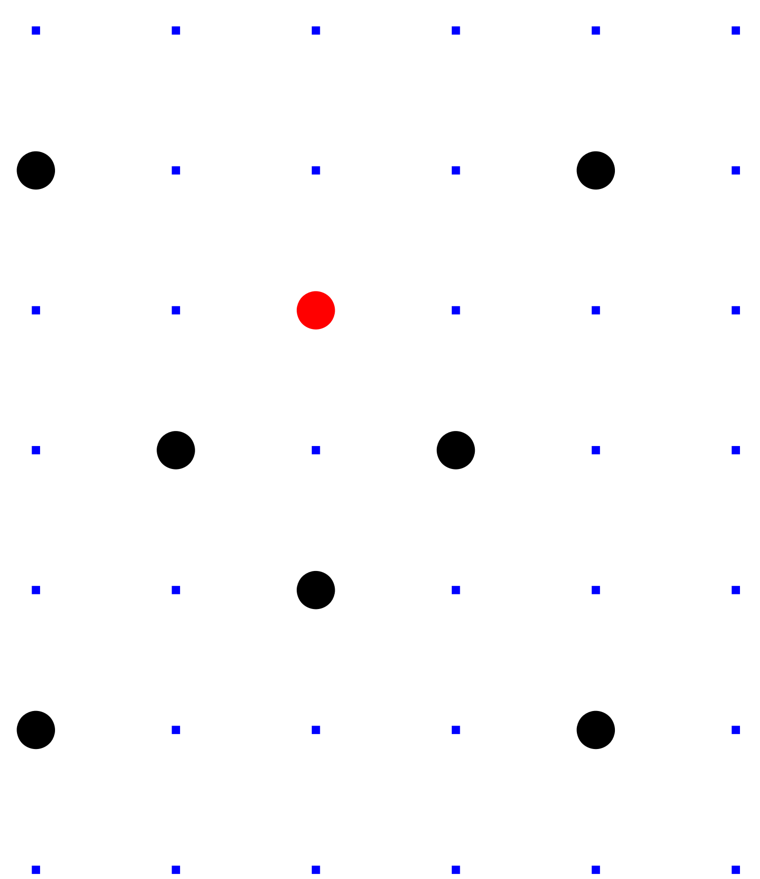

$_7) =$

$\in \mathbf{F}_7 \times \mathbf{F}_7 : x^2 + y^2 = 1\}.$

$= \{0, 1, 2, 3, 4, 5, 6\}$

$2, 3, -3, -2, -1\}$

$-, \times$ modulo 7.

$5 = 3$ and $3/2 = 5$ in $\mathbf{F}_7$.

```
>>> for x in range(7):
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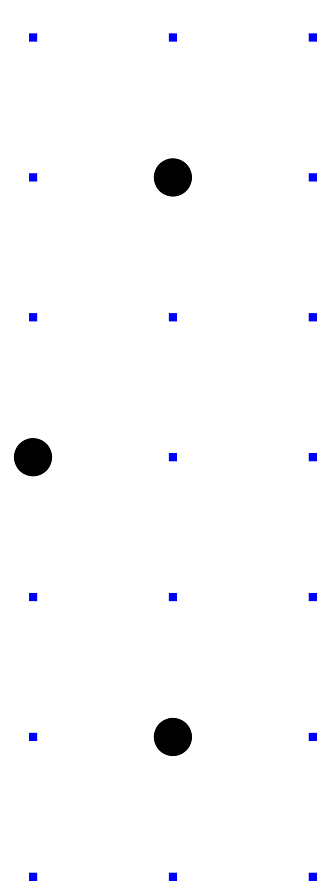
...       print (x,y)
...
(0, 1)
(0, 6)
(1, 0)
(2, 2)
(2, 5)
(5, 2)
(5, 5)
(6, 0)
>>>
```

```
>>> clas
...   de
...     
...   de
...     
...   __
...
>>> prin
2
>>> prin
6
>>> prin
0
>>> prin
3
```

fields



$: x^2 + y^2 = 1\}.$

$, 3, 4, 5, 6\}$

$2, -1\}$

ulo 7.

$3/2 = 5$ in $\mathbf{F}_7$.

```
>>> for x in range(7):
...   for y in range(7):
...     if (x*x+y*y) % 7 == 1:
...       print (x,y)
...
(0, 1)
(0, 6)
(1, 0)
(2, 2)
(2, 5)
(5, 2)
(5, 5)
(6, 0)
>>>
```

```
>>> class F7:
...   def __init__
...     self.int =
...   def __str__(
...     return str
...   __repr__ = _
...
>>> print F7(2)
2
>>> print F7(6)
6
>>> print F7(7)
0
>>> print F7(10)
3
```

$= 1\}$.

n $\mathbf{F}_7$.

```
>>> for x in range(7):
...   for y in range(7):
...     if (x*x+y*y) % 7 == 1:
...       print (x,y)
...
(0, 1)
(0, 6)
(1, 0)
(2, 2)
(2, 5)
(5, 2)
(5, 5)
(6, 0)
>>>
```

```
>>> class F7:
...   def __init__(self,x):
...     self.int = x % 7
...   def __str__(self):
...     return str(self.int)
...   __repr__ = __str__
...
>>> print F7(2)
2
>>> print F7(6)
6
>>> print F7(7)
0
>>> print F7(10)
3
```

```
>>> for x in range(7):
...   for y in range(7):
...     if (x*x+y*y) % 7 == 1:
...       print (x,y)
...
(0, 1)
(0, 6)
(1, 0)
(2, 2)
(2, 5)
(5, 2)
(5, 5)
(6, 0)
>>>
```

```
>>> class F7:
...   def __init__(self,x):
...     self.int = x % 7
...   def __str__(self):
...     return str(self.int)
...   __repr__ = __str__
...
>>> print F7(2)
2
>>> print F7(6)
6
>>> print F7(7)
0
>>> print F7(10)
3
```

```
x in range(7):
r y in range(7):
if (x*x+y*y) % 7 == 1:
  print (x,y)
```

```
>>> class F7:
...   def __init__(self,x):
...     self.int = x % 7
...   def __str__(self):
...     return str(self.int)
...   __repr__ = __str__
...
>>> print F7(2)
2
>>> print F7(6)
6
>>> print F7(7)
0
>>> print F7(10)
3
```

```
>>> F7._
...   a..
>>>
>>> prin
True
>>> prin
True
>>> prin
True
>>> prin
False
>>> prin
False
>>> prin
False
```

```
(7):
ge(7):
y) % 7 == 1:
,y)
```

```
>>> class F7:
...   def __init__(self,x):
...     self.int = x % 7
...   def __str__(self):
...     return str(self.int)
...   __repr__ = __str__
...
>>> print F7(2)
2
>>> print F7(6)
6
>>> print F7(7)
0
>>> print F7(10)
3
```

```
>>> F7.__eq__ = la
...     a.int == b.i
>>>
>>> print F7(7) ==
True
>>> print F7(10) =
True
>>> print F7(-3) =
True
>>> print F7(0) ==
False
>>> print F7(0) ==
False
>>> print F7(0) ==
False
```

1:

```
>>> class F7:
...   def __init__(self,x):
...     self.int = x % 7
...   def __str__(self):
...     return str(self.int)
...   __repr__ = __str__
...
>>> print F7(2)
2
>>> print F7(6)
6
>>> print F7(7)
0
>>> print F7(10)
3
```

```
>>> F7.__eq__ = lambda a,b:
...    a.int == b.int
>>>
>>> print F7(7) == F7(0)
True
>>> print F7(10) == F7(3)
True
>>> print F7(-3) == F7(4)
True
>>> print F7(0) == F7(1)
False
>>> print F7(0) == F7(2)
False
>>> print F7(0) == F7(3)
False
```

```
>>> class F7:
...   def __init__(self,x):
...     self.int = x % 7
...   def __str__(self):
...     return str(self.int)
...   __repr__ = __str__
...
>>> print F7(2)
2
>>> print F7(6)
6
>>> print F7(7)
0
>>> print F7(10)
3
```

```
>>> F7.__eq__ = lambda a,b: \
...    a.int == b.int
>>>
>>> print F7(7) == F7(0)
True
>>> print F7(10) == F7(3)
True
>>> print F7(-3) == F7(4)
True
>>> print F7(0) == F7(1)
False
>>> print F7(0) == F7(2)
False
>>> print F7(0) == F7(3)
False
```

```
s F7:
f __init__(self,x):
self.int = x % 7
f __str__(self):
return str(self.int)
repr__ = __str__

t F7(2)

t F7(6)

t F7(7)

t F7(10)
```

```
>>> F7.__eq__ = lambda a,b: \
...   a.int == b.int
>>>
>>> print F7(7) == F7(0)
True
>>> print F7(10) == F7(3)
True
>>> print F7(-3) == F7(4)
True
>>> print F7(0) == F7(1)
False
>>> print F7(0) == F7(2)
False
>>> print F7(0) == F7(3)
False
```

```
>>> F7._
...   F7
>>> F7._
...   F7
>>> F7._
...   F7
>>>
>>> prin
0
>>> prin
4
>>> prin
3
>>>
```

```
(self,x):
 x % 7
self):
(self.int)
_str__
```

```
>>> F7.__eq__ = lambda a,b: \
...   a.int == b.int
>>>
>>> print F7(7) == F7(0)
True
>>> print F7(10) == F7(3)
True
>>> print F7(-3) == F7(4)
True
>>> print F7(0) == F7(1)
False
>>> print F7(0) == F7(2)
False
>>> print F7(0) == F7(3)
False
```

```
>>> F7.__add__ = l
...   F7(a.int + b
>>> F7.__sub__ = l
...   F7(a.int - b
>>> F7.__mul__ = l
...   F7(a.int * b
>>>
>>> print F7(2) +
0
>>> print F7(2) -
4
>>> print F7(2) *
3
>>>
```

```
>>> F7.__eq__ = lambda a,b: \
...   a.int == b.int
>>>
>>> print F7(7) == F7(0)
True
>>> print F7(10) == F7(3)
True
>>> print F7(-3) == F7(4)
True
>>> print F7(0) == F7(1)
False
>>> print F7(0) == F7(2)
False
>>> print F7(0) == F7(3)
False
```

```
>>> F7.__add__ = lambda a,b:
...   F7(a.int + b.int)
>>> F7.__sub__ = lambda a,b:
...   F7(a.int - b.int)
>>> F7.__mul__ = lambda a,b:
...   F7(a.int * b.int)
>>>
>>> print F7(2) + F7(5)
0
>>> print F7(2) - F7(5)
4
>>> print F7(2) * F7(5)
3
>>>
```

```
>>> F7.__eq__ = lambda a,b: \
...   a.int == b.int
>>>
>>> print F7(7) == F7(0)
True
>>> print F7(10) == F7(3)
True
>>> print F7(-3) == F7(4)
True
>>> print F7(0) == F7(1)
False
>>> print F7(0) == F7(2)
False
>>> print F7(0) == F7(3)
False
```

```
>>> F7.__add__ = lambda a,b: \
...   F7(a.int + b.int)
>>> F7.__sub__ = lambda a,b: \
...   F7(a.int - b.int)
>>> F7.__mul__ = lambda a,b: \
...   F7(a.int * b.int)
>>>
>>> print F7(2) + F7(5)
0
>>> print F7(2) - F7(5)
4
>>> print F7(2) * F7(5)
3
>>>
```

```
_eq__ = lambda a,b: \
int == b.int

t F7(7) == F7(0)

t F7(10) == F7(3)

t F7(-3) == F7(4)

t F7(0) == F7(1)

t F7(0) == F7(2)

t F7(0) == F7(3)
```

```
>>> F7.__add__ = lambda a,b: \
...   F7(a.int + b.int)
>>> F7.__sub__ = lambda a,b: \
...   F7(a.int - b.int)
>>> F7.__mul__ = lambda a,b: \
...   F7(a.int * b.int)
>>>
>>> print F7(2) + F7(5)
0
>>> print F7(2) - F7(5)
4
>>> print F7(2) * F7(5)
3
>>>
```

Larger e

```
p = 1000
class Fp
  ...

def cloc
  x1,y1 =
  x2,y2 =
  x3 = x
  y3 = y
  return
```

```
mbda a,b: \
nt

 F7(0)

= F7(3)

= F7(4)

 F7(1)

 F7(2)

 F7(3)
```

```
>>> F7.__add__ = lambda a,b: \
...   F7(a.int + b.int)
>>> F7.__sub__ = lambda a,b: \
...   F7(a.int - b.int)
>>> F7.__mul__ = lambda a,b: \
...   F7(a.int * b.int)
>>>
>>> print F7(2) + F7(5)
0
>>> print F7(2) - F7(5)
4
>>> print F7(2) * F7(5)
3
>>>
```

Larger example: C

```
p = 1000003
class Fp:
  ...

def clockadd(P1,P2
  x1,y1 = P1
  x2,y2 = P2
  x3 = x1*y2+y1*x2
  y3 = y1*y2-x1*x2
  return x3,y3
```

\

```
>>> F7.__add__ = lambda a,b: \
...   F7(a.int + b.int)
>>> F7.__sub__ = lambda a,b: \
...   F7(a.int - b.int)
>>> F7.__mul__ = lambda a,b: \
...   F7(a.int * b.int)
>>>
>>> print F7(2) + F7(5)
0
>>> print F7(2) - F7(5)
4
>>> print F7(2) * F7(5)
3
>>>
```

Larger example: Clock($\mathbf{F}_{100}$

```
p = 1000003
class Fp:
  ...

def clockadd(P1,P2):
  x1,y1 = P1
  x2,y2 = P2
  x3 = x1*y2+y1*x2
  y3 = y1*y2-x1*x2
  return x3,y3
```

```
>>> F7.__add__ = lambda a,b: \
...   F7(a.int + b.int)
>>> F7.__sub__ = lambda a,b: \
...   F7(a.int - b.int)
>>> F7.__mul__ = lambda a,b: \
...   F7(a.int * b.int)
>>>
>>> print F7(2) + F7(5)
0
>>> print F7(2) - F7(5)
4
>>> print F7(2) * F7(5)
3
>>>
```

Larger example: Clock($\mathbf{F}_{1000003}$).

```
p = 1000003
class Fp:
  ...

def clockadd(P1,P2):
  x1,y1 = P1
  x2,y2 = P2
  x3 = x1*y2+y1*x2
  y3 = y1*y2-x1*x2
  return x3,y3
```

```
_add__ = lambda a,b: \
(a.int + b.int)
_sub__ = lambda a,b: \
(a.int - b.int)
_mul__ = lambda a,b: \
(a.int * b.int)

t F7(2) + F7(5)

t F7(2) - F7(5)

t F7(2) * F7(5)
```

Larger example: Clock($\mathbf{F}_{1000003}$).

```
p = 1000003
class Fp:
  ...

def clockadd(P1,P2):
  x1,y1 = P1
  x2,y2 = P2
  x3 = x1*y2+y1*x2
  y3 = y1*y2-x1*x2
  return x3,y3
```

```
>>> P =
>>> P2 =
>>> prin
(4000, 7
>>> P3 =
>>> prin
(15000,
>>> P4 =
>>> P5 =
>>> P6 =
>>> prin
(780000,
>>> prin
(780000,
>>>
```

```
ambda a,b: \
.int)
ambda a,b: \
.int)
ambda a,b: \
.int)

F7(5)

F7(5)

F7(5)
```

Larger example: Clock($\mathbf{F}_{1000003}$).

```
p = 1000003
class Fp:
  ...

def clockadd(P1,P2):
  x1,y1 = P1
  x2,y2 = P2
  x3 = x1*y2+y1*x2
  y3 = y1*y2-x1*x2
  return x3,y3
```

```
>>> P = (Fp(1000),
>>> P2 = clockadd(
>>> print P2
(4000, 7)
>>> P3 = clockadd(
>>> print P3
(15000, 26)
>>> P4 = clockadd(
>>> P5 = clockadd(
>>> P6 = clockadd(
>>> print P6
(780000, 1351)
>>> print clockadd
(780000, 1351)
>>>
```

\

\

\

Larger example: Clock($\mathbf{F}_{1000003}$).

```
p = 1000003
class Fp:
  ...

def clockadd(P1,P2):
  x1,y1 = P1
  x2,y2 = P2
  x3 = x1*y2+y1*x2
  y3 = y1*y2-x1*x2
  return x3,y3
```

```
>>> P = (Fp(1000),Fp(2))
>>> P2 = clockadd(P,P)
>>> print P2
(4000, 7)
>>> P3 = clockadd(P2,P)
>>> print P3
(15000, 26)
>>> P4 = clockadd(P3,P)
>>> P5 = clockadd(P4,P)
>>> P6 = clockadd(P5,P)
>>> print P6
(780000, 1351)
>>> print clockadd(P3,P3)
(780000, 1351)
>>>
```

Larger example: Clock($\mathbf{F}_{1000003}$).

```
p = 1000003
class Fp:
  ...

def clockadd(P1,P2):
  x1,y1 = P1
  x2,y2 = P2
  x3 = x1*y2+y1*x2
  y3 = y1*y2-x1*x2
  return x3,y3
```

```
>>> P = (Fp(1000),Fp(2))
>>> P2 = clockadd(P,P)
>>> print P2
(4000, 7)
>>> P3 = clockadd(P2,P)
>>> print P3
(15000, 26)
>>> P4 = clockadd(P3,P)
>>> P5 = clockadd(P4,P)
>>> P6 = clockadd(P5,P)
>>> print P6
(780000, 1351)
>>> print clockadd(P3,P3)
(780000, 1351)
>>>
```

xample: Clock($\mathbf{F}_{1000003}$).

```
003
:

kadd(P1,P2):
= P1
= P2
1*y2+y1*x2
1*y2-x1*x2
 x3,y3
```

```
>>> P = (Fp(1000),Fp(2))
>>> P2 = clockadd(P,P)
>>> print P2
(4000, 7)
>>> P3 = clockadd(P2,P)
>>> print P3
(15000, 26)
>>> P4 = clockadd(P3,P)
>>> P5 = clockadd(P4,P)
>>> P6 = clockadd(P5,P)
>>> print P6
(780000, 1351)
>>> print clockadd(P3,P3)
(780000, 1351)
>>>
```

```
>>> def
...   if
...   re
...   if
...   Q
...   Q
...   if
...   re
...
>>> n =
>>> scal
(947472,
>>>
```

Can you

Clock($\mathbf{F}_{1000003}$).

):

```
>>> P = (Fp(1000),Fp(2))
>>> P2 = clockadd(P,P)
>>> print P2
(4000, 7)
>>> P3 = clockadd(P2,P)
>>> print P3
(15000, 26)
>>> P4 = clockadd(P3,P)
>>> P5 = clockadd(P4,P)
>>> P6 = clockadd(P5,P)
>>> print P6
(780000, 1351)
>>> print clockadd(P3,P3)
(780000, 1351)
>>>
```

```
>>> def scalarmult
...   if n == 0: \
...   return (Fp(0
...   if n == 1: r
...   Q = scalarmu
...   Q = clockadd
...   if n % 2: Q
...   return Q
...
>>> n = oursixdigi
>>> scalarmult(n,P
(947472, 736284)
>>>
```

Can you figure out

0003).

```
>>> P = (Fp(1000),Fp(2))
>>> P2 = clockadd(P,P)
>>> print P2
(4000, 7)
>>> P3 = clockadd(P2,P)
>>> print P3
(15000, 26)
>>> P4 = clockadd(P3,P)
>>> P5 = clockadd(P4,P)
>>> P6 = clockadd(P5,P)
>>> print P6
(780000, 1351)
>>> print clockadd(P3,P3)
(780000, 1351)
>>>
```

```
>>> def scalarmult(n,P):
...   if n == 0: \
...   return (Fp(0),Fp(1))
...   if n == 1: return P
...   Q = scalarmult(n//2,P)
...   Q = clockadd(Q,Q)
...   if n % 2: Q = clockadd
...   return Q
...
>>> n = oursixdigitsecret
>>> scalarmult(n,P)
(947472, 736284)
>>>
```

Can you figure out our secre

```
>>> P = (Fp(1000),Fp(2))
>>> P2 = clockadd(P,P)
>>> print P2
(4000, 7)
>>> P3 = clockadd(P2,P)
>>> print P3
(15000, 26)
>>> P4 = clockadd(P3,P)
>>> P5 = clockadd(P4,P)
>>> P6 = clockadd(P5,P)
>>> print P6
(780000, 1351)
>>> print clockadd(P3,P3)
(780000, 1351)
>>>
```

```
>>> def scalarmult(n,P):
...   if n == 0: \
...   return (Fp(0),Fp(1))
...   if n == 1: return P
...   Q = scalarmult(n//2,P)
...   Q = clockadd(Q,Q)
...   if n % 2: Q = clockadd(P,Q)
...   return Q
...
>>> n = oursixdigitsecret
>>> scalarmult(n,P)
(947472, 736284)
>>>
```

Can you figure out our secret n?

```
(Fp(1000),Fp(2))
 clockadd(P,P)
t P2
)
 clockadd(P2,P)
t P3
26)
 clockadd(P3,P)
 clockadd(P4,P)
 clockadd(P5,P)
t P6
 1351)
t clockadd(P3,P3)
 1351)
```

```
>>> def scalarmult(n,P):
...   if n == 0: \
...   return (Fp(0),Fp(1))
...   if n == 1: return P
...   Q = scalarmult(n//2,P)
...   Q = clockadd(Q,Q)
...   if n % 2: Q = clockadd(P,Q)
...   return Q
...
>>> n = oursixdigitsecret
>>> scalarmult(n,P)
(947472, 736284)
>>>
```

Can you figure out our secret n?

Clock cr

The "Cl
protocol

Standard
base po

Alice ch
compute

Bob cho
compute

Alice co
Bob com
They us
to encry

```
Fp(2))
P,P)



P2,P)



P3,P)
P4,P)
P5,P)



(P3,P3)
```

```
>>> def scalarmult(n,P):
...   if n == 0: \
...   return (Fp(0),Fp(1))
...   if n == 1: return P
...   Q = scalarmult(n//2,P)
...   Q = clockadd(Q,Q)
...   if n % 2: Q = clockadd(P,Q)
...   return Q
...
>>> n = oursixdigitsecret
>>> scalarmult(n,P)
(947472, 736284)
>>>
```

Can you figure out our secret n?

Clock cryptograph

The "Clock Diffie-
protocol":

Standardize large
base point (x, y)

Alice chooses big s
computes her publ

Bob chooses big se
computes his publ

Alice computes $a($
Bob computes $b(a$
They use this shar
to encrypt with AE

```
>>> def scalarmult(n,P):
...     if n == 0: \
...     return (Fp(0),Fp(1))
...     if n == 1: return P
...     Q = scalarmult(n//2,P)
...     Q = clockadd(Q,Q)
...     if n % 2: Q = clockadd(P,Q)
...     return Q
...
>>> n = oursixdigitsecret
>>> scalarmult(n,P)
(947472, 736284)
>>>
```

Can you figure out our secret n?

Clock cryptography

The "Clock Diffie–Hellman
protocol":

Standardize large prime p &
base point $(x, y) \in$ Clock(**F**

Alice chooses big secret a,
computes her public key $a(x$

Bob chooses big secret b,
computes his public key $b(x$

Alice computes $a(b(x, y))$.
Bob computes $b(a(x, y))$.
They use this shared secret
to encrypt with AES-GCM e

```
>>> def scalarmult(n,P):
...   if n == 0: \
...   return (Fp(0),Fp(1))
...   if n == 1: return P
...   Q = scalarmult(n//2,P)
...   Q = clockadd(Q,Q)
...   if n % 2: Q = clockadd(P,Q)
...   return Q
...
>>> n = oursixdigitsecret
>>> scalarmult(n,P)
(947472, 736284)
>>>
```

Can you figure out our secret n?

Clock cryptography

The "Clock Diffie–Hellman protocol":

Standardize large prime p & **base point** $(x, y) \in \text{Clock}(\mathbf{F}_p)$.

Alice chooses big secret a, computes her public key $a(x, y)$.

Bob chooses big secret b, computes his public key $b(x, y)$.

Alice computes $a(b(x, y))$.
Bob computes $b(a(x, y))$.
They use this shared secret to encrypt with AES-GCM etc.

```
scalarmult(n,P):
 n == 0: \
turn (Fp(0),Fp(1))
 n == 1: return P
= scalarmult(n//2,P)
= clockadd(Q,Q)
 n % 2: Q = clockadd(P,Q)
turn Q

oursixdigitsecret
armult(n,P)
 736284)
```

figure out our secret n?

Clock cryptography

The "Clock Diffie–Hellman protocol":

Standardize large prime p & **base point** $(x, y) \in \mathrm{Clock}(\mathbf{F}_p)$.

Alice chooses big secret a, computes her public key $a(x, y)$.

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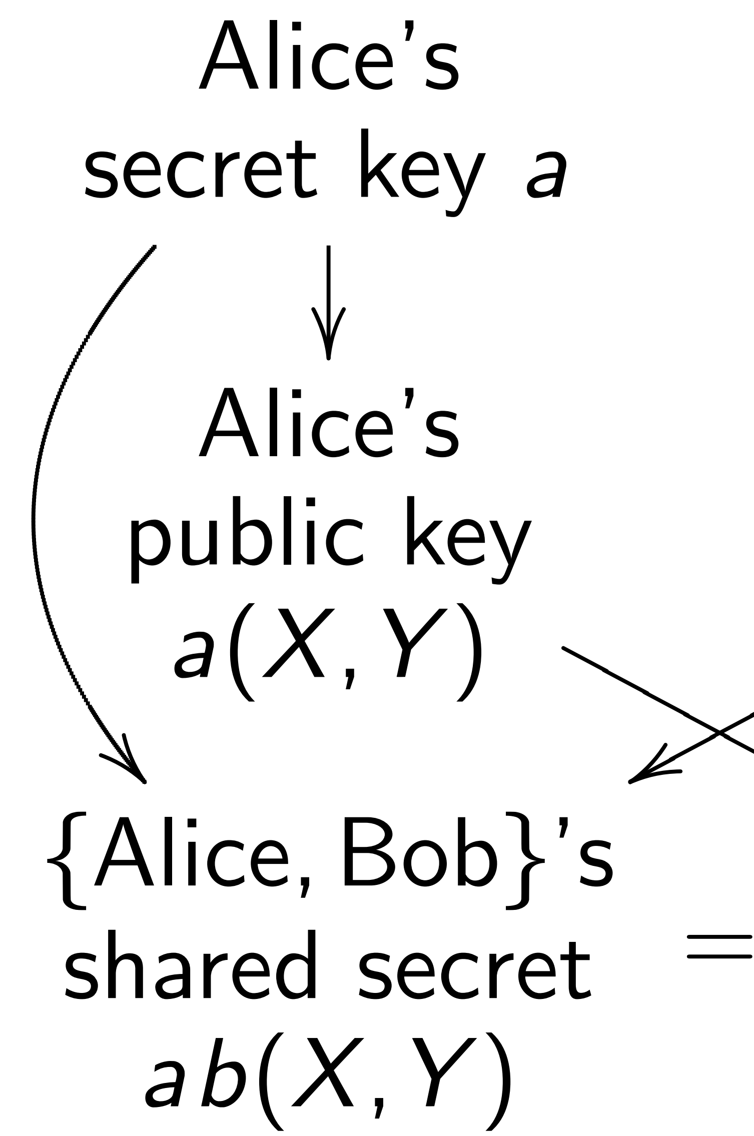

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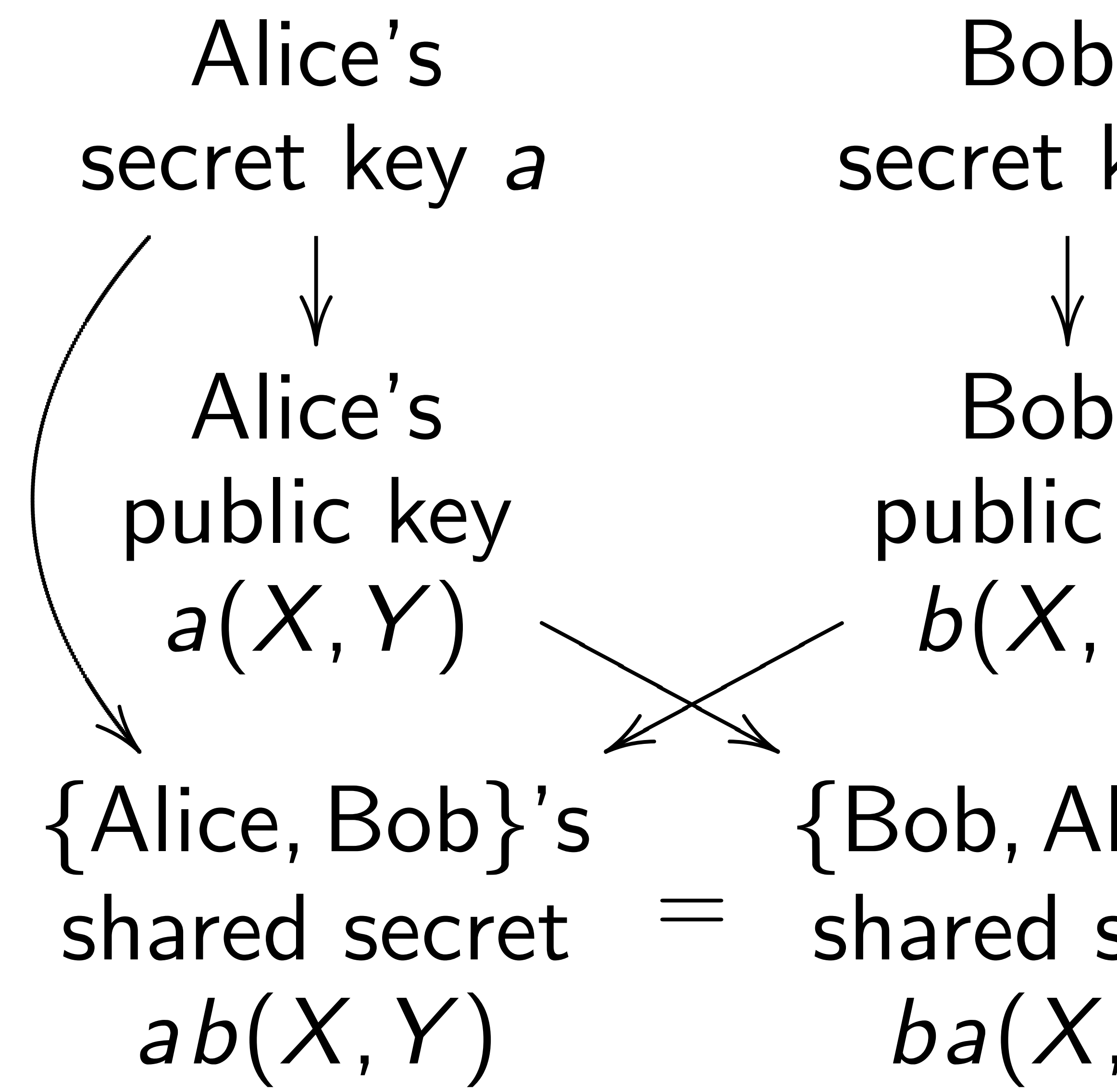

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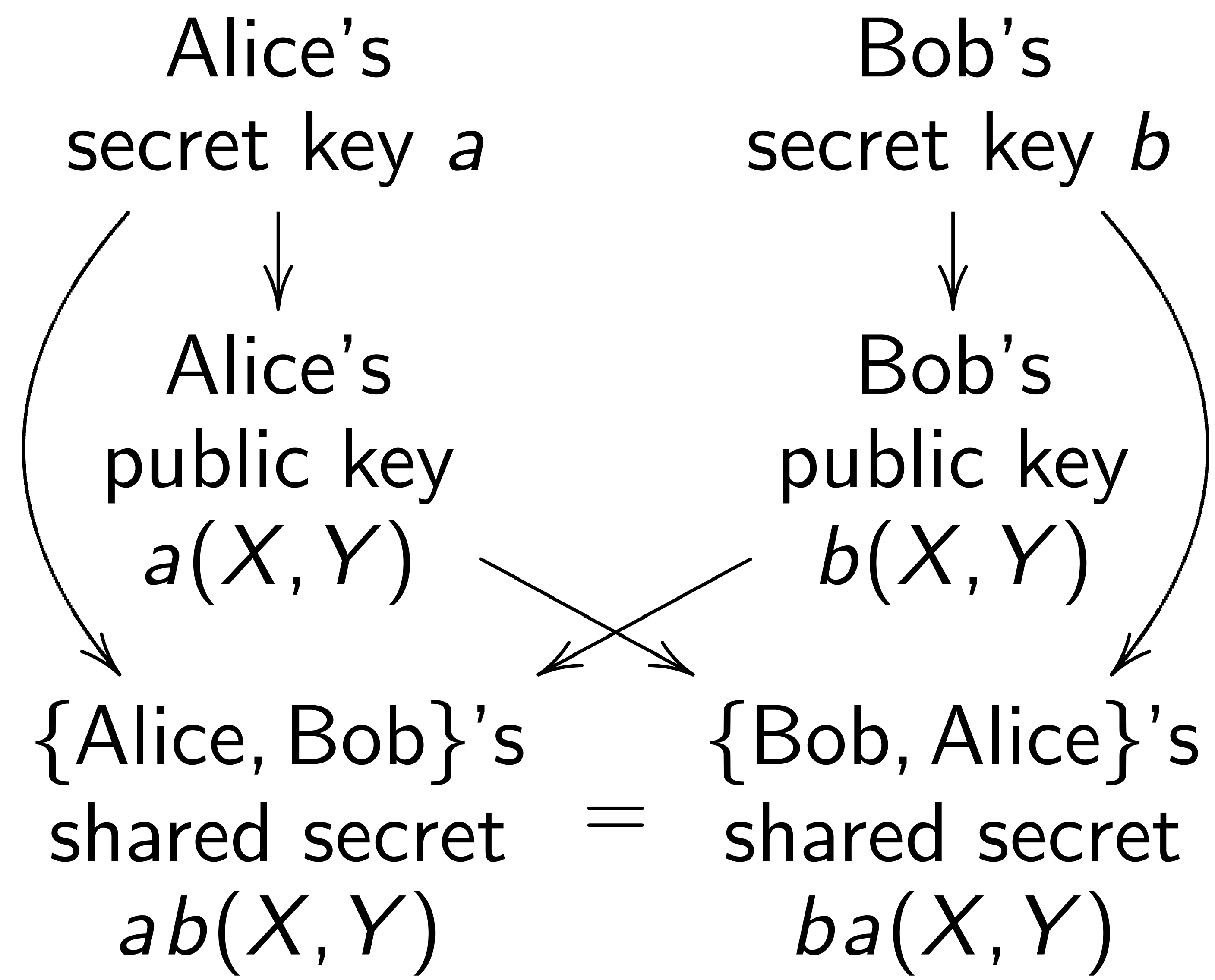

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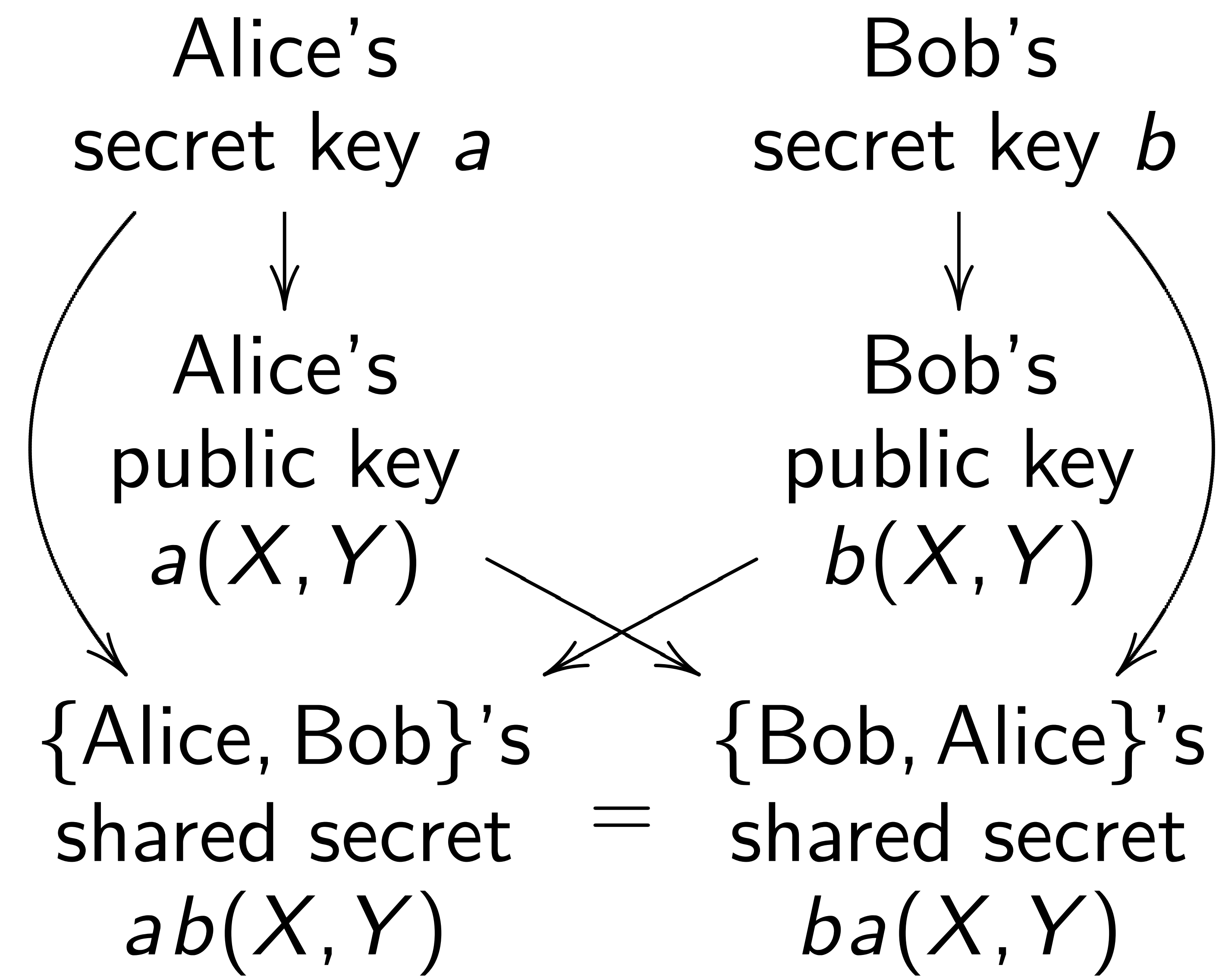

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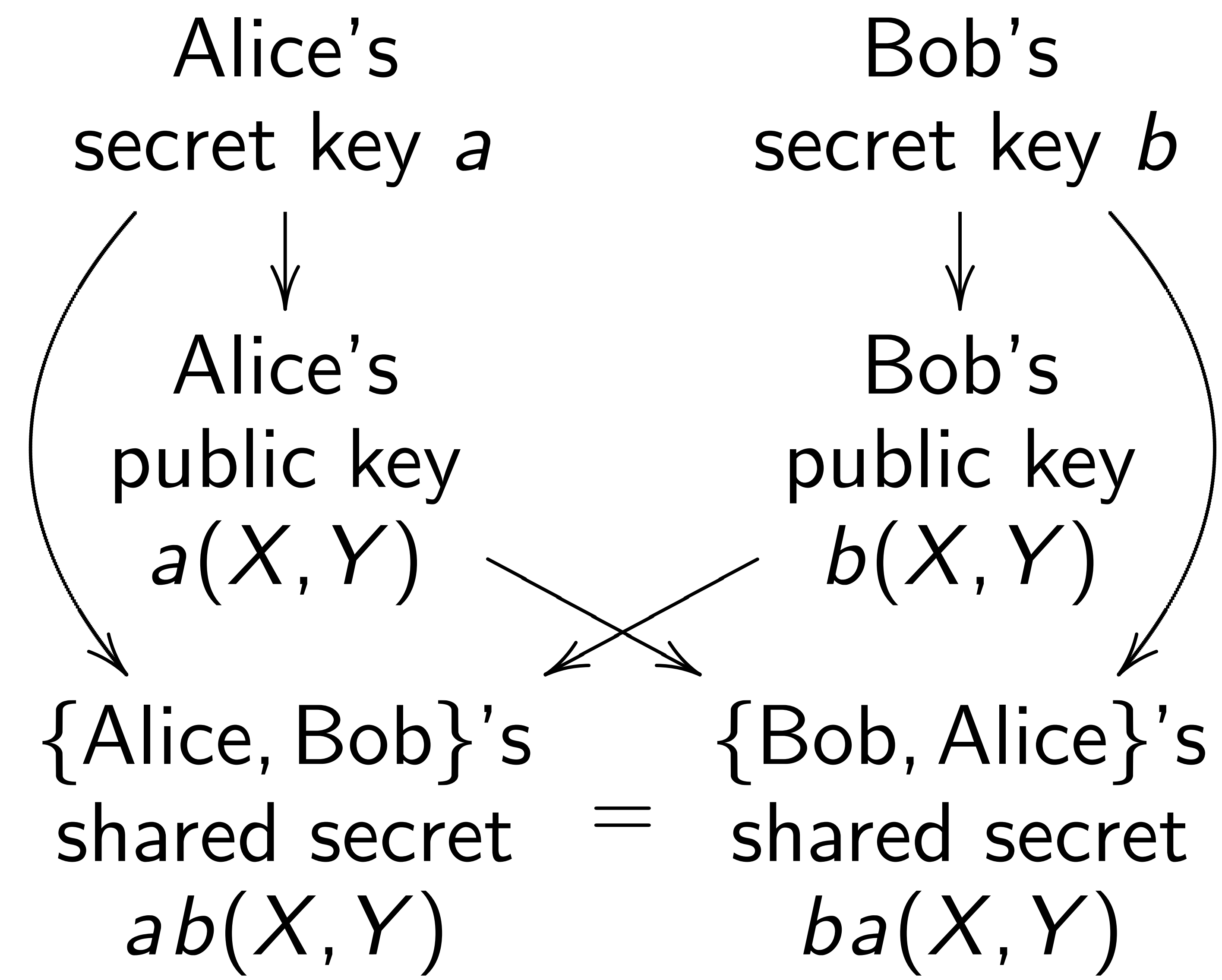

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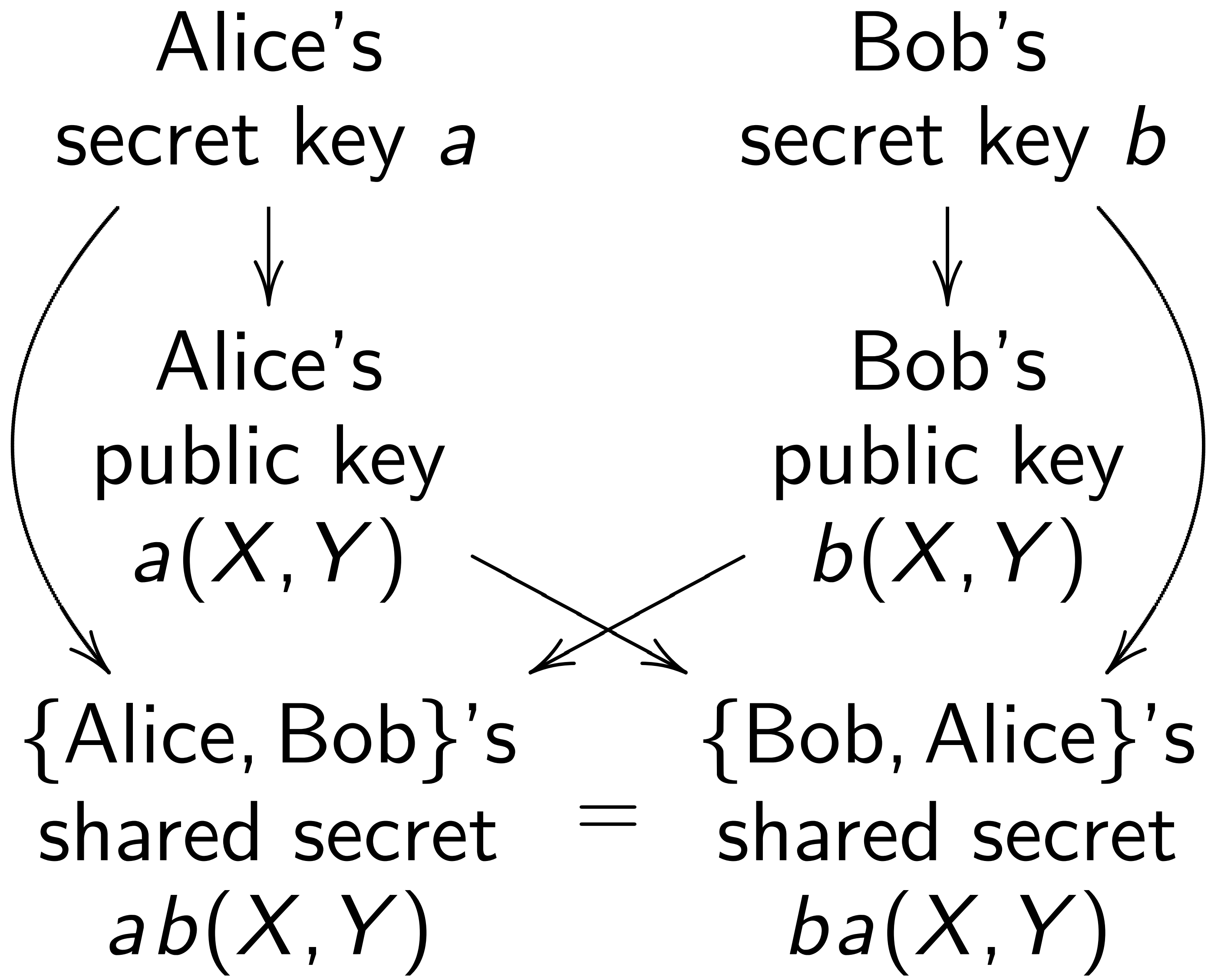

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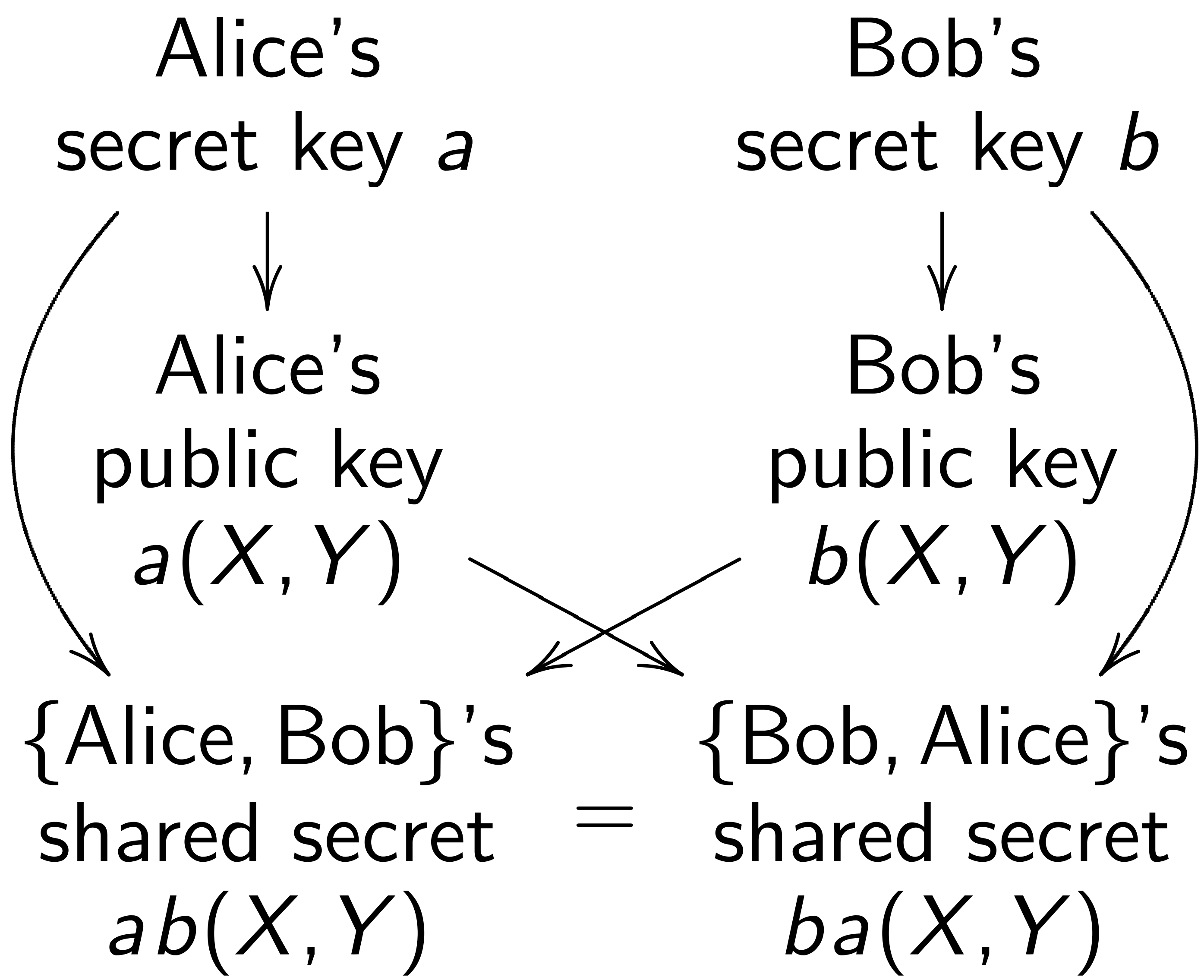

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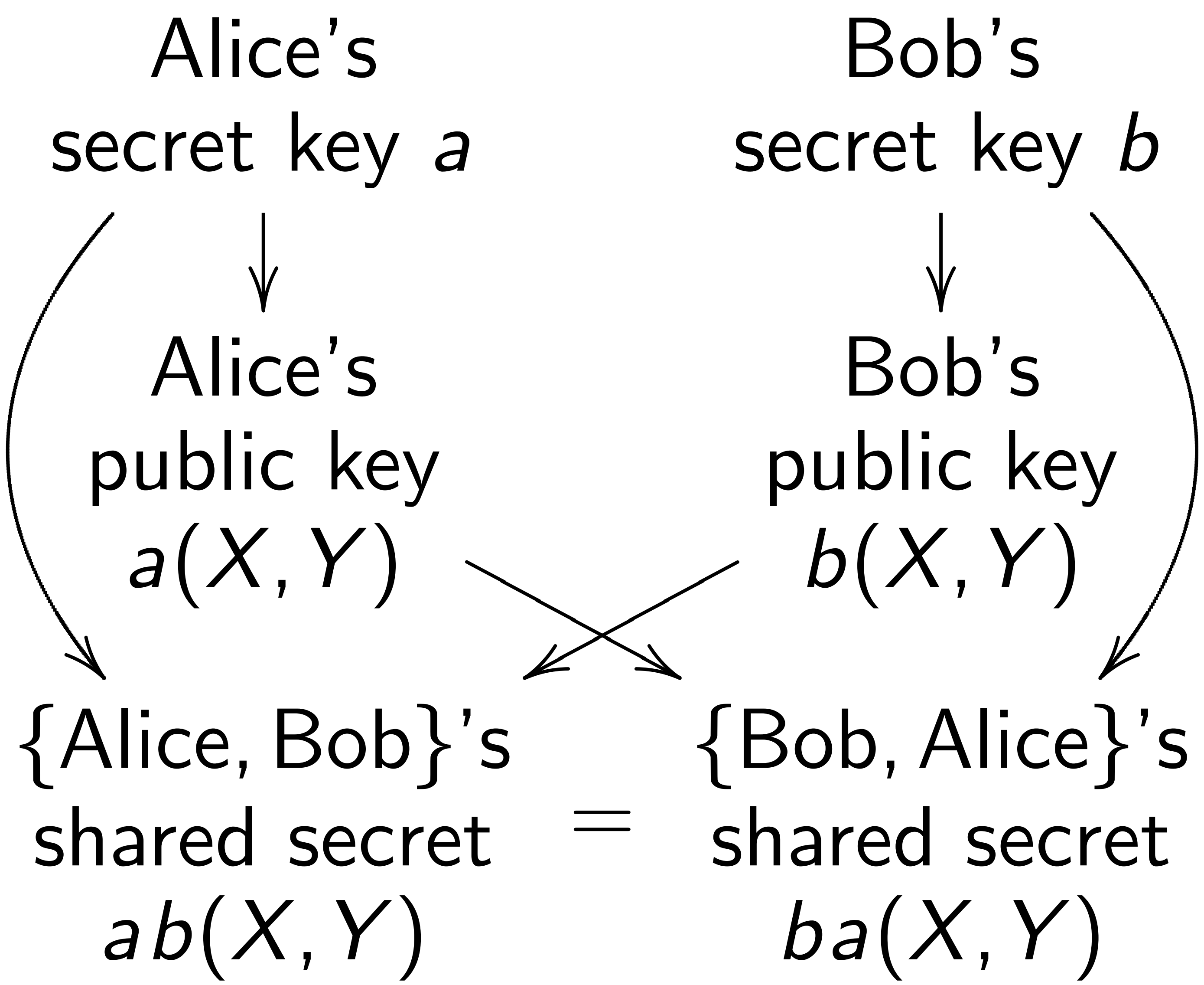

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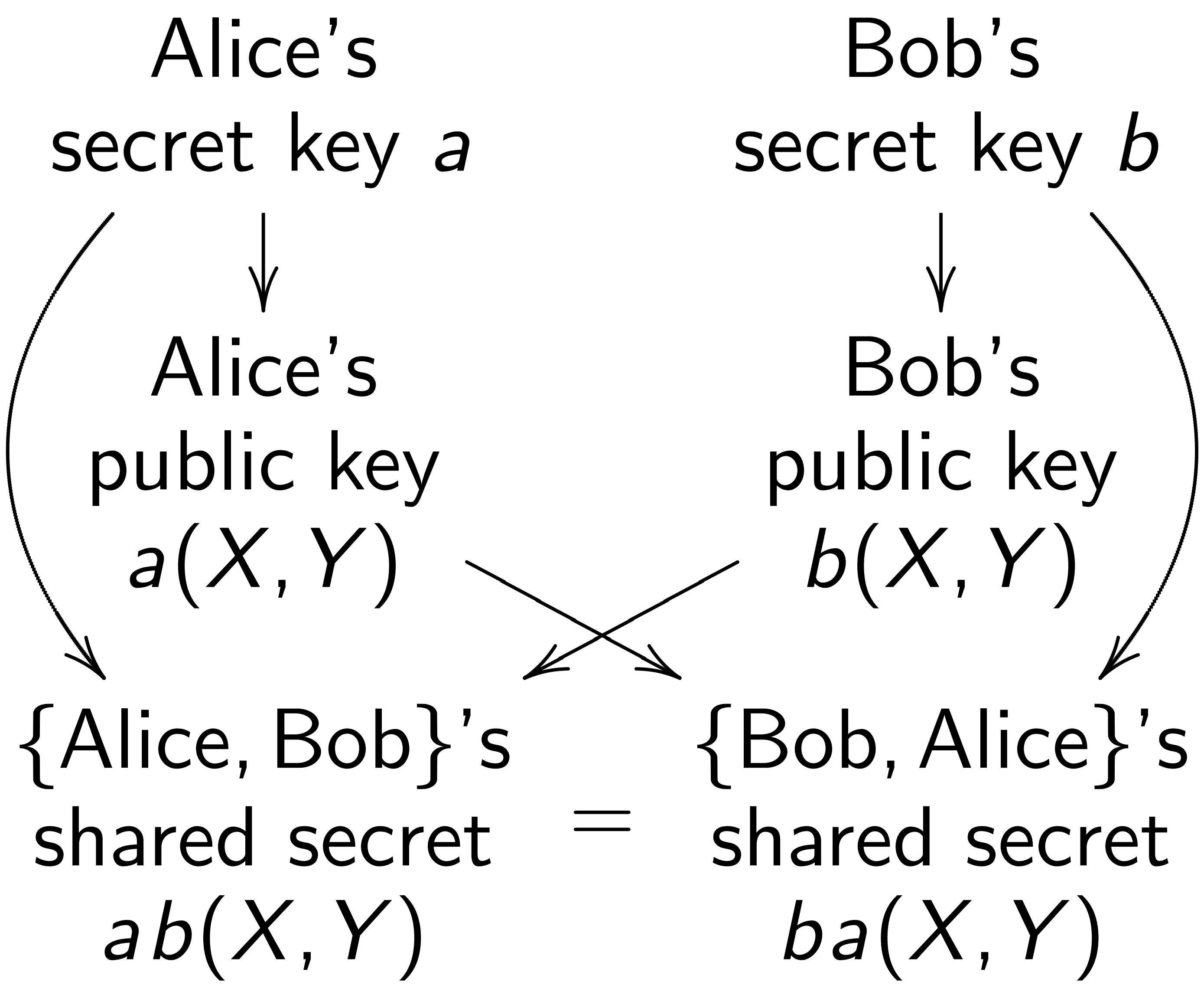

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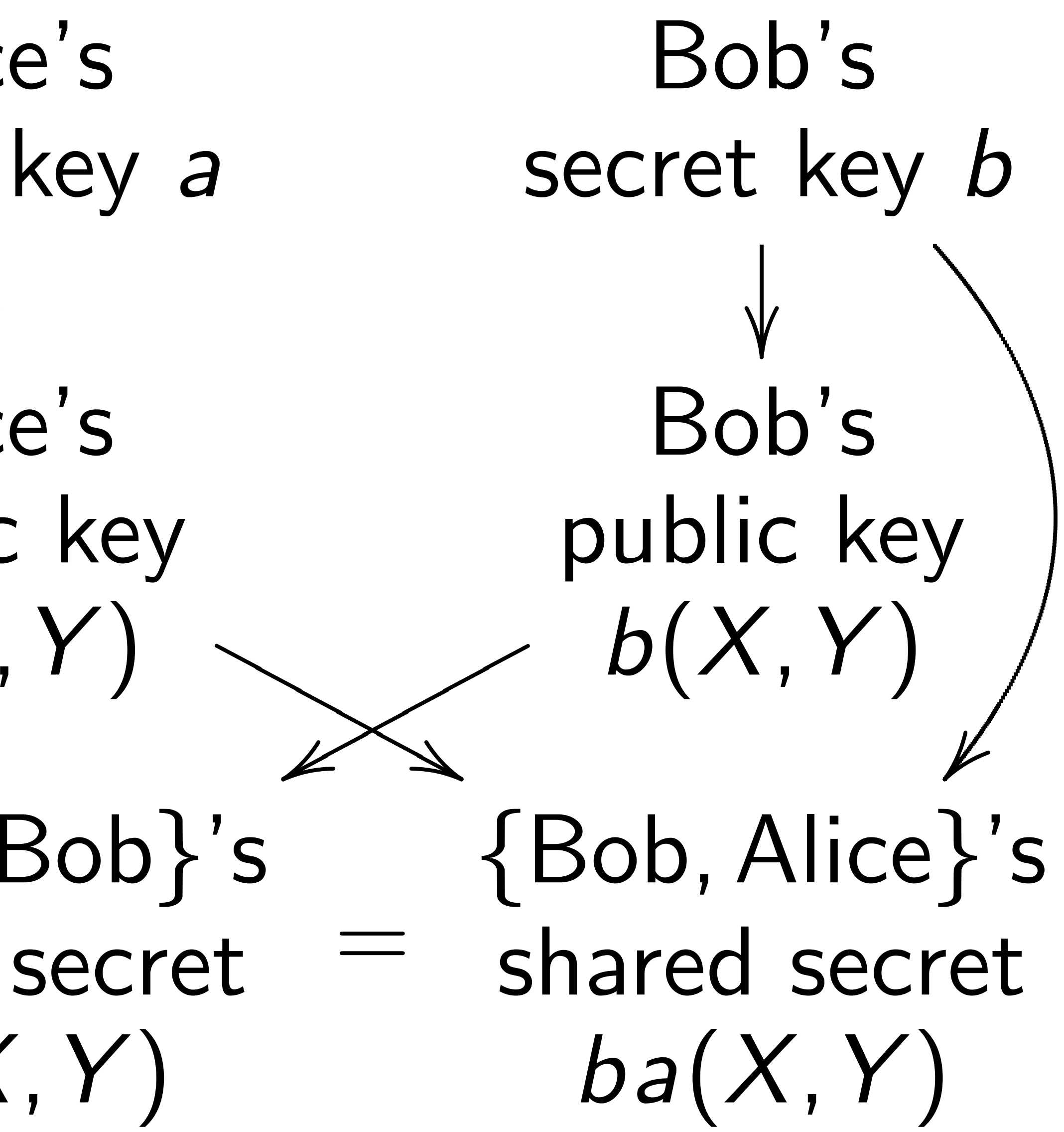

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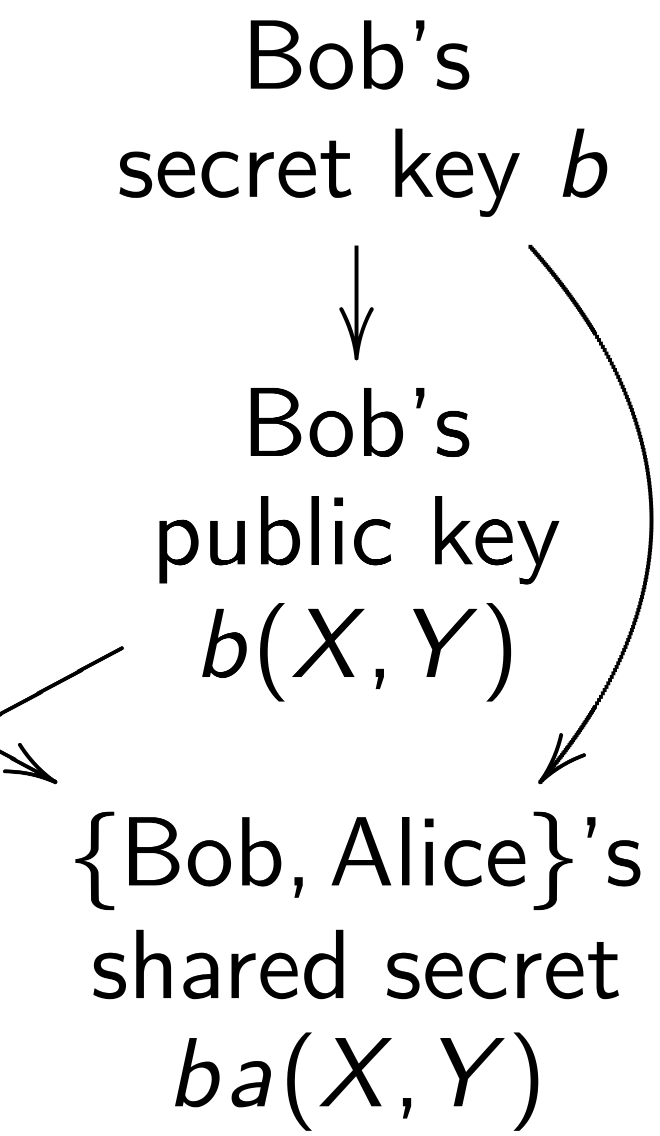

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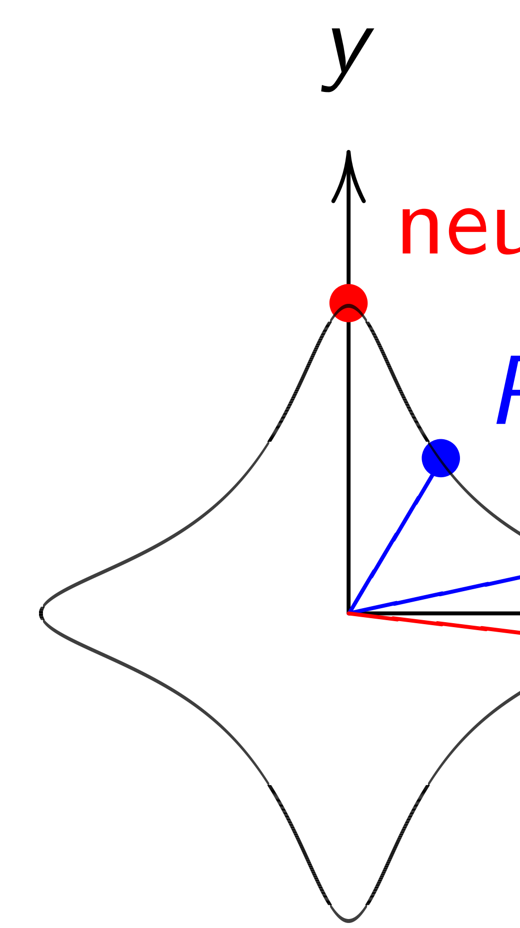

$x^2 + y^2 = 1 - 30x$
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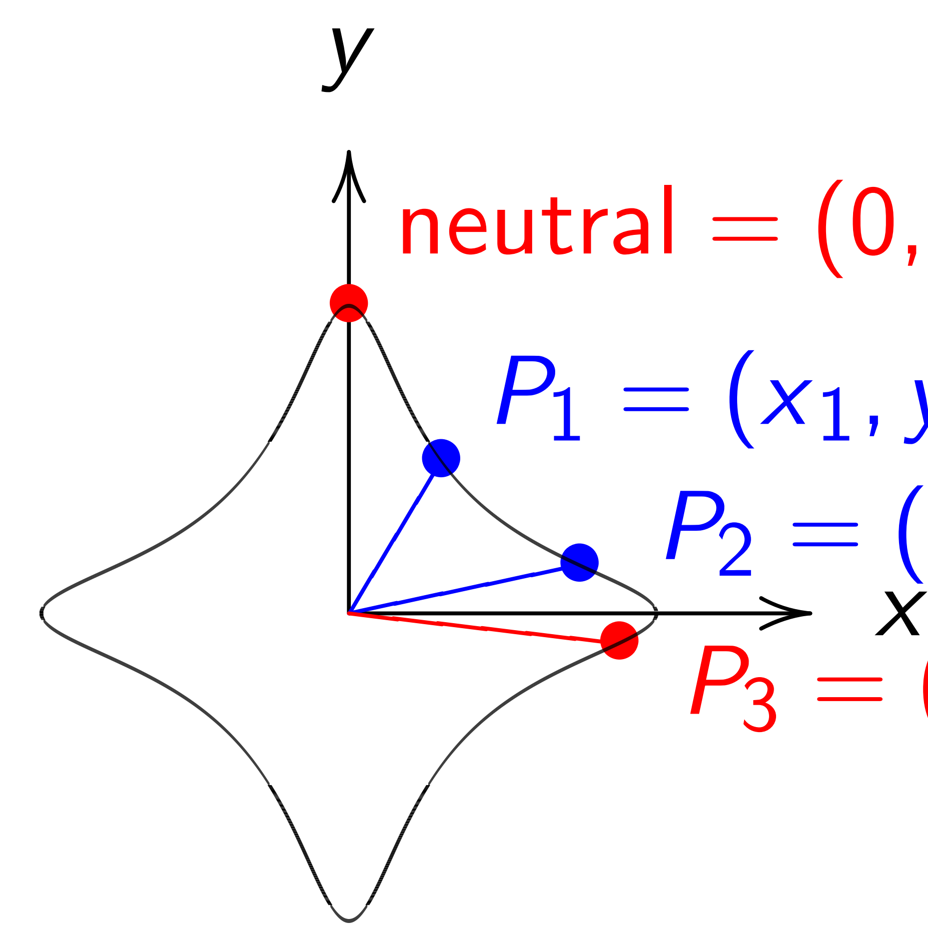

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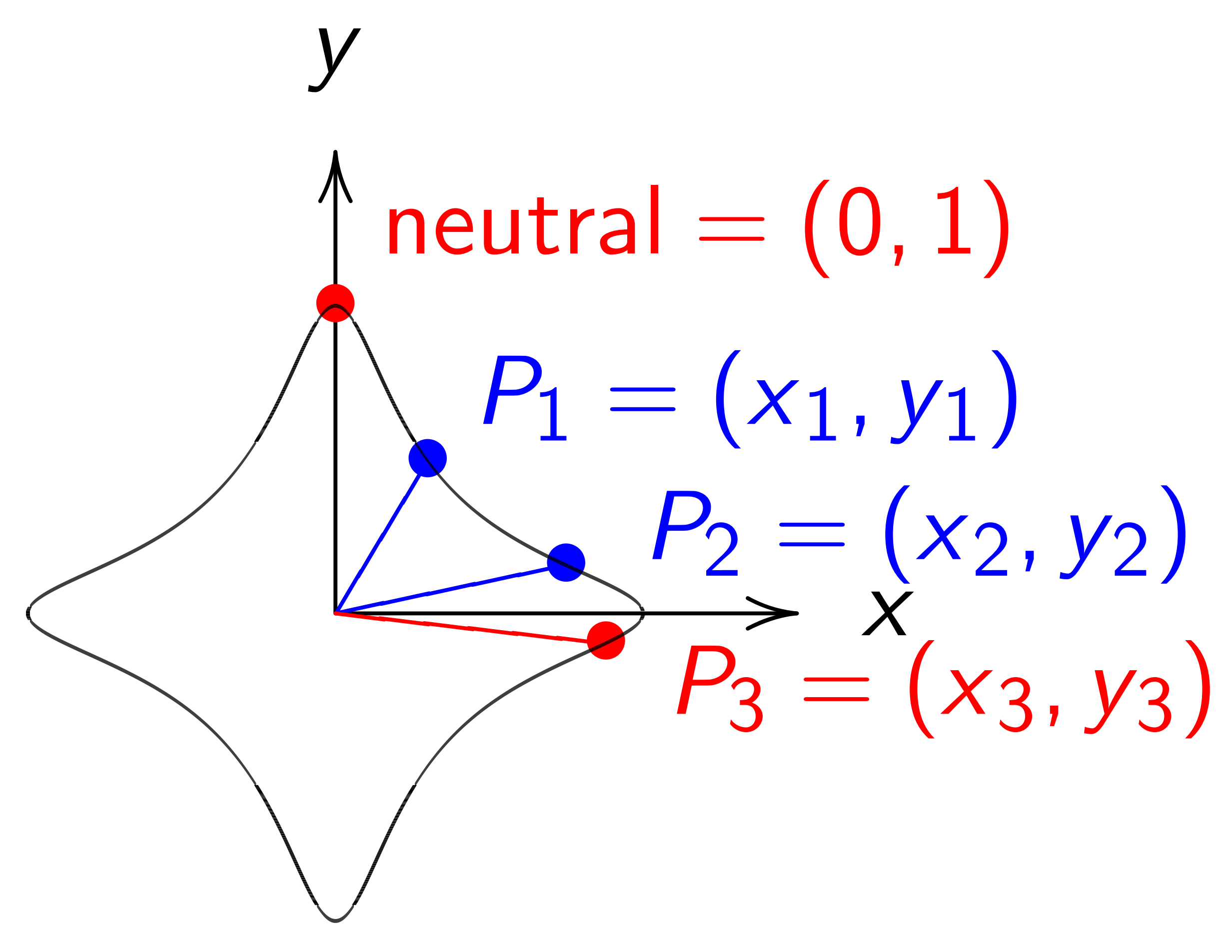

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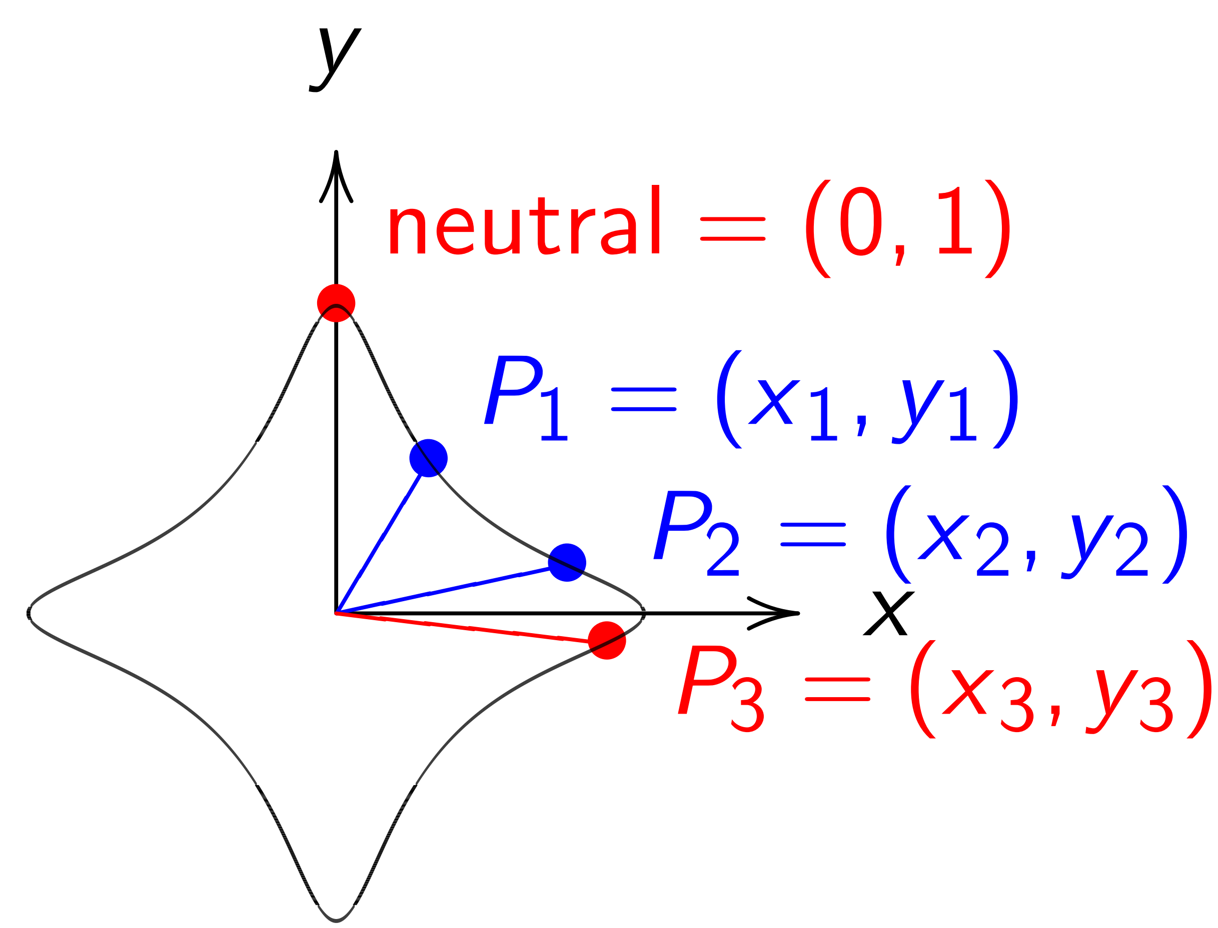

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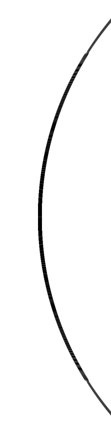

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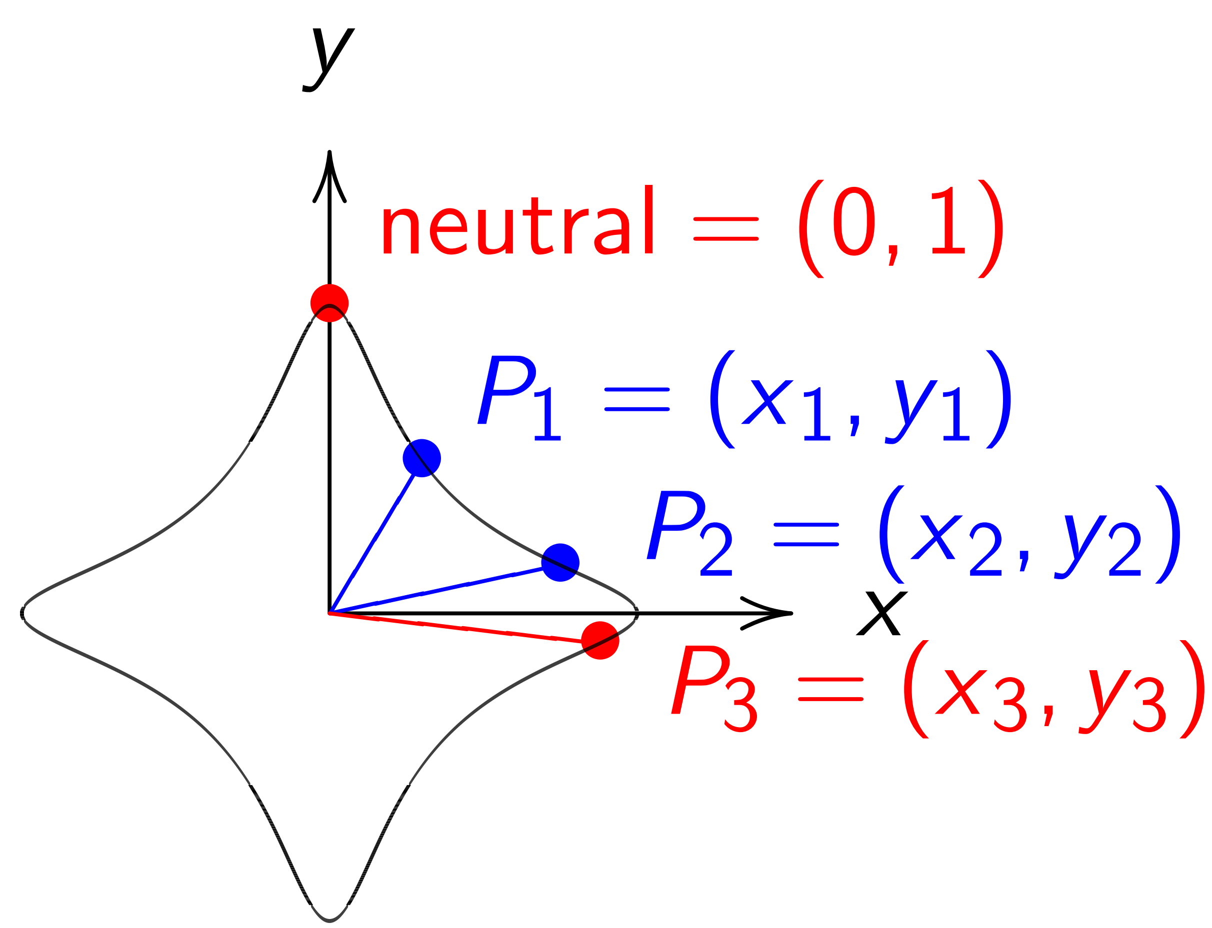

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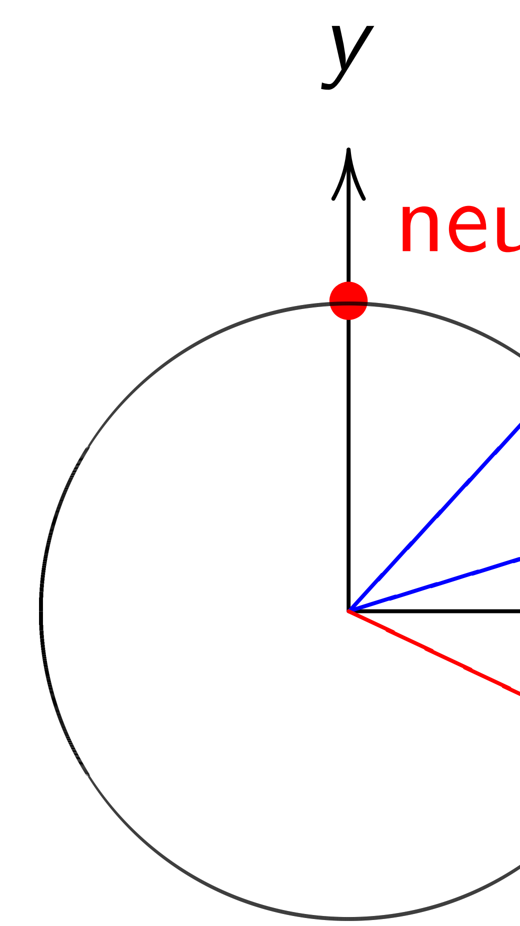

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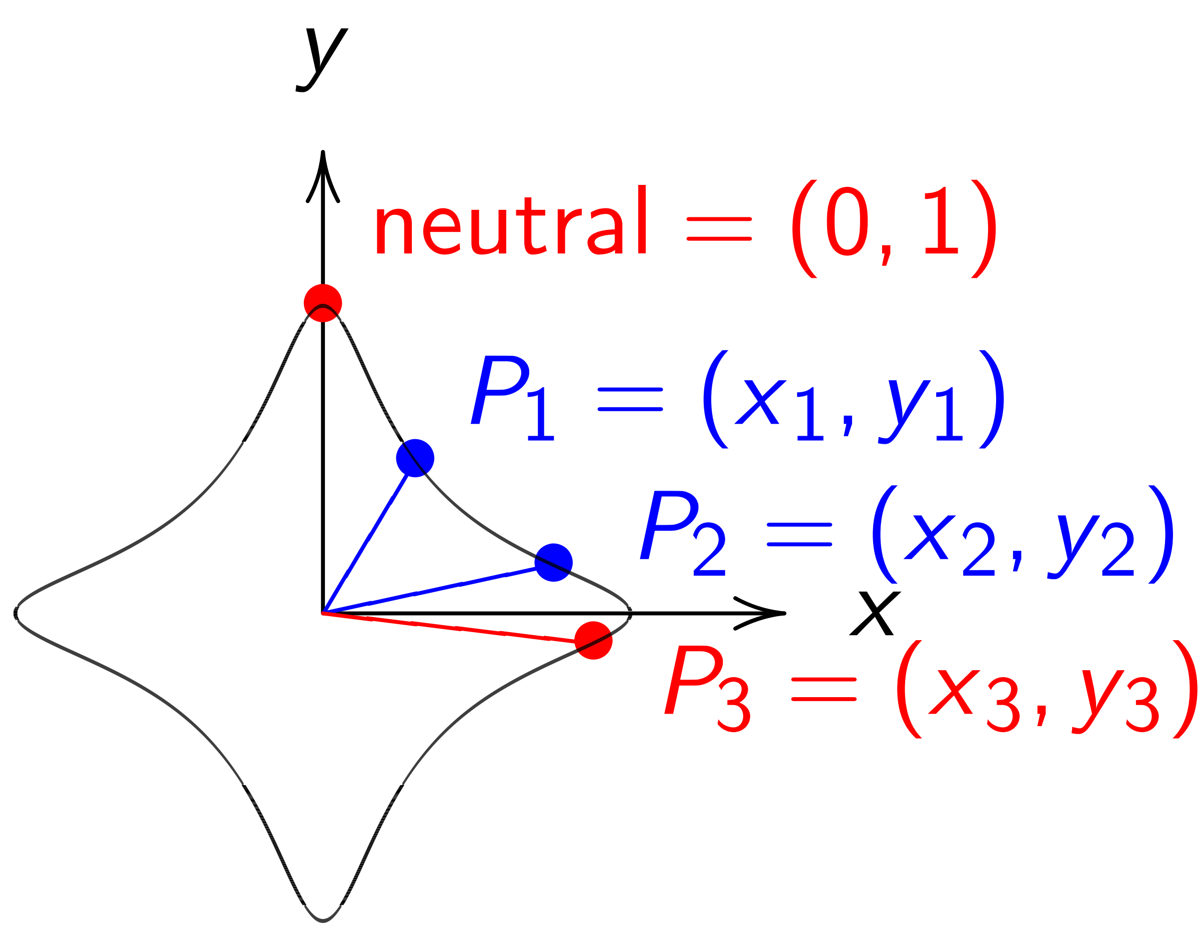

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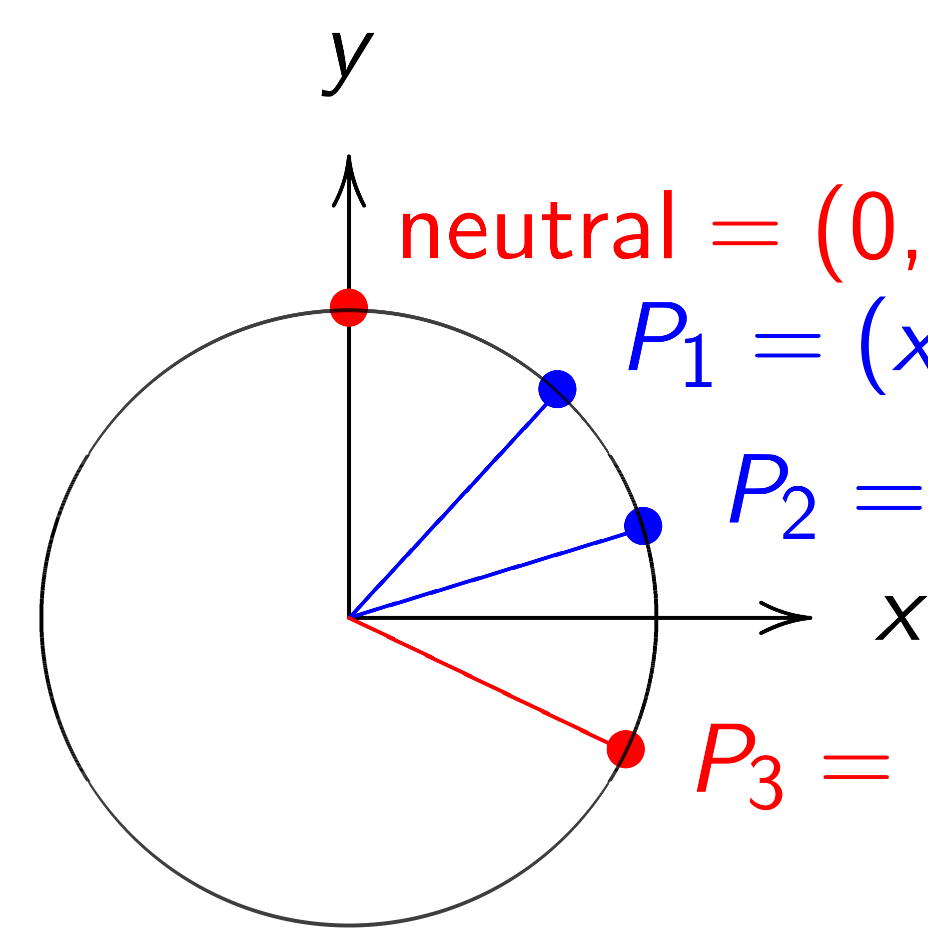

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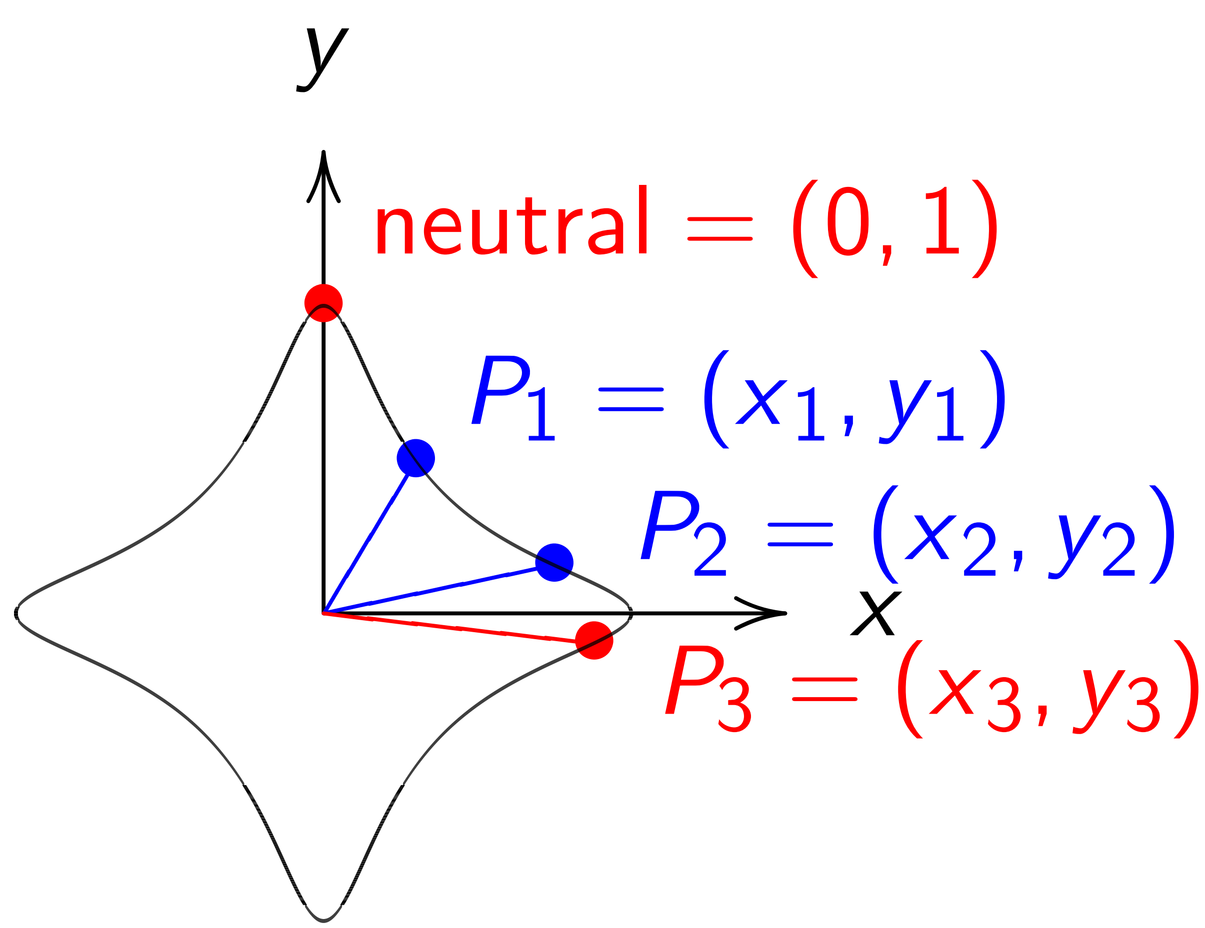

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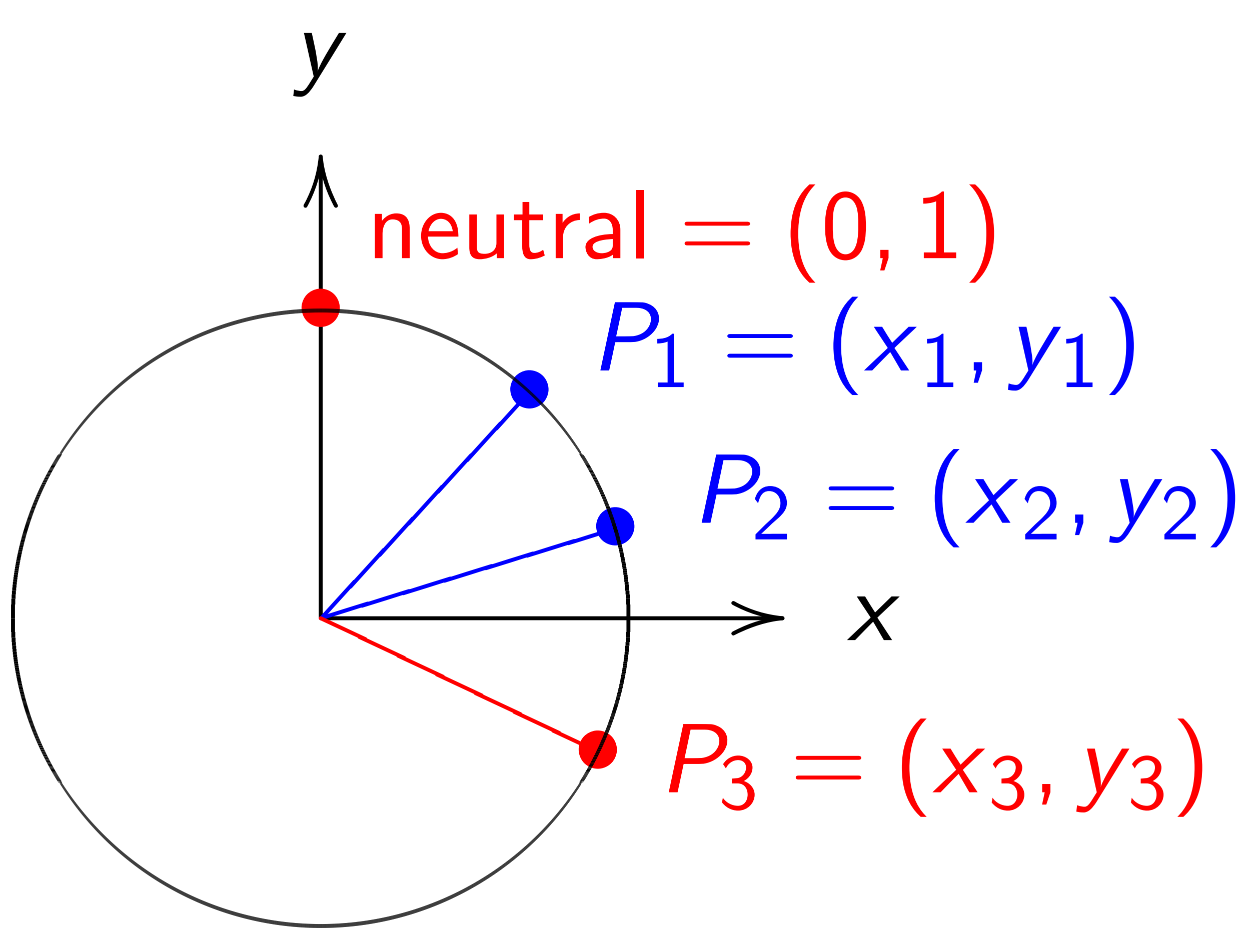

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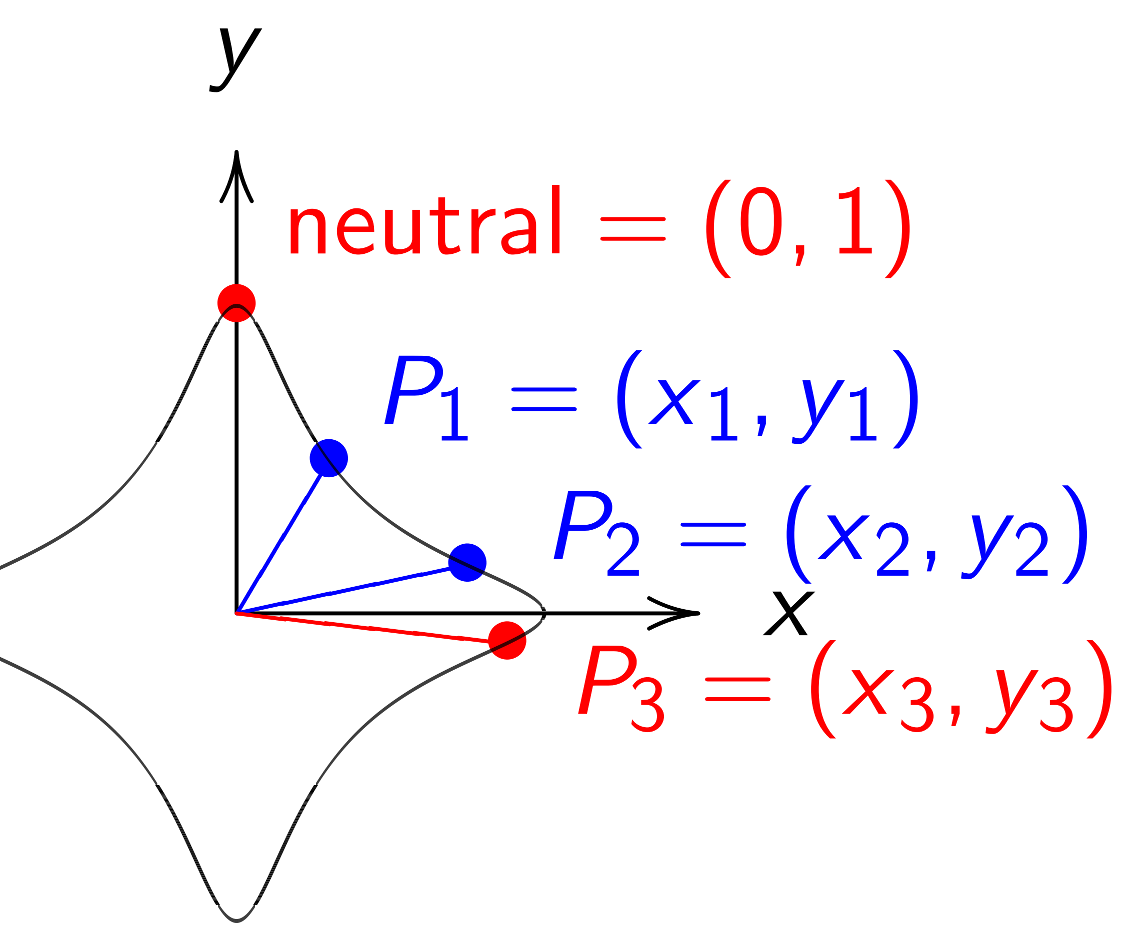

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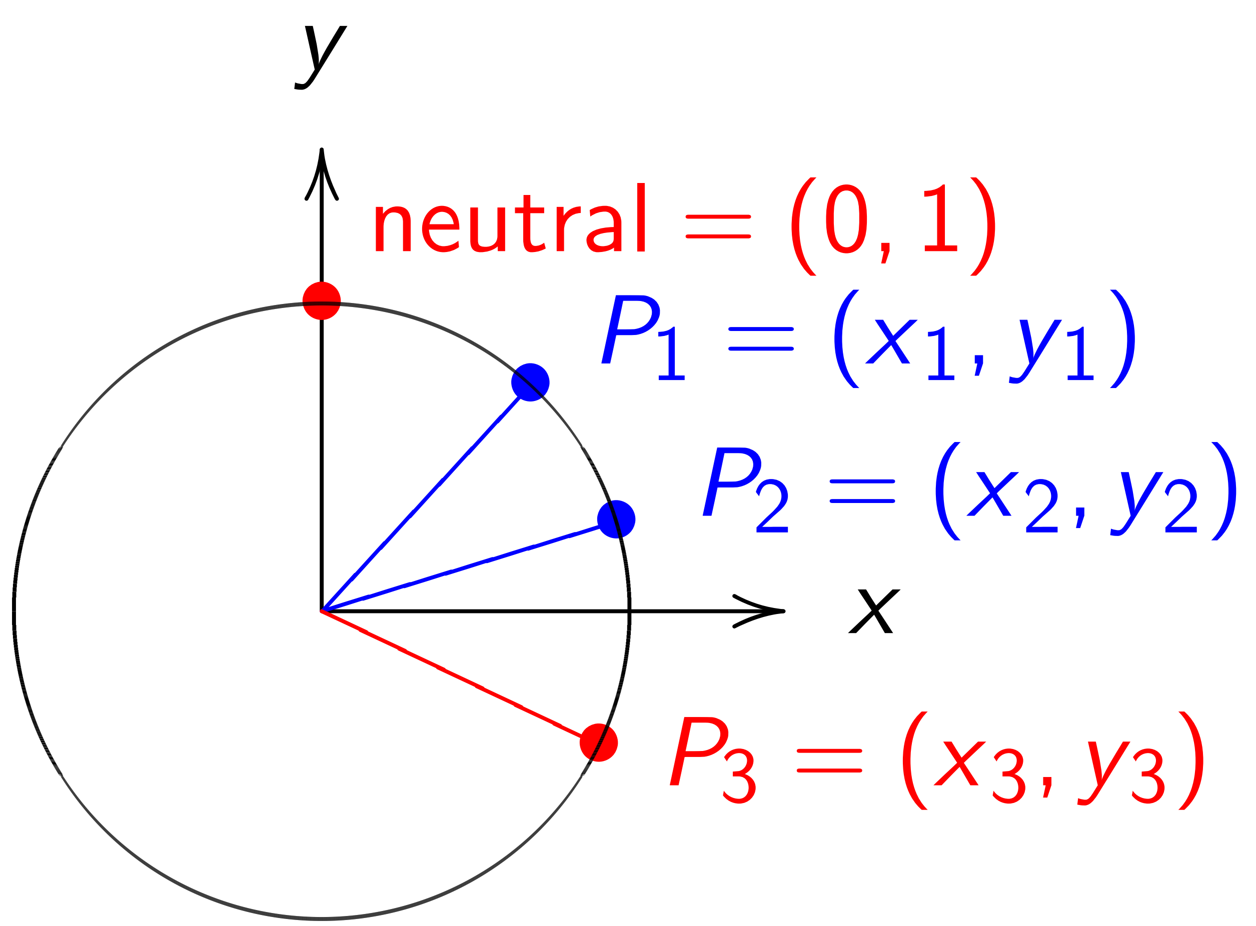

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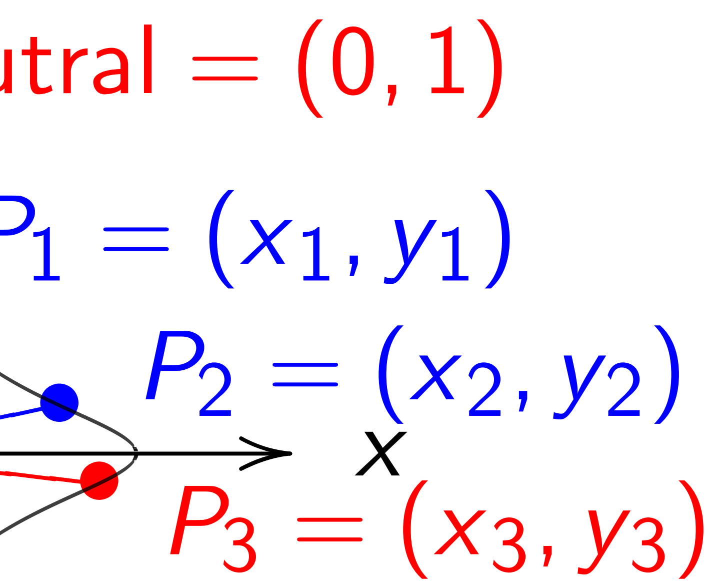

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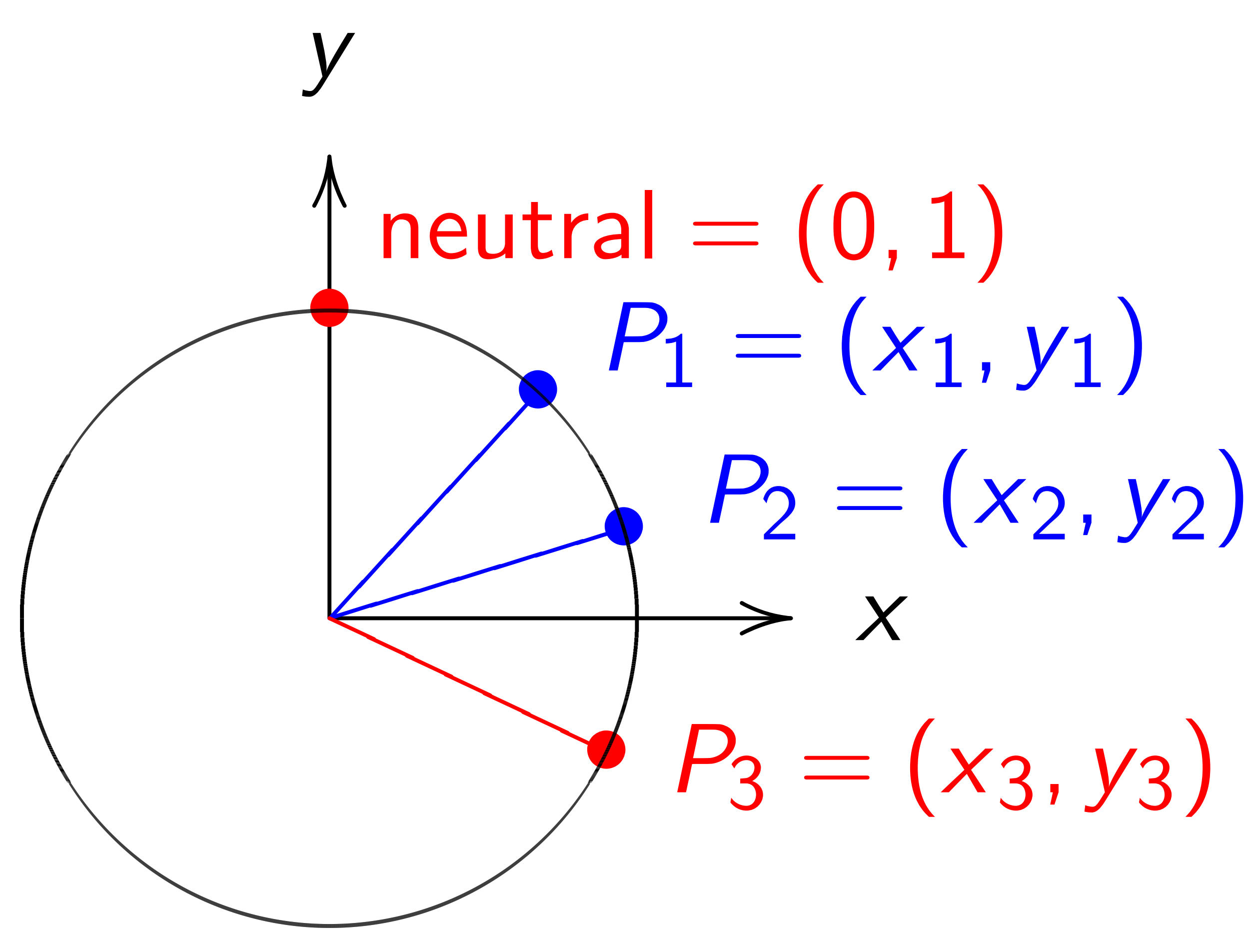

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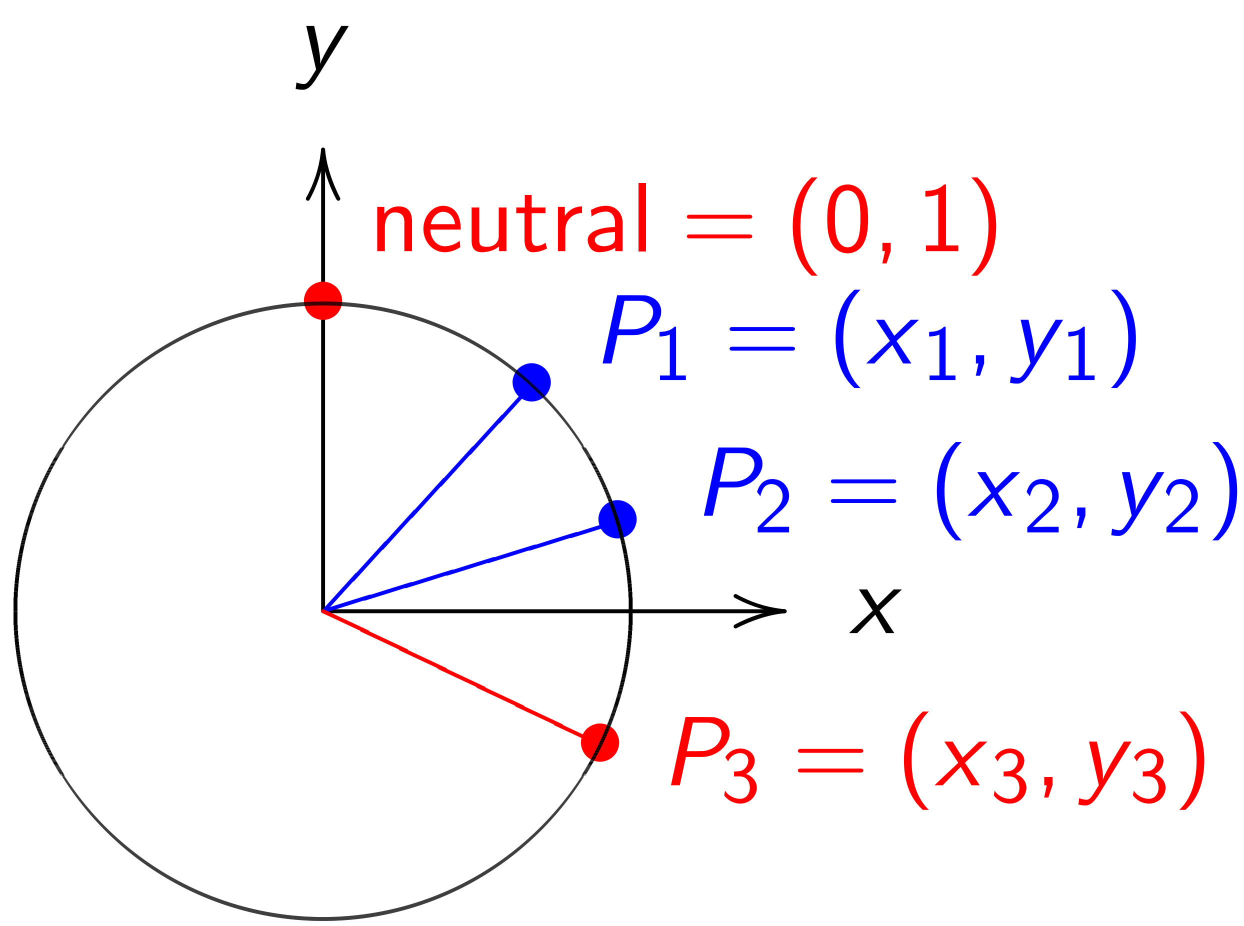

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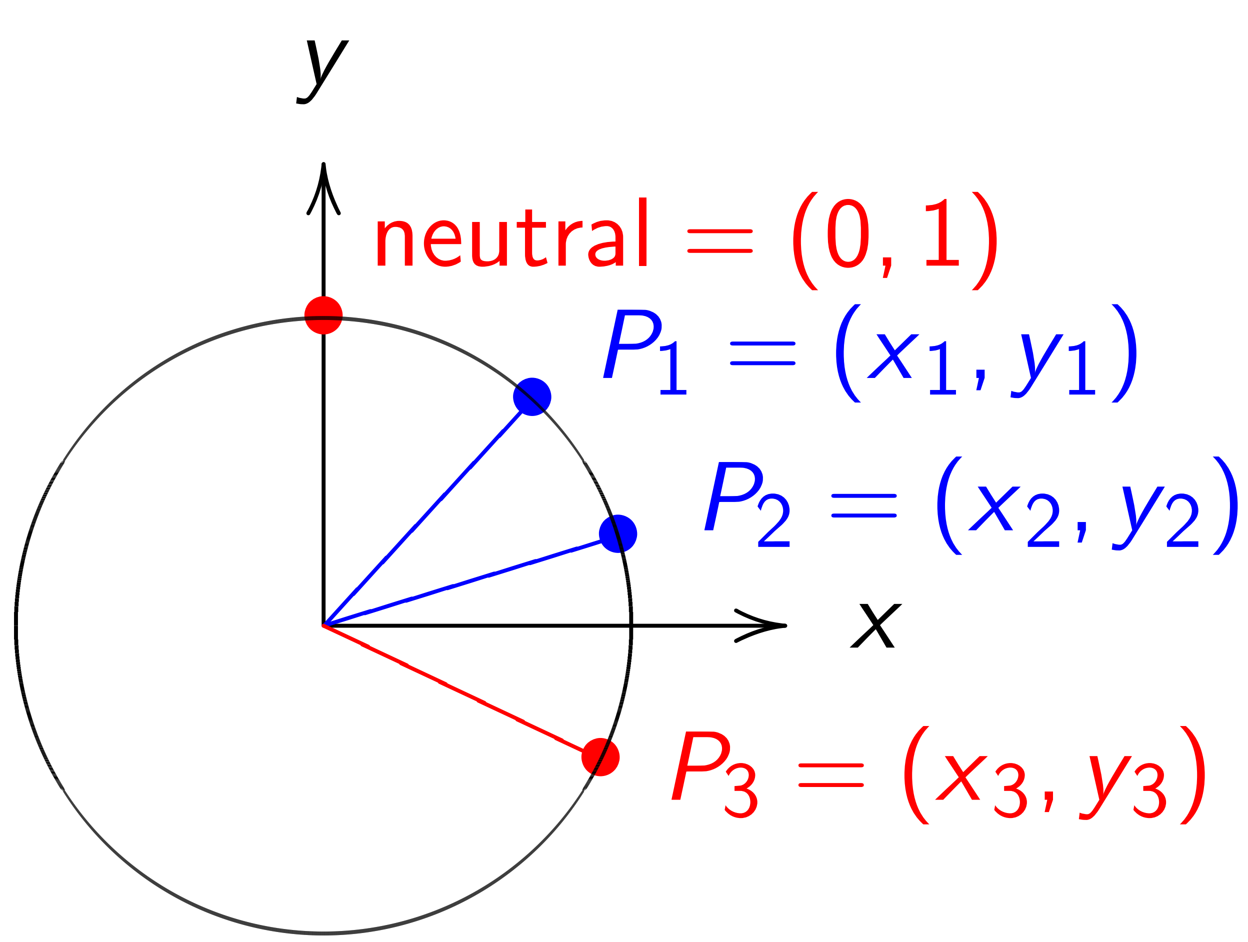

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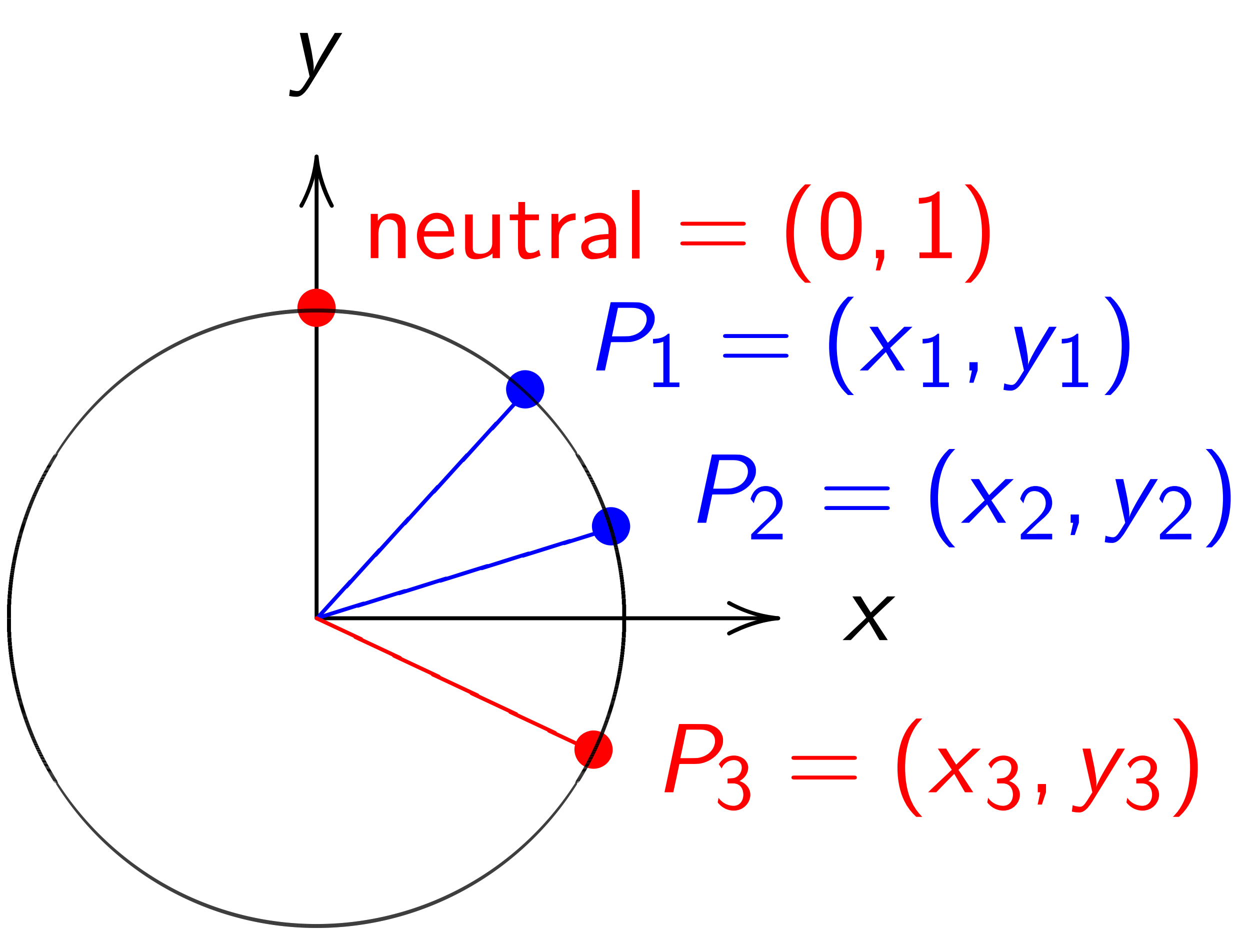

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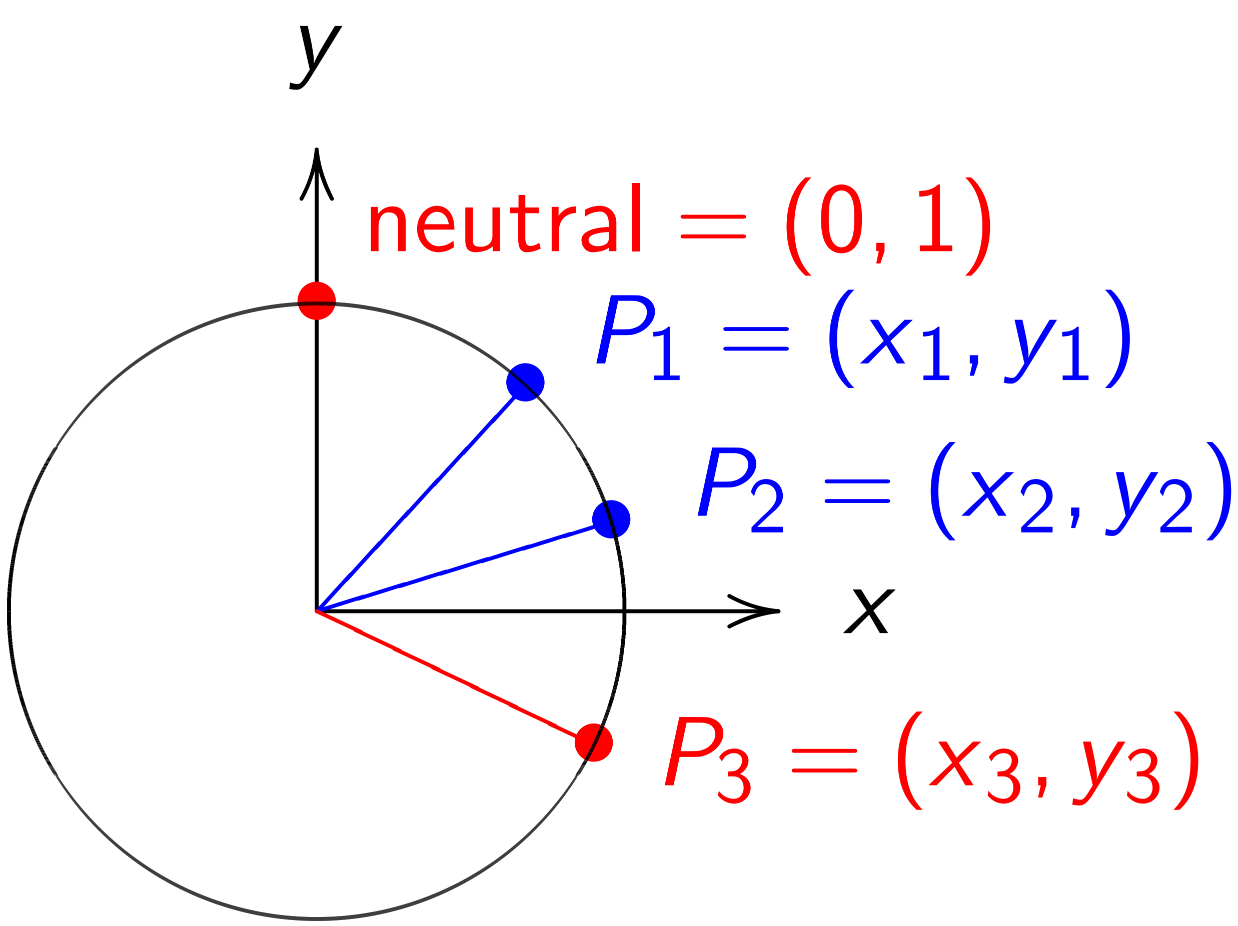

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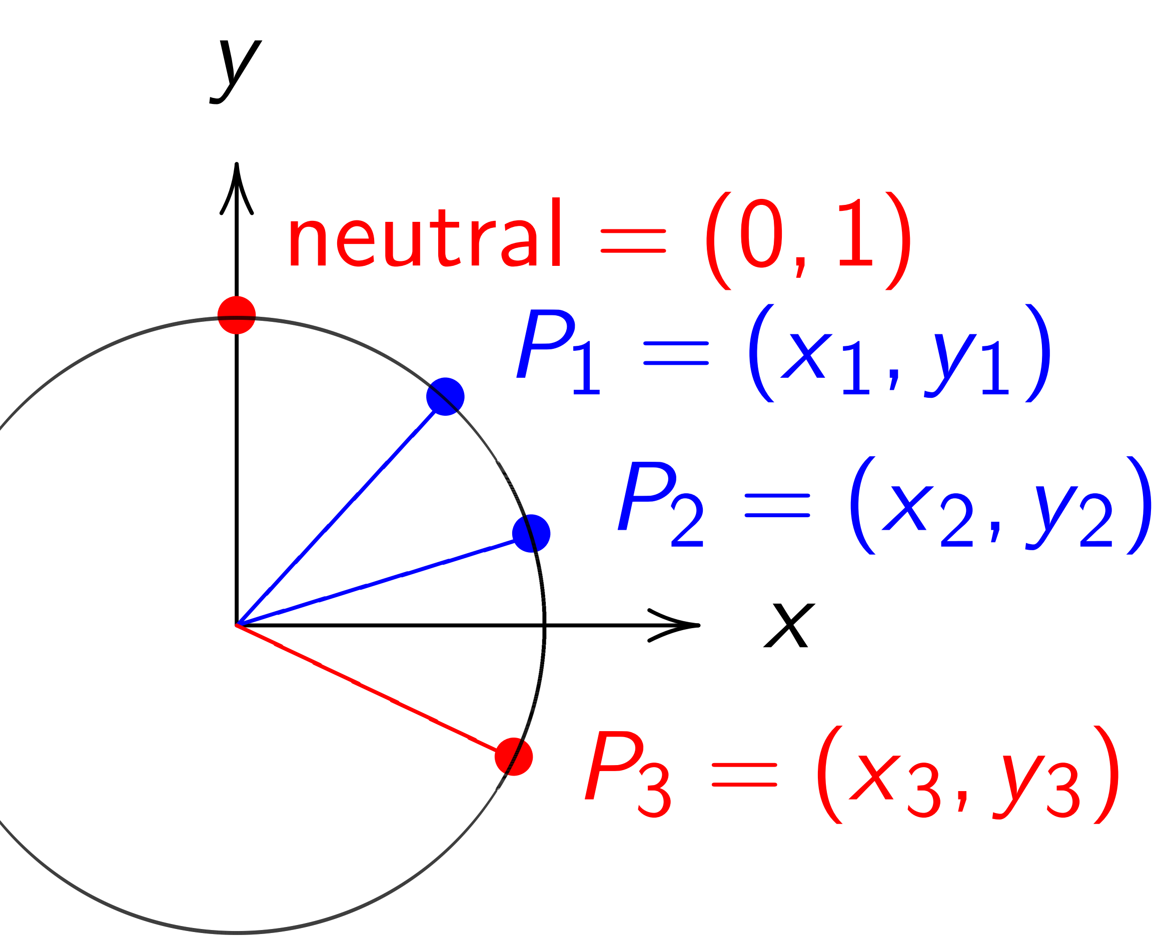

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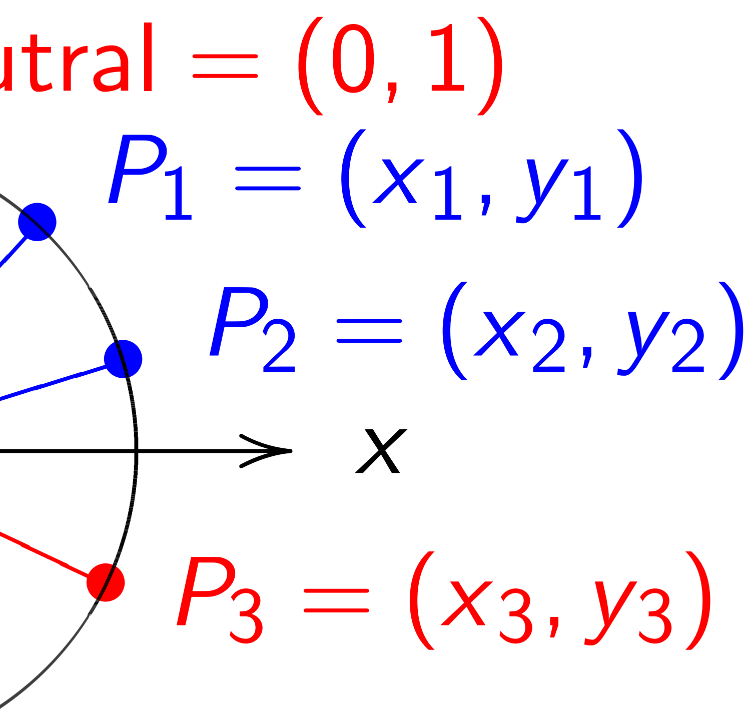

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  return
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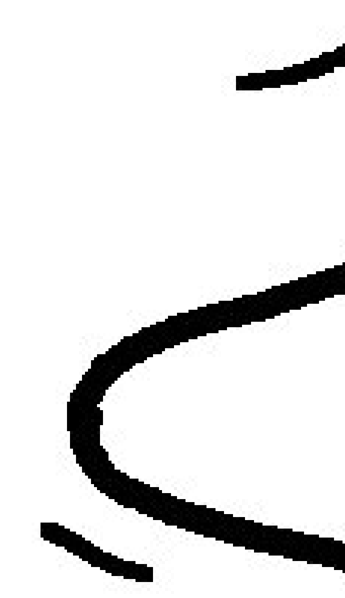

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curve is still elliptic, and
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but there are failure cases,
often exploitable by attackers.
Safe code is more complicated.

Edwards curves are

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Denominators are never 0.
But need different proof;
“$x^2 + y^2 > 0$” doesn’t work.

Answer: Can prove that
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Addition law is **complete**.

This proof relies on
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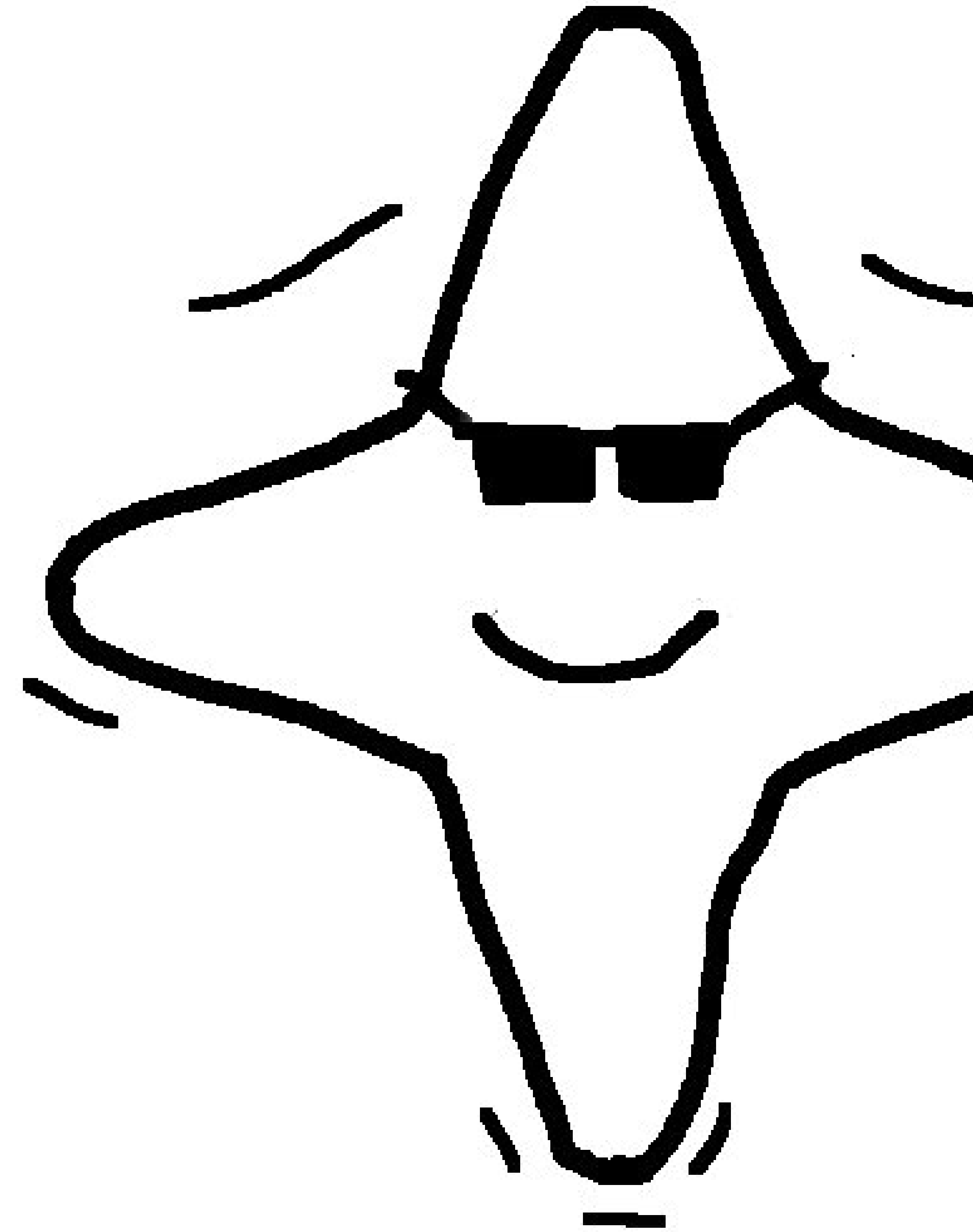

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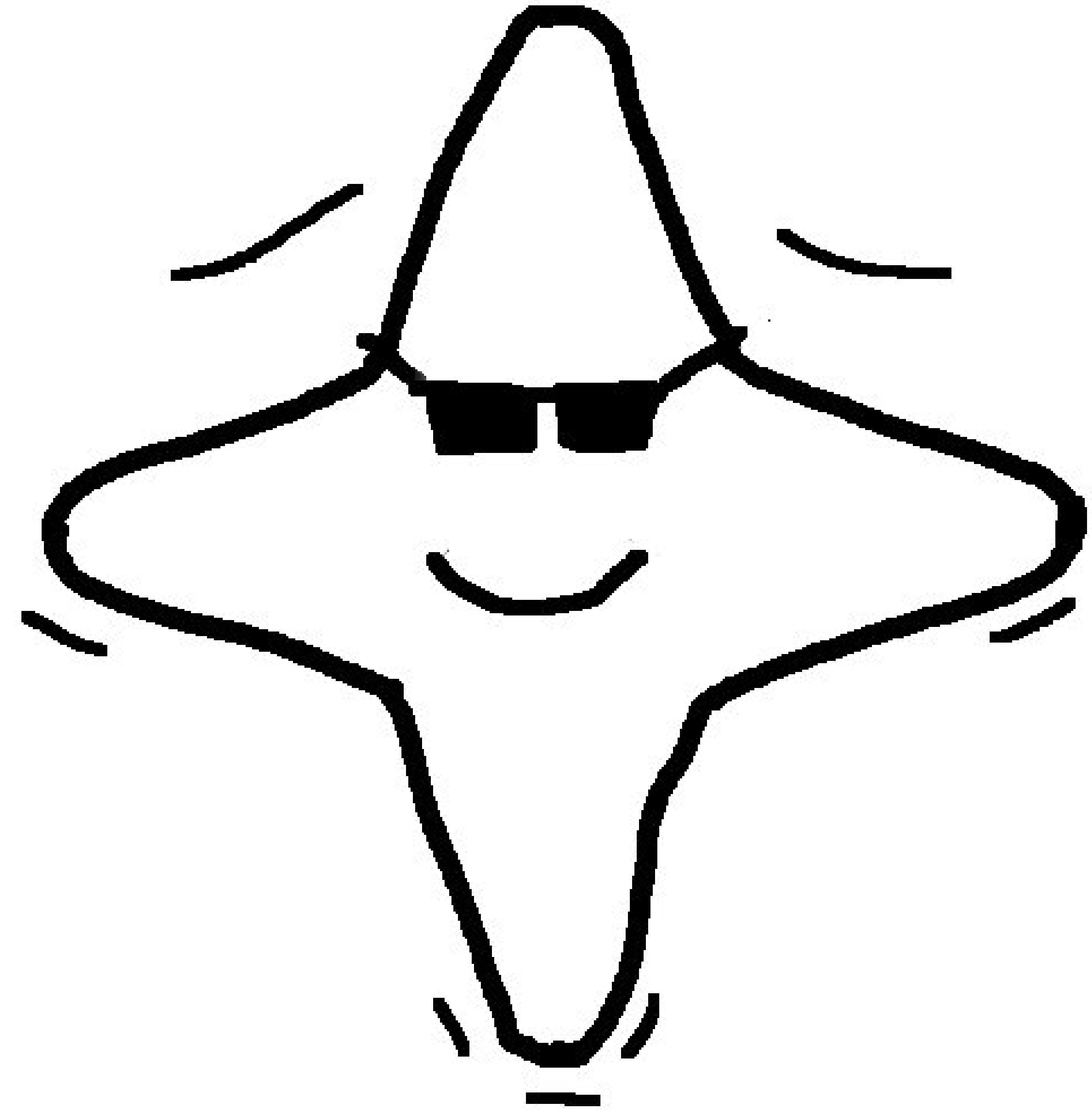

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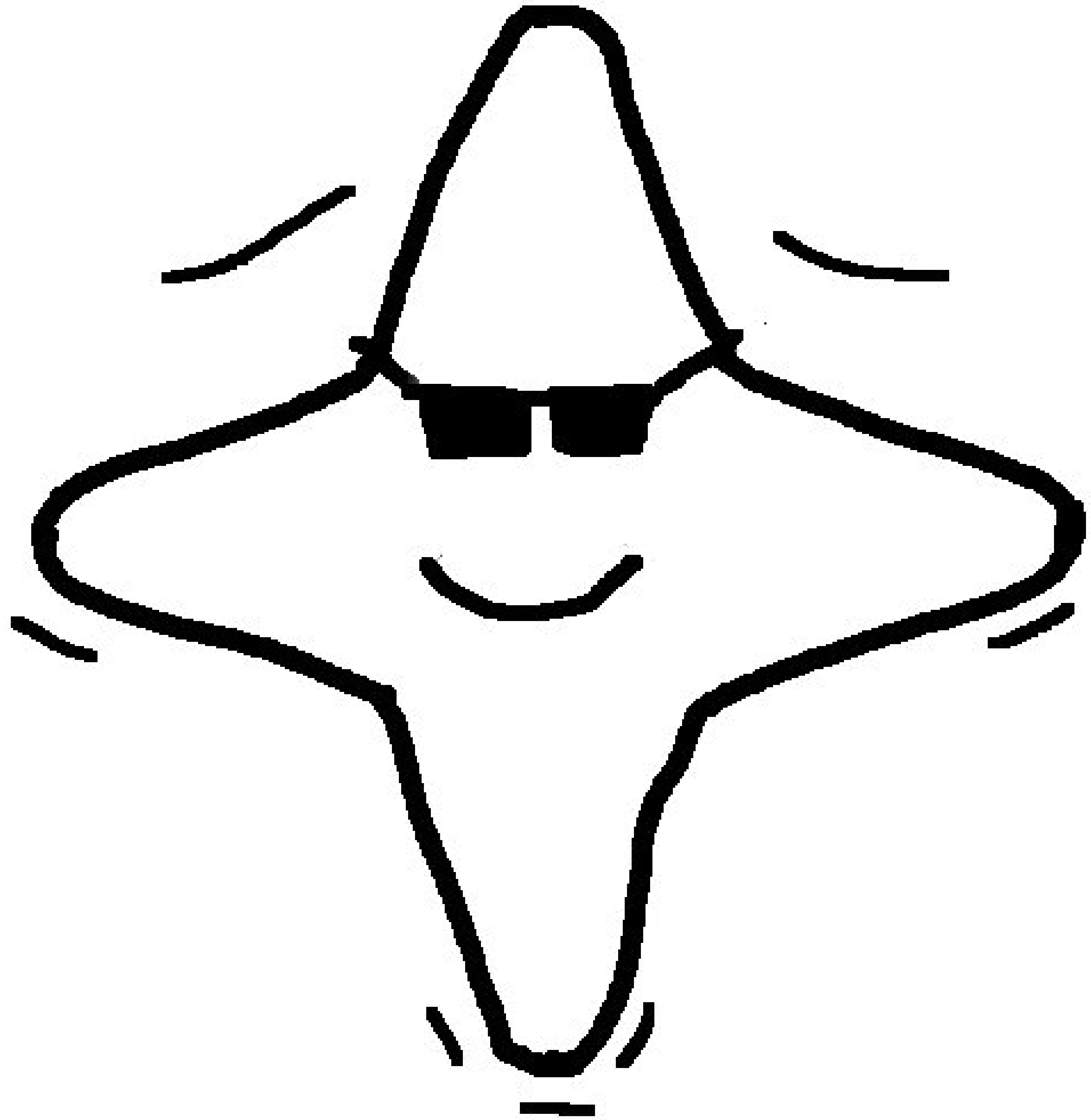

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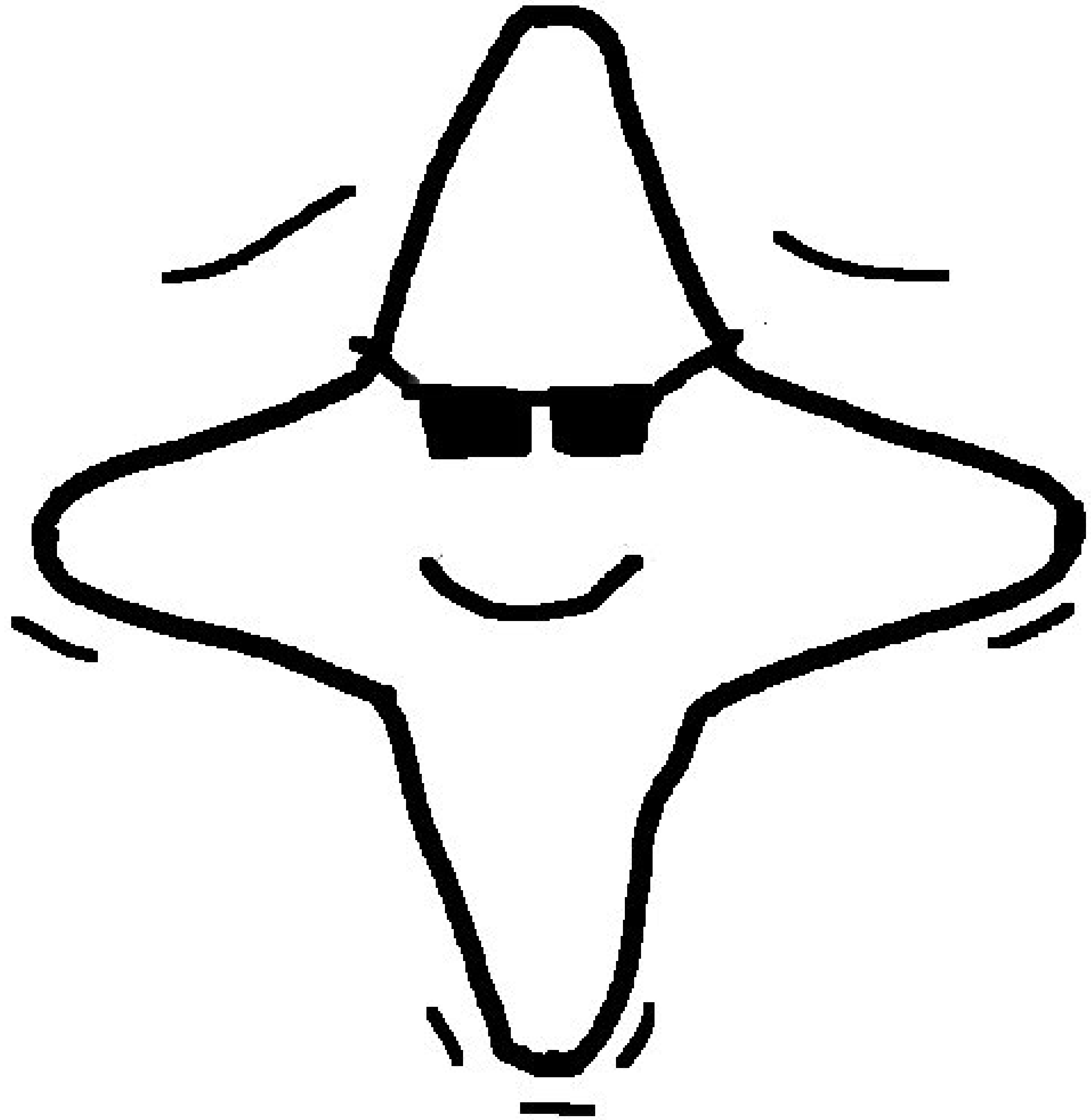

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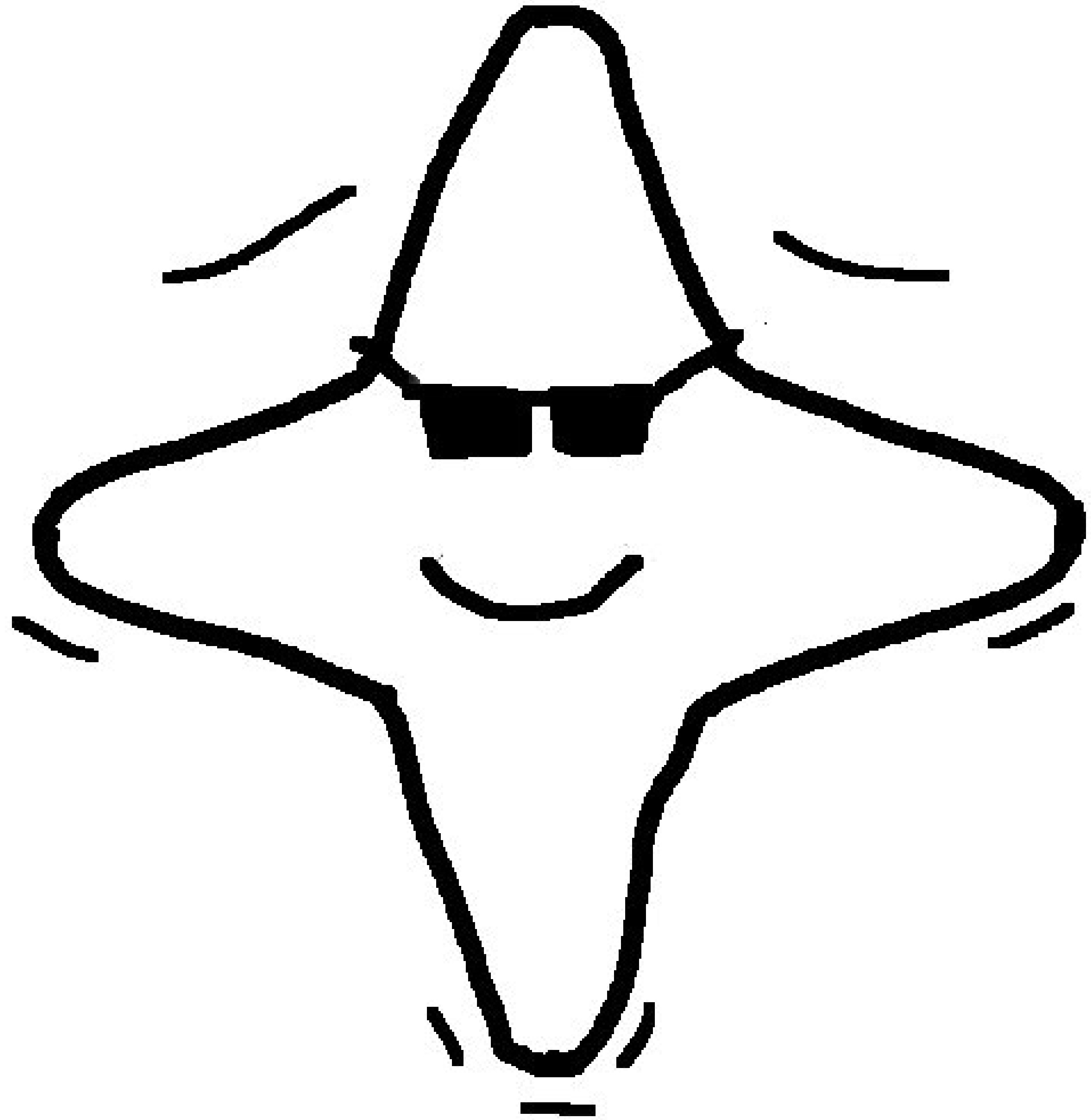

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Users can sign messages using Edwards curves.

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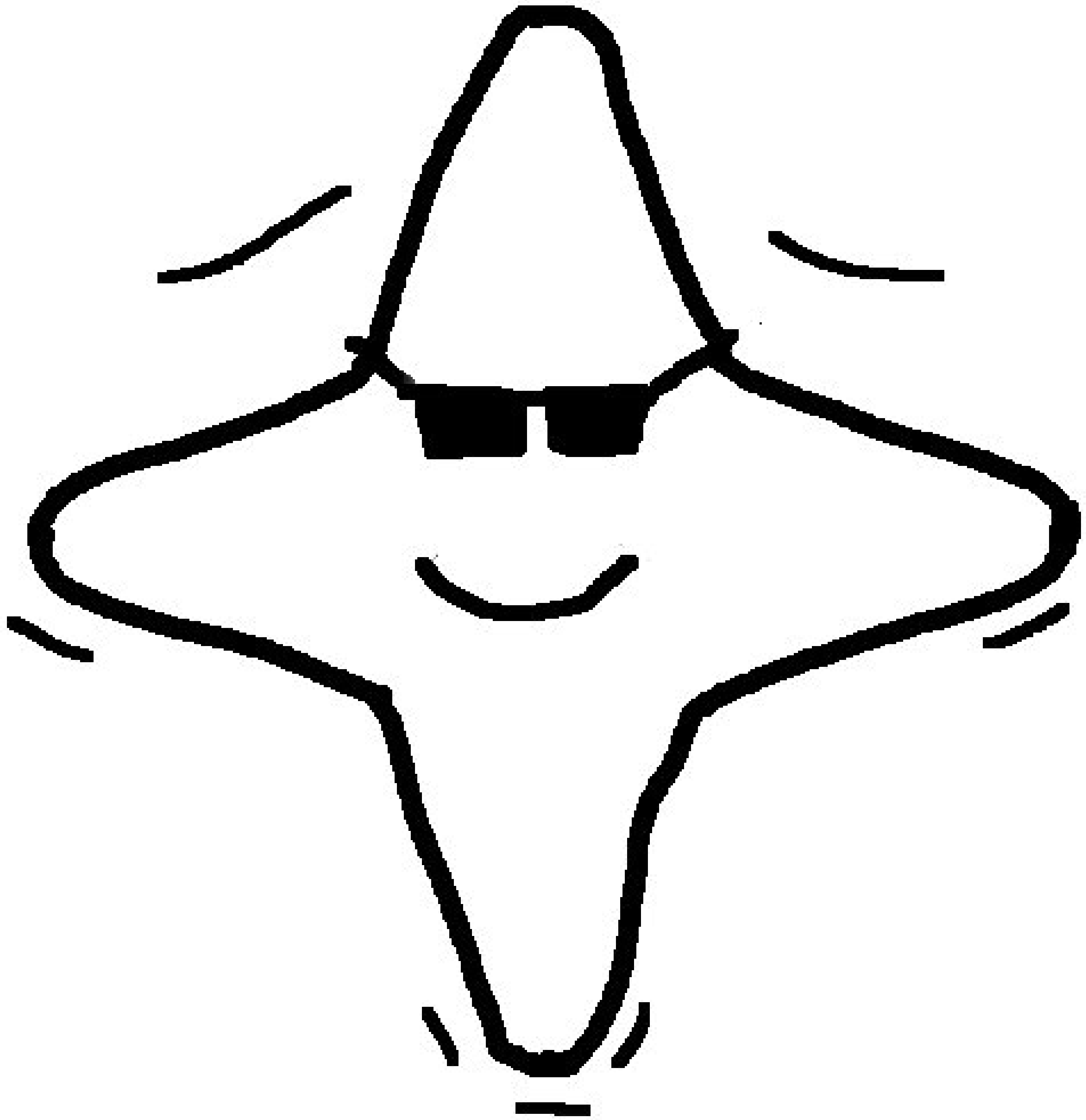

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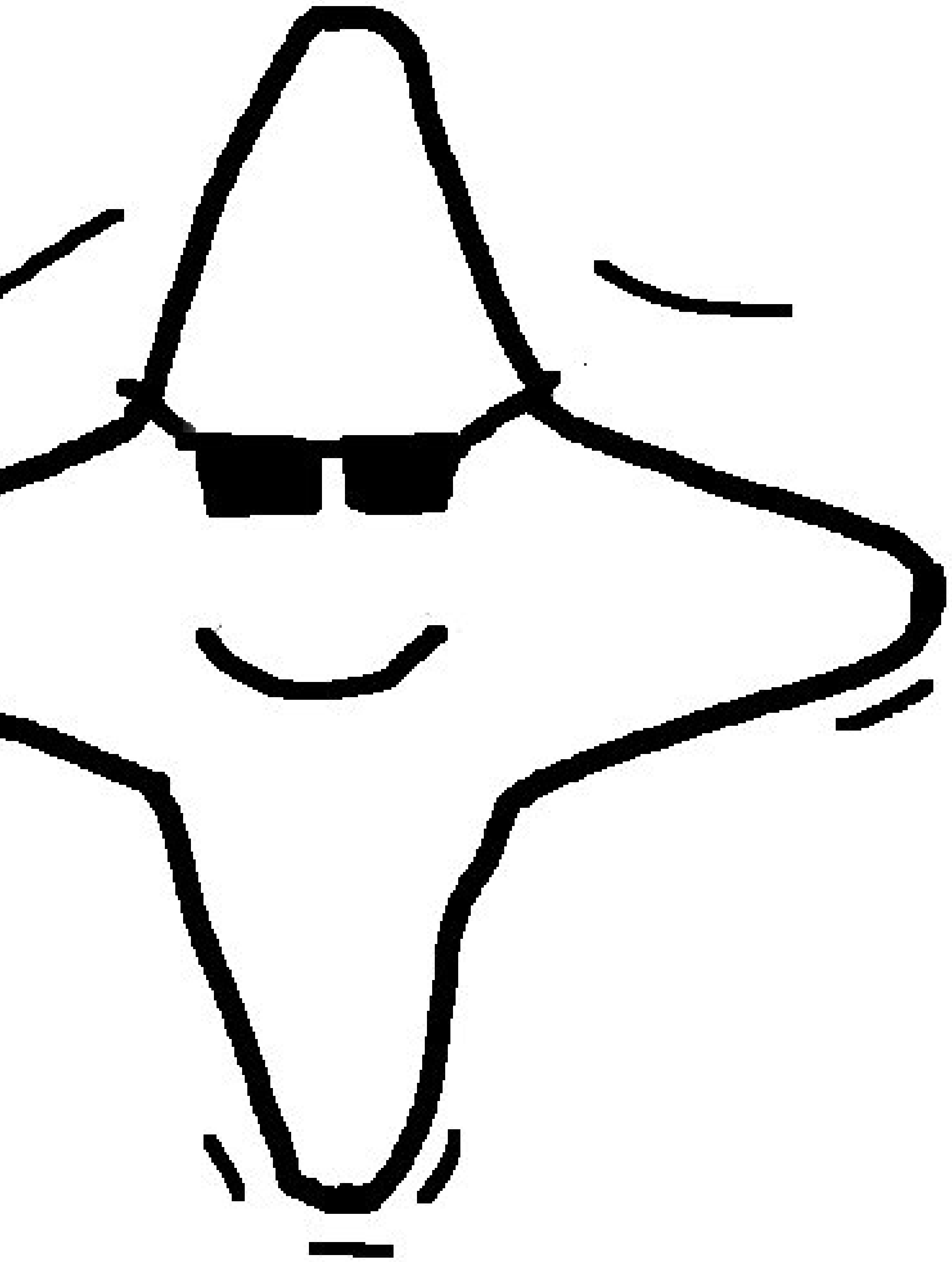

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To sign message m
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puts $r \equiv y_1$ mod ℓ
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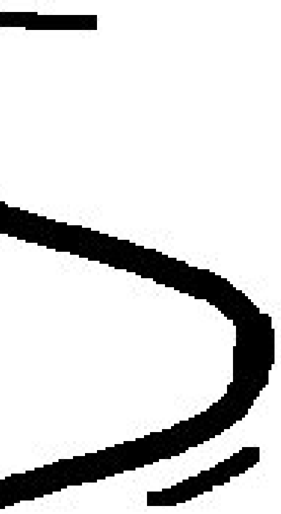

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To sign message m,
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Compute $w_1 \equiv s^{-}$
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Anybody can verify signatur
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Compute $w_1 \equiv s^{-1}h(m)$ m
and $w_2 \equiv s^{-1} \cdot r \bmod \ell$.
Check whether the y-coordi
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and if so, accept signature.

Alice's signatures are valid:
$w_1P + w_2P_A =$
$(s^{-1}h(m))P + (s^{-1} \cdot r$
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Check whether the y-coordinate of $w_1P + w_2P_A$ equals r modulo ℓ and if so, accept signature.

Alice's signatures are valid:

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\begin{aligned}
w_1P + w_2P_A &= \\
&(s^{-1}h(m))P + (s^{-1} \cdot r)P_A = \\
&(s^{-1}(h(m) + ra))P = kP
\end{aligned}
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and so the y-coordinate of this expression equals r,
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$$w_1P + w_2P_A = (s^{-1}h(m))P + (s^{-1} \cdot r)P_A = (s^{-1}(h(m) + ra))P = kP$$

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Attacker's view on

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Attacker's view on signature

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and so the y-coordinate of this expression equals r,
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Attacker's view on signatures

Anybody can produce an $R = kP$.
Alice's private key is only used in $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.

Can fake signatures if one can break the DLP, i.e., if one can compute a from P_A.
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If k is known for s
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If k is known for some $m, ($

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$s_1 - s_2 = k_1^{-1}(h(m_1) + r$

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Continue as above.

If bits of many k's are know

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If bits of many k's are known (biased PRNG) can attack $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$ as hidden number problem using lattice basis reduction.

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Malicious signer

Alice can set up h

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Malicious signer

Alice can set up her public k

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i.e., she can claim to have

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$R = (x_1, y_1)$ and $-R = (-x$

have the same y-coordinate.

Thus, (r, s) fits $R = kP$,

$s \equiv k^{-1}(h(m_1) + ra) \bmod$

$-R = (-k)P$,

$s \equiv -k^{-1}(h(m_2) + ra) \bmod$

$a \equiv -(h(m_1)+h(m_2))/2r$

If k is known for some $m, (r, s)$ then $a \equiv (sk - h(m))/r \bmod \ell$.

If two signatures $m_1, (r, s_1)$ and $m_2, (r, s_2)$ have the same value for r: assume $k_1 = k_2$; observe $s_1 - s_2 = k_1^{-1}(h(m_1) + ra - (h(m_2) + ra))$; compute $k = (s_1 - s_2)/(h(m_1) - h(m_2))$. Continue as above.

If bits of many k's are known (biased PRNG) can attack $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$ as hidden number problem using lattice basis reduction.

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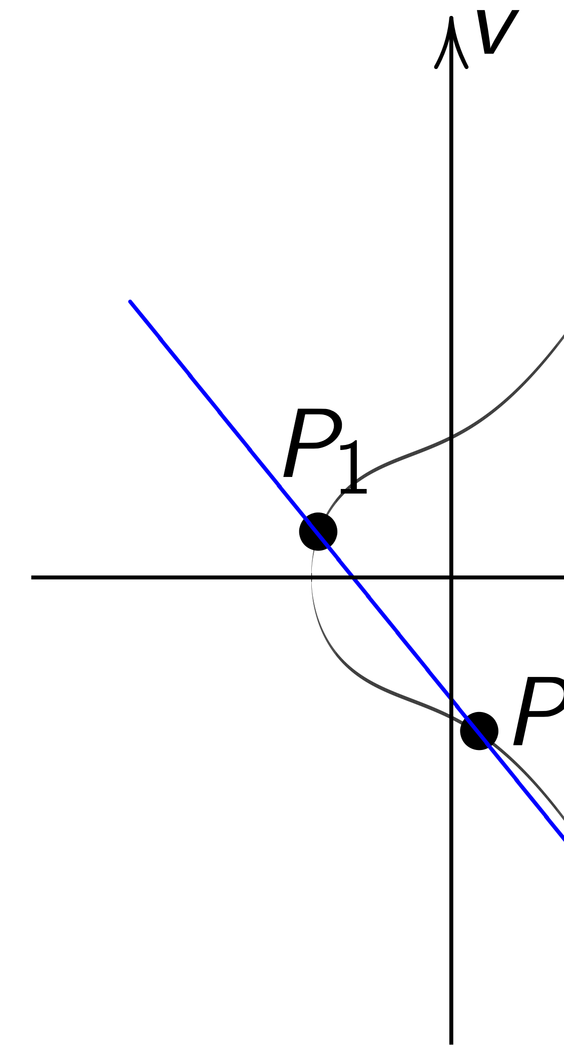

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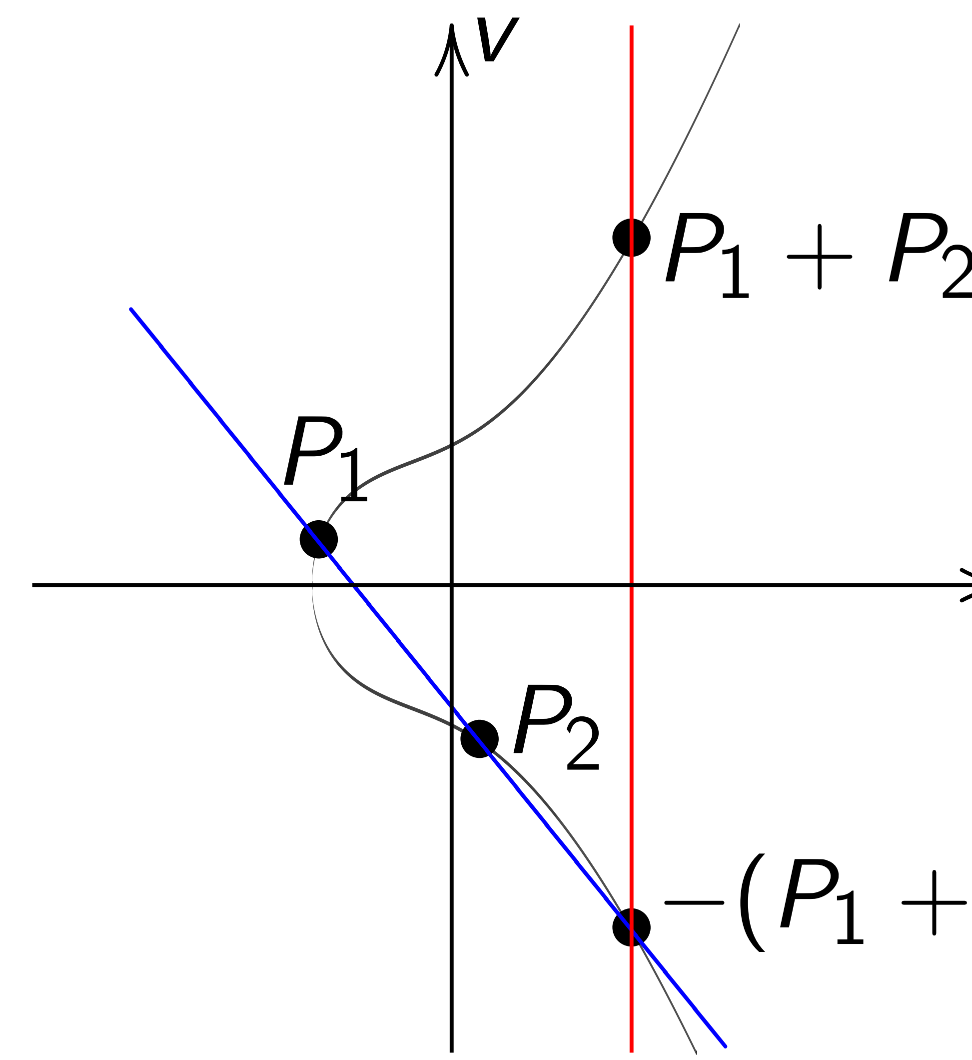

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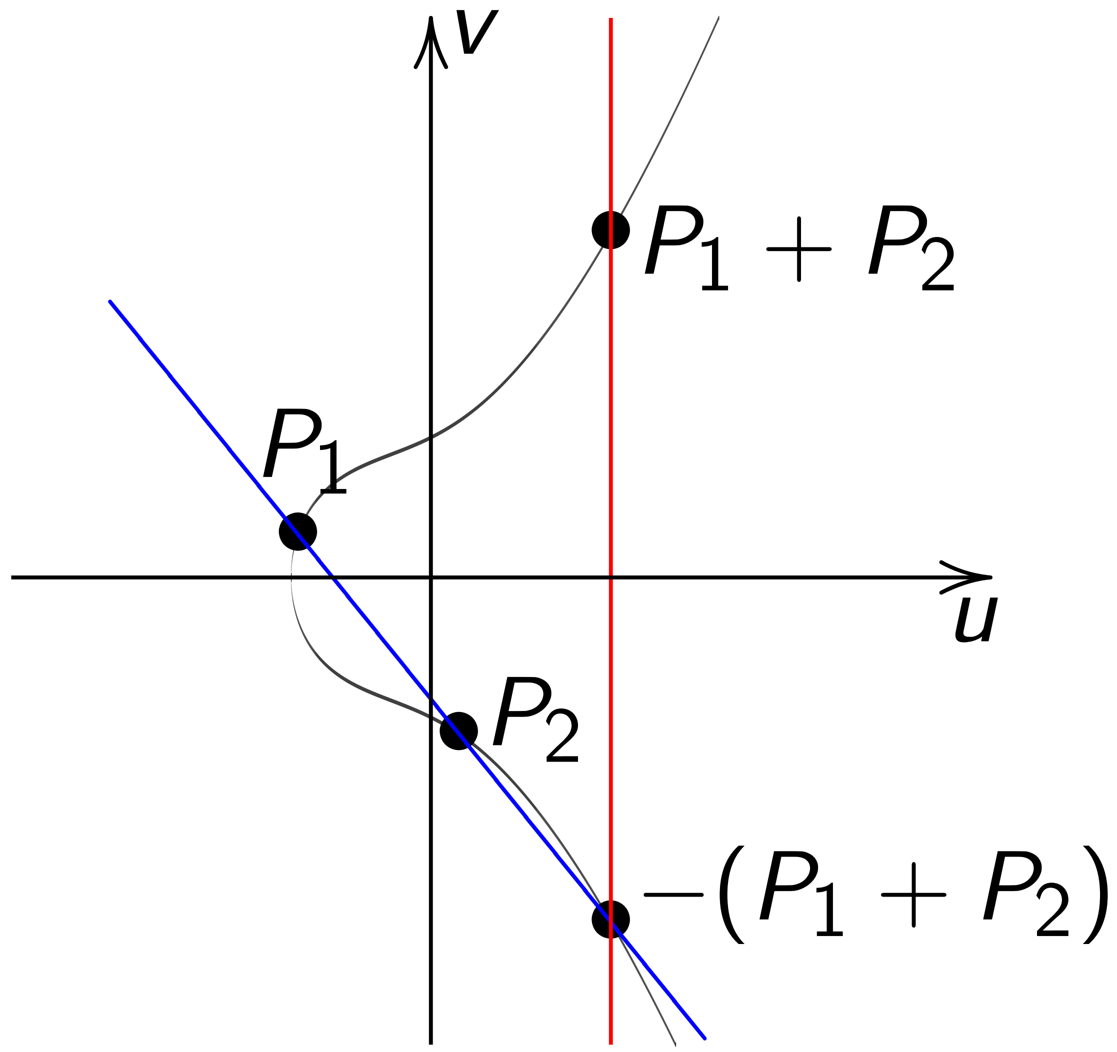

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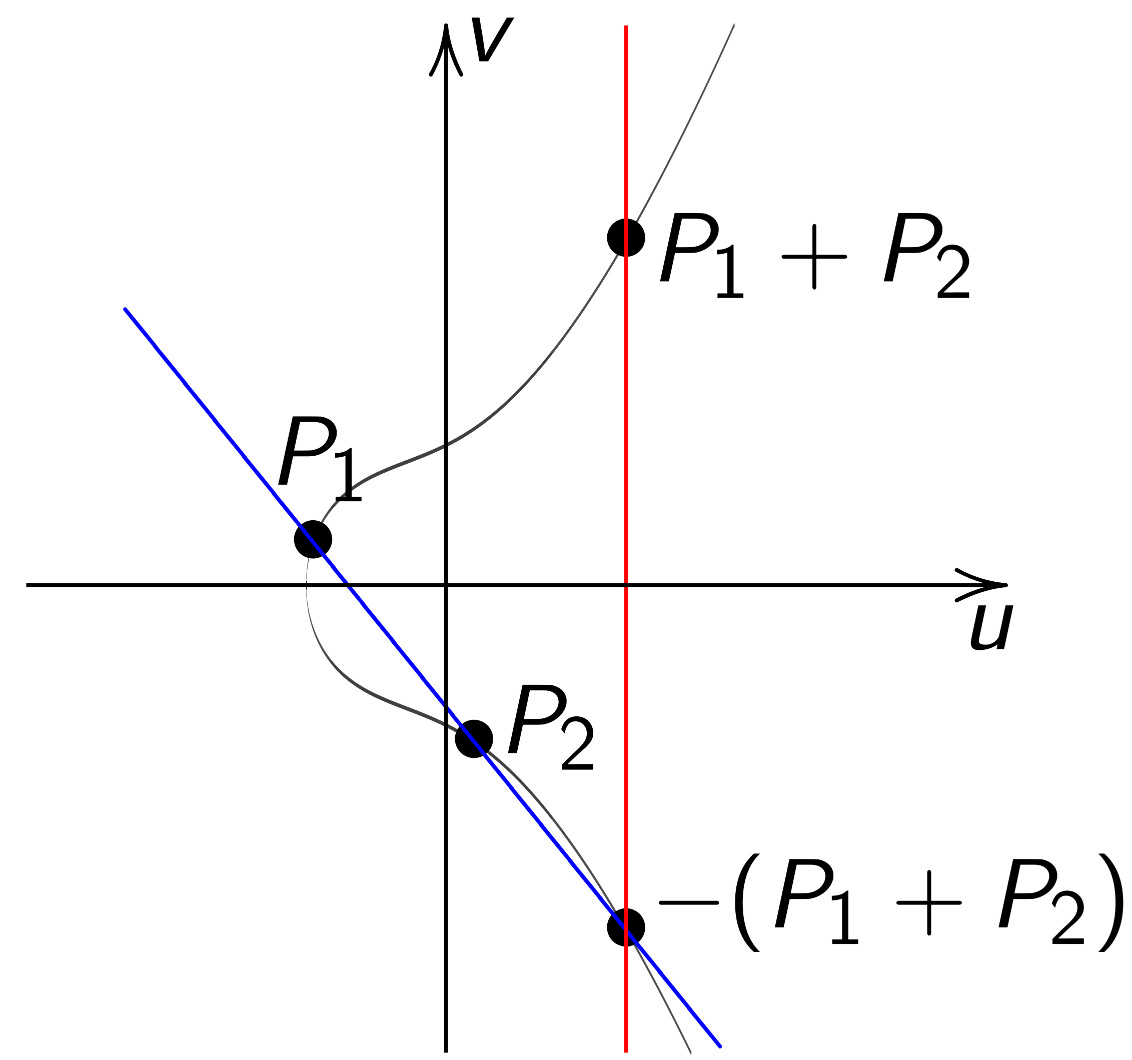

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Doubling

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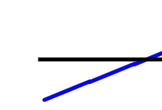

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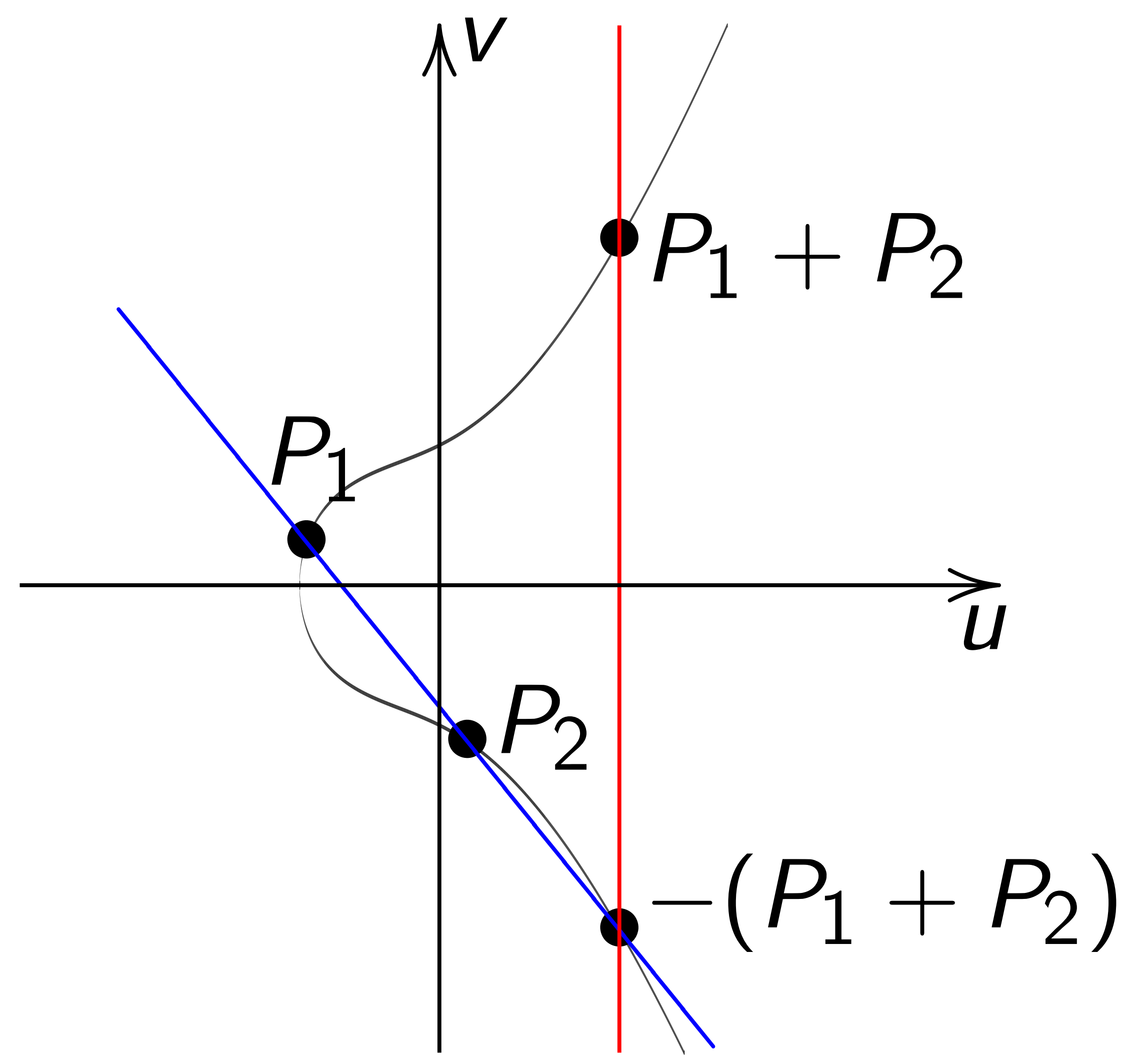

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Doubling on Weier

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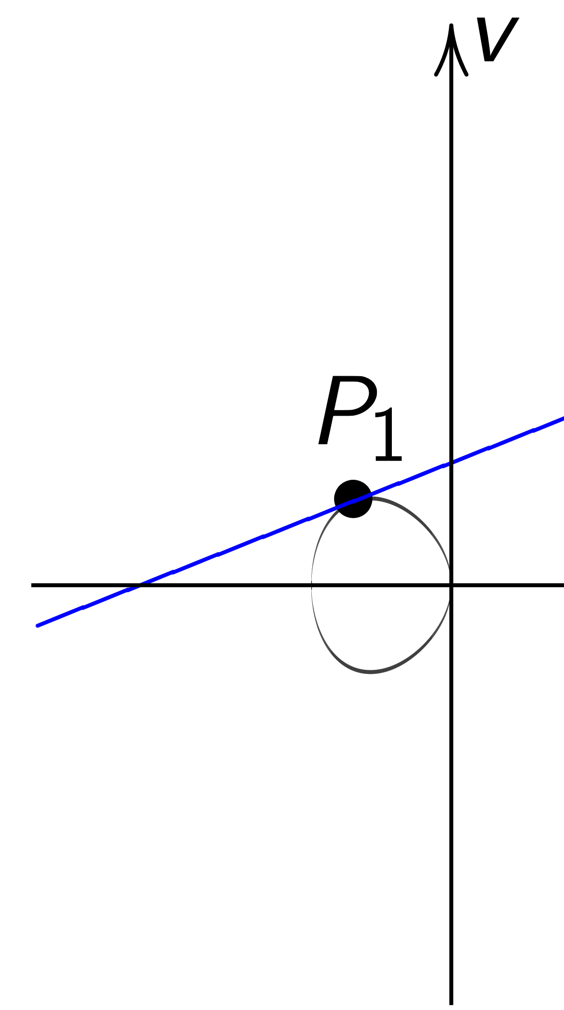

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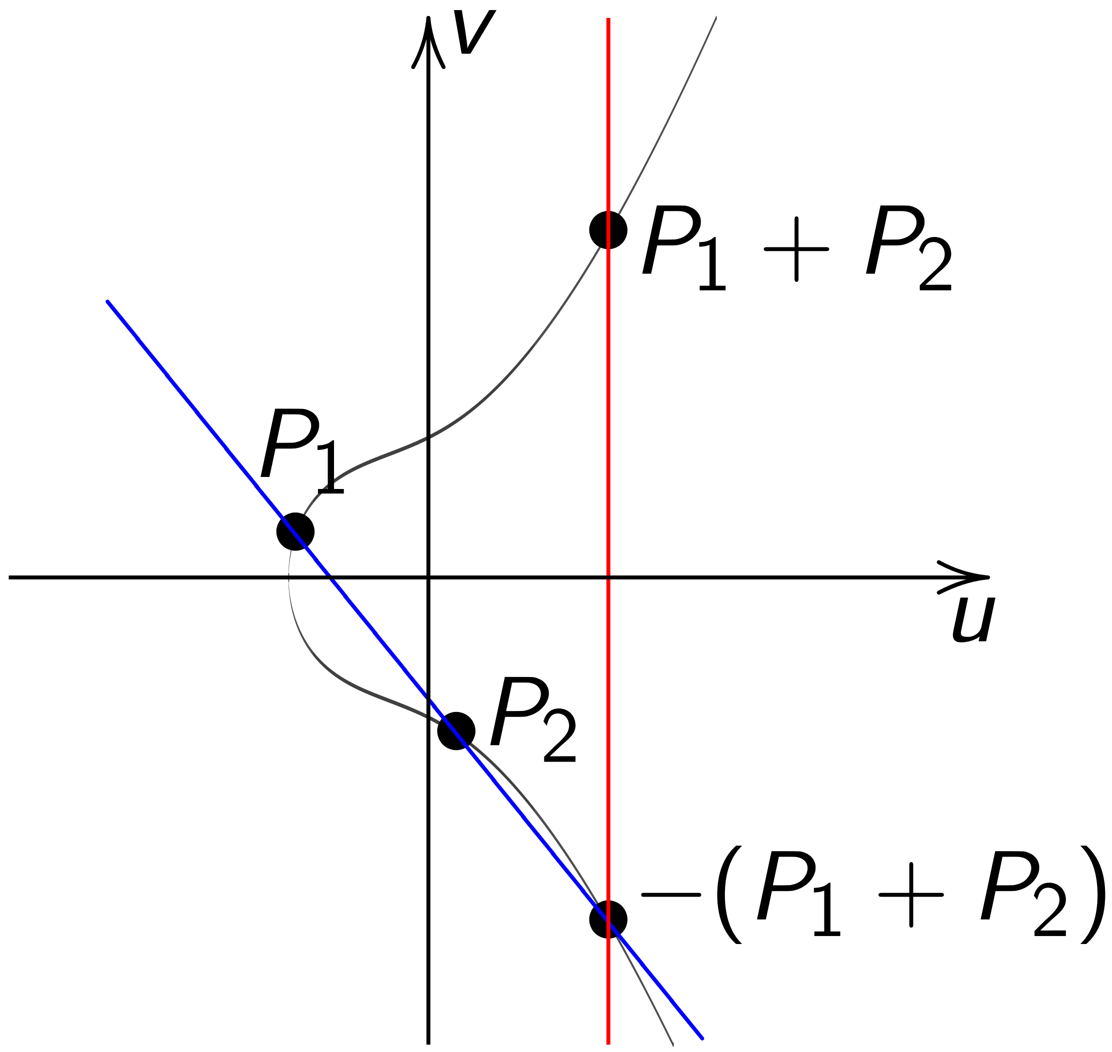

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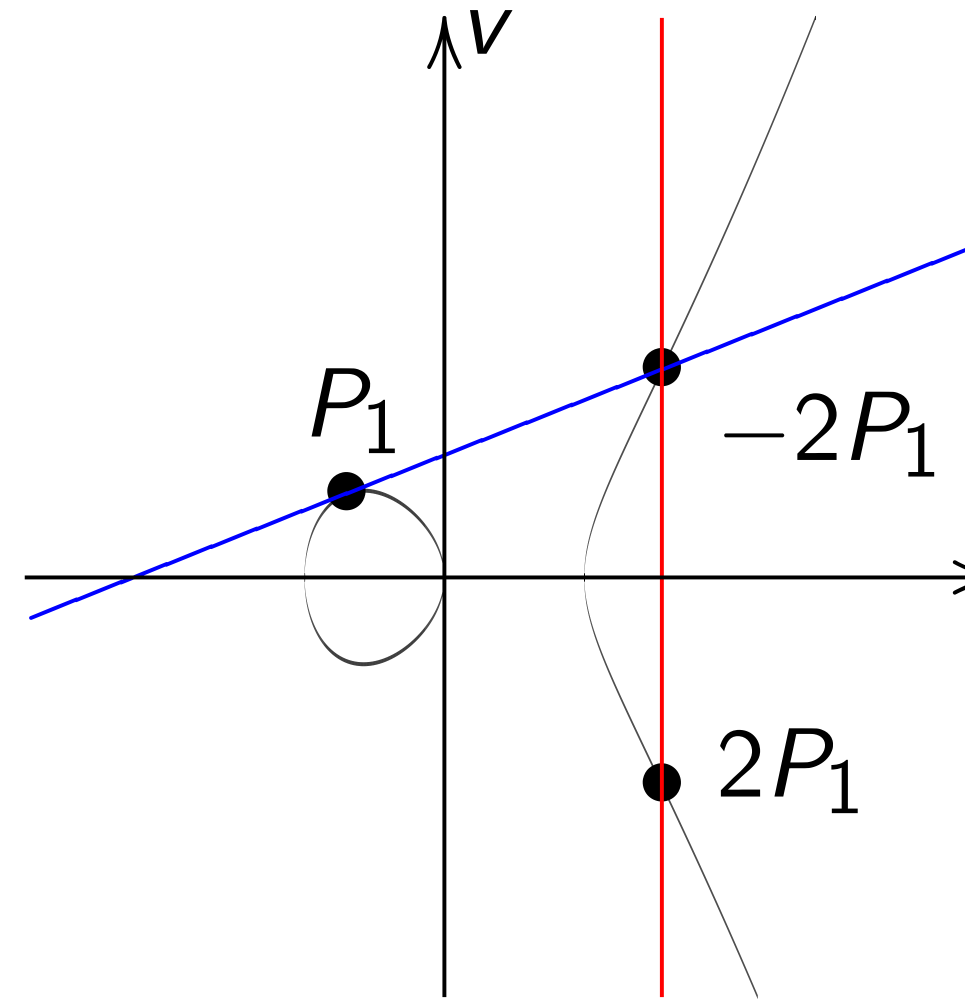

Slope $\lambda = (3u_1^2 - 1)/(2v_1)$.

Addition on Weierstrass curve

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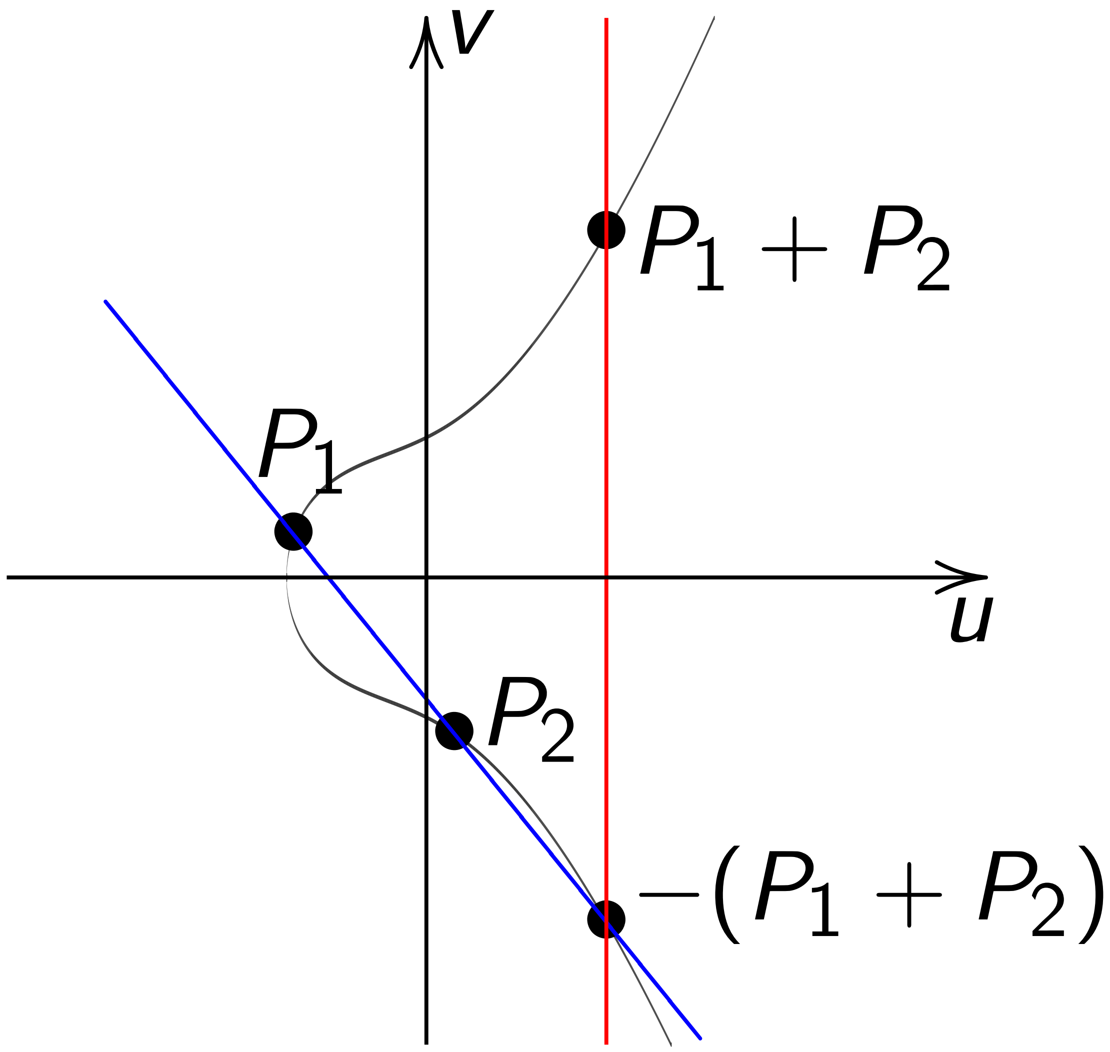

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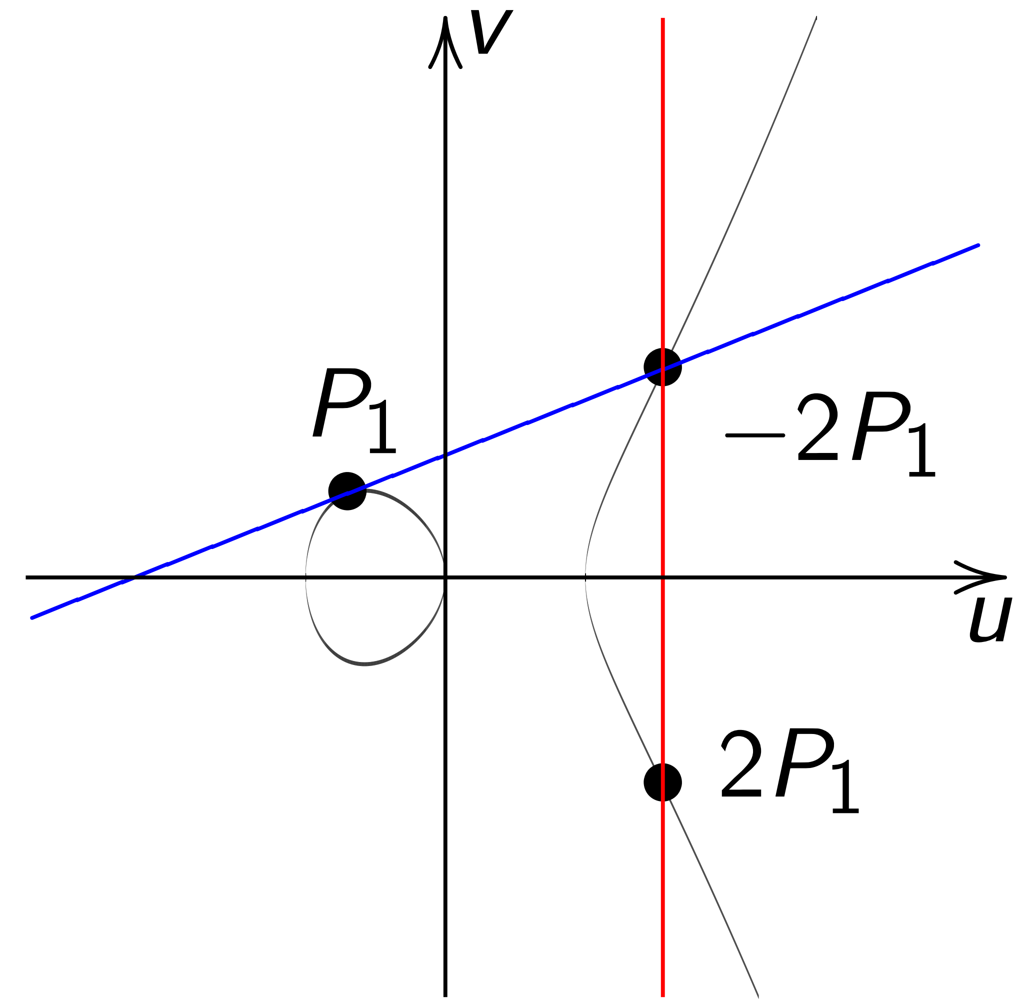

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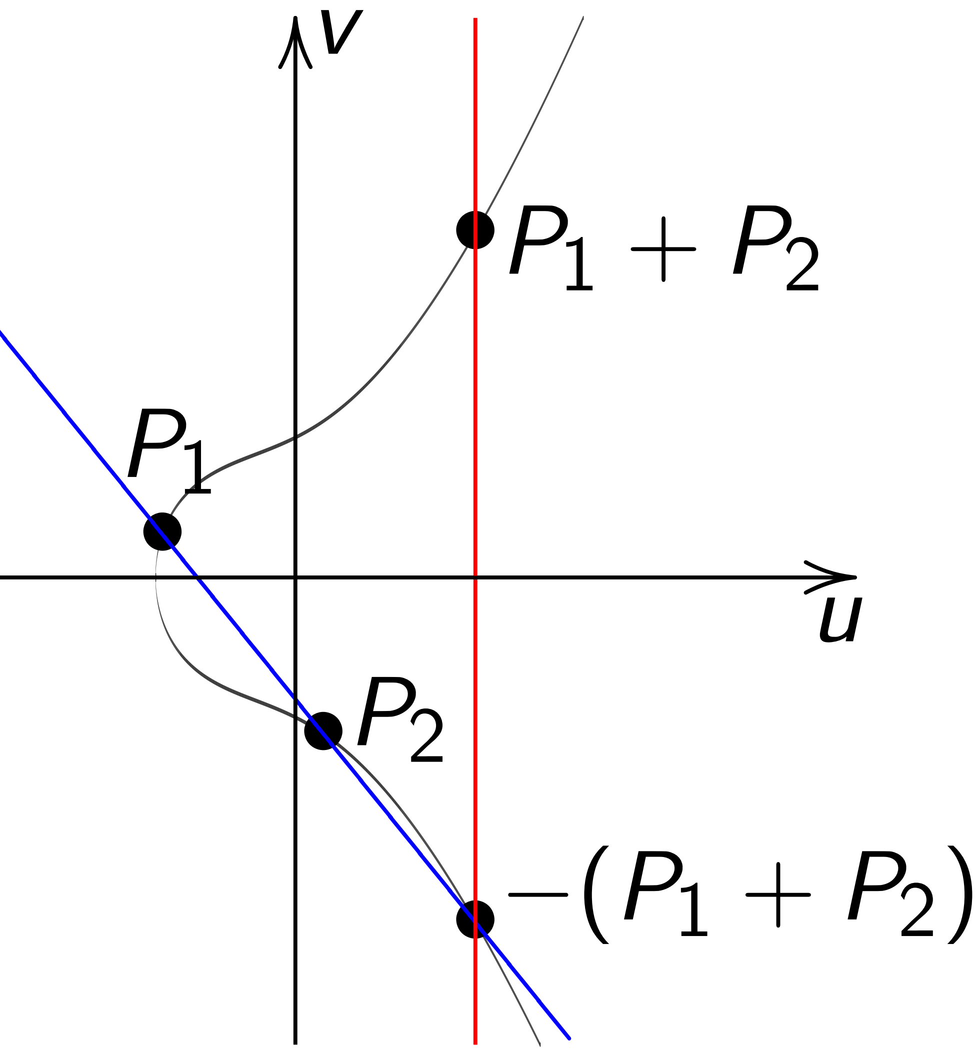

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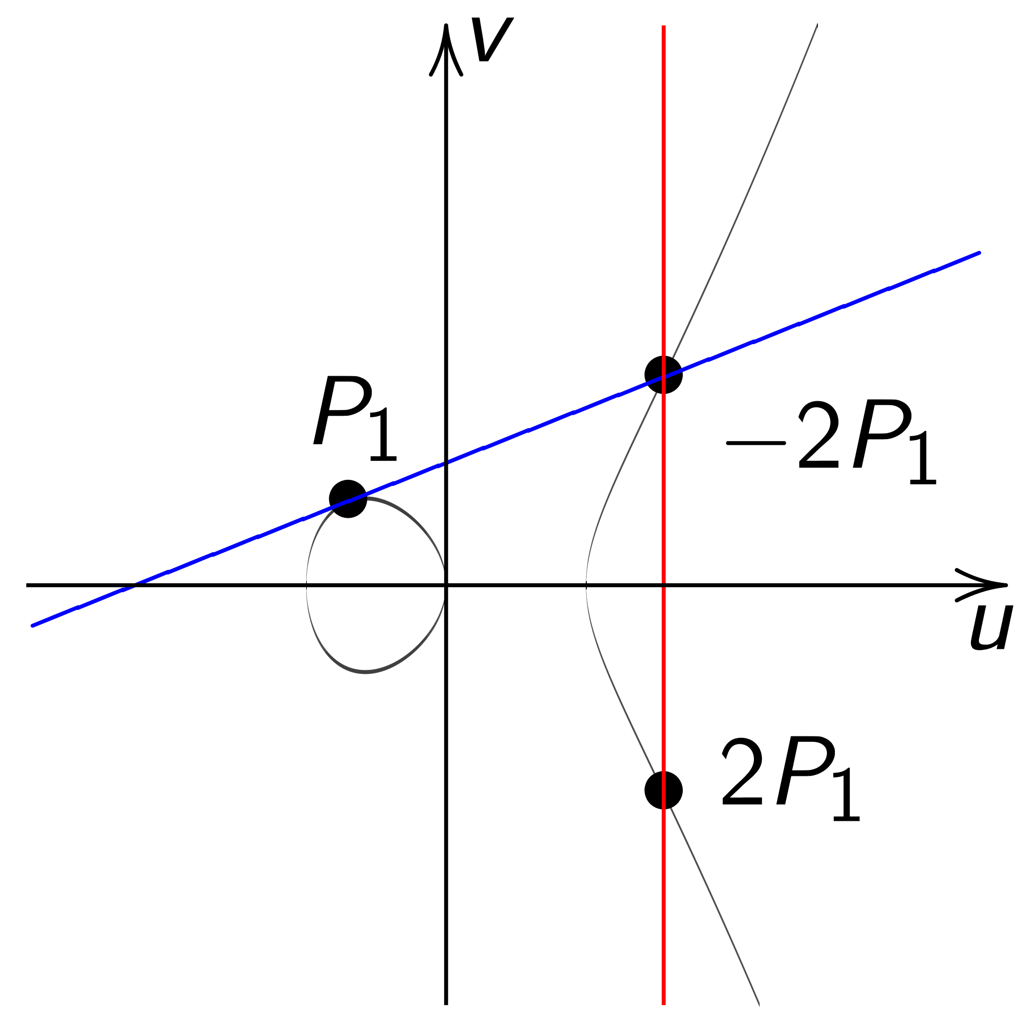

Slope $\lambda = (3u_1^2 - 1)/(2v_1)$.

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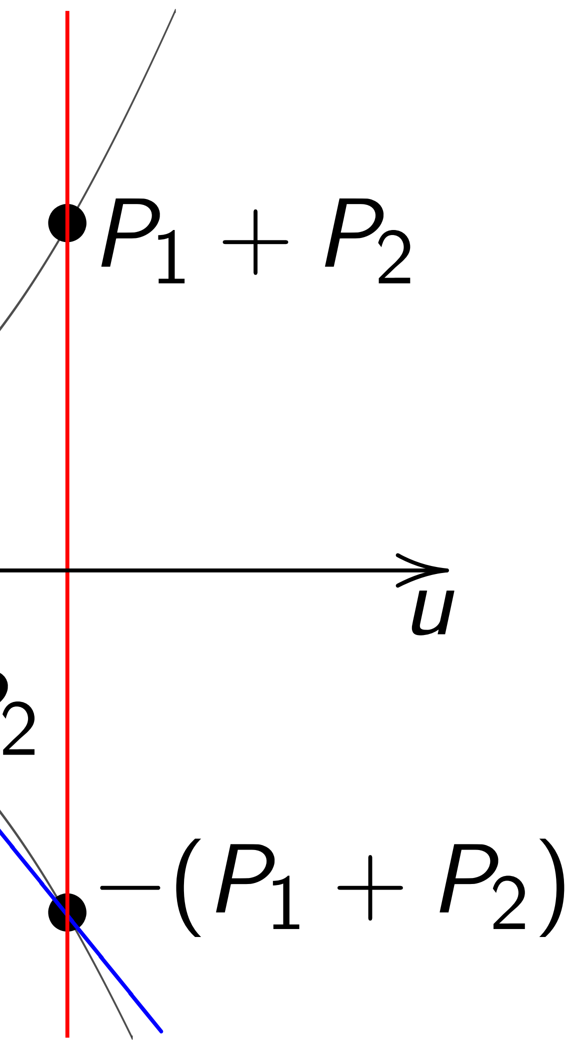

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Doubling on Weierstrass curve

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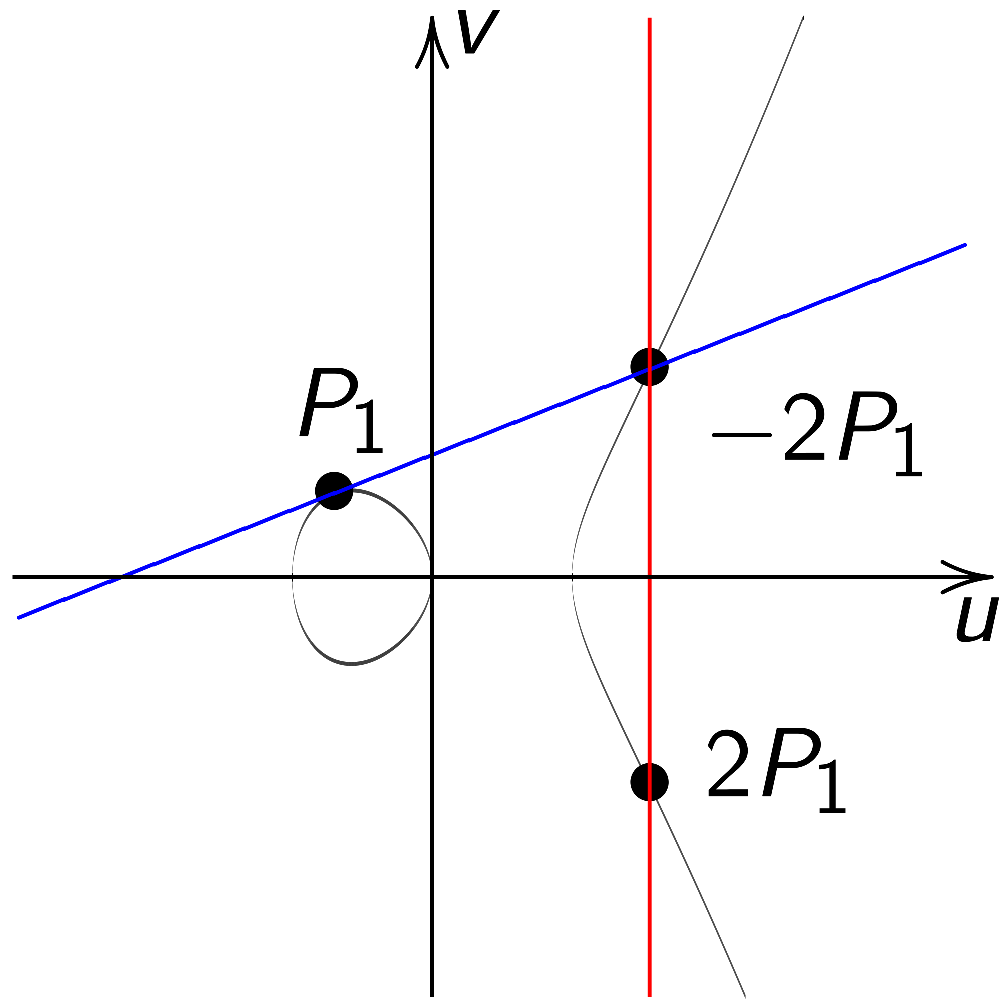

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Doubling on Weierstrass curve

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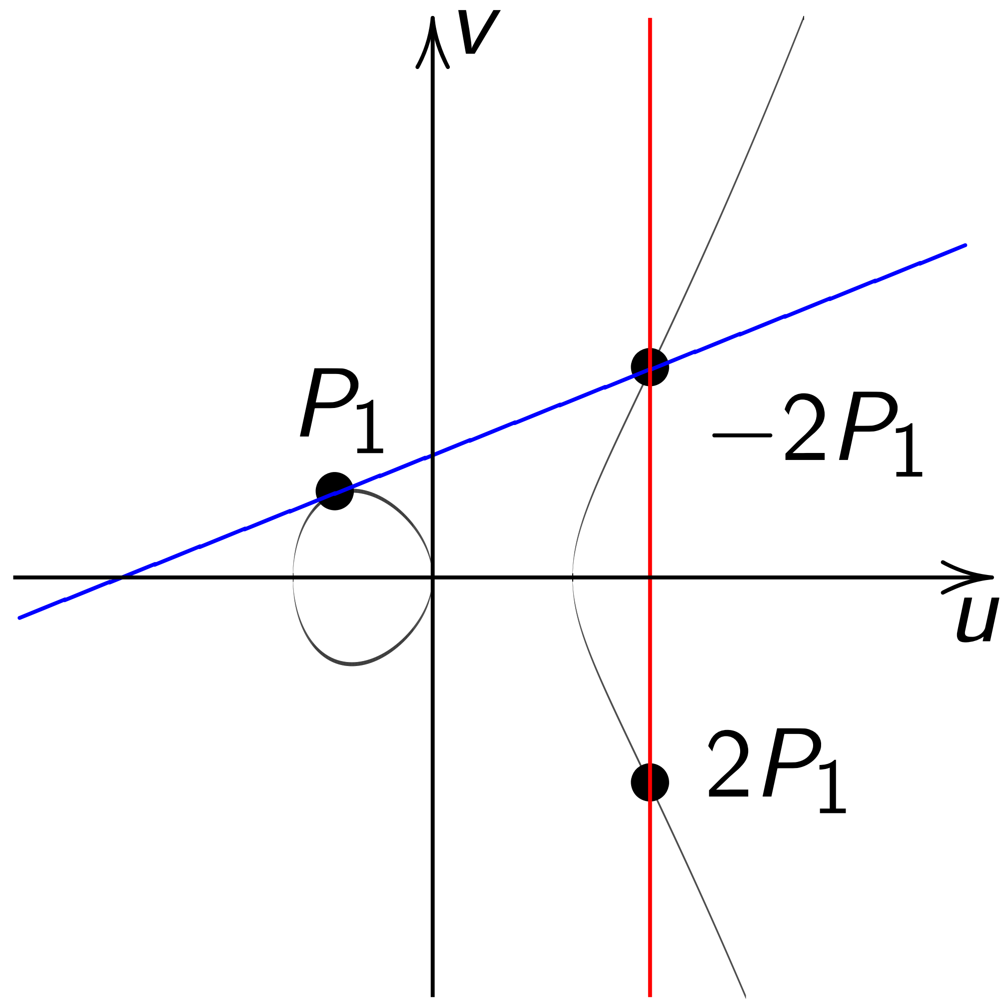

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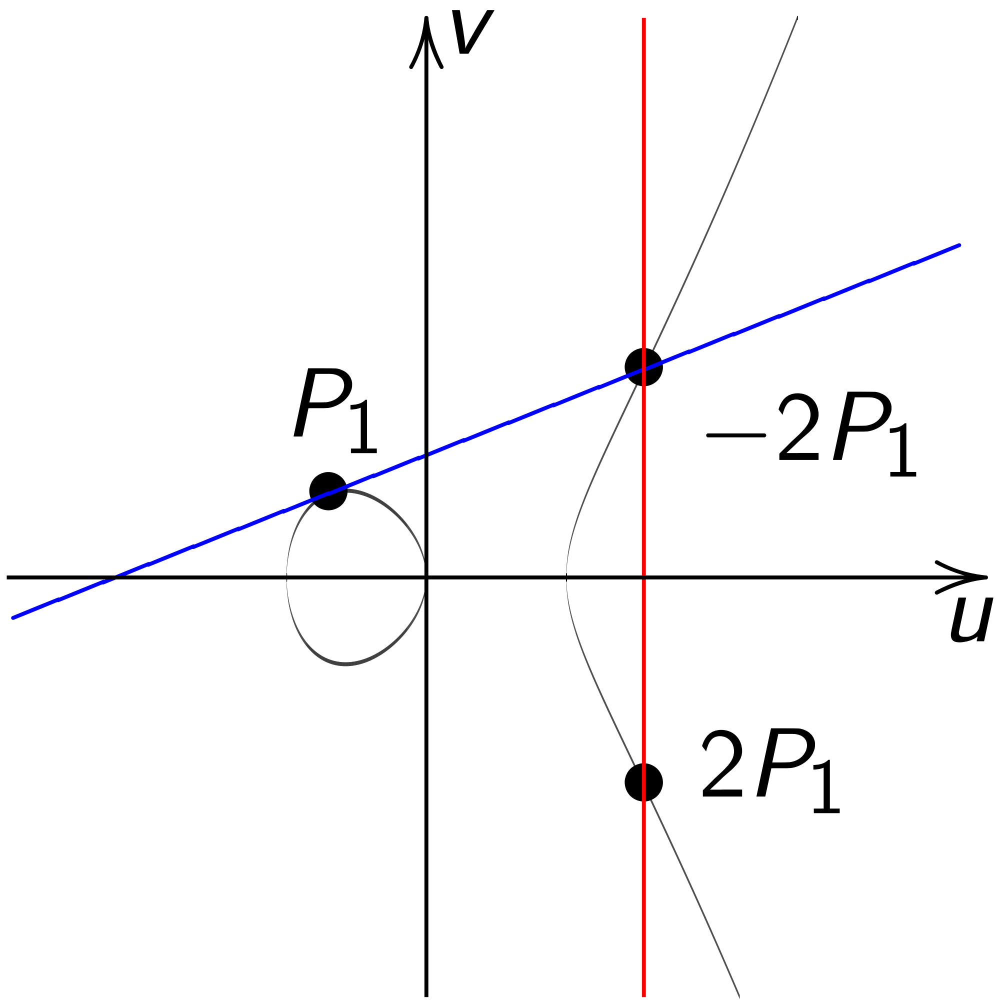

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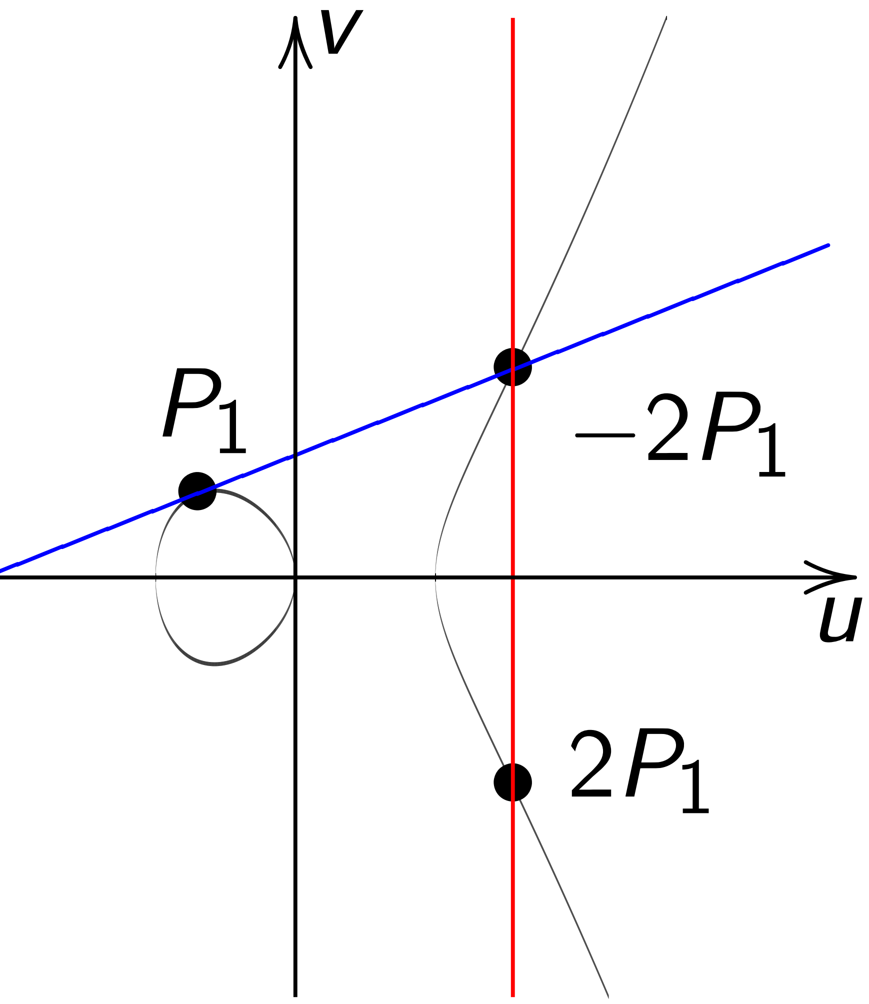

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Biration

Starting
on $x^2 +$

Define A
$B = 4/($
$u = (1 -$
$v = u/x$
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$v^2 = u^3$

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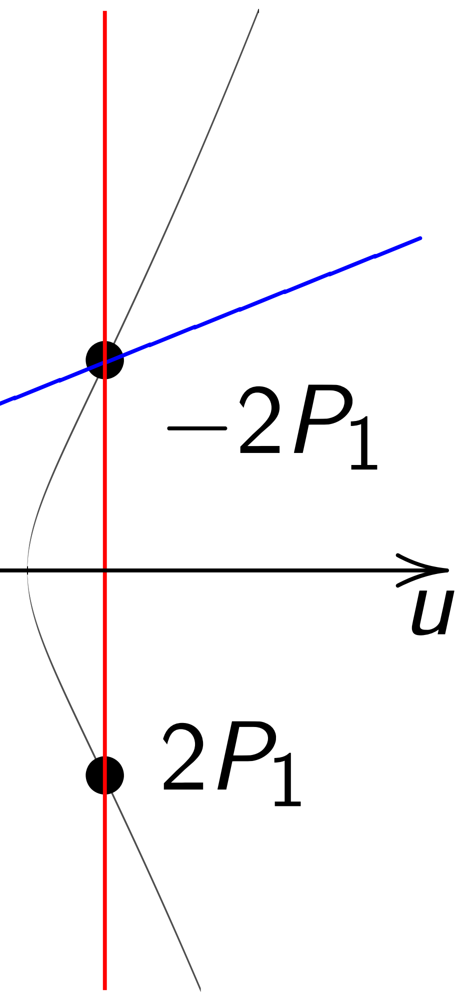

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Messy to implement and test.

Birational equivale

Starting from poin
on $x^2 + y^2 = 1 +$

Define $A = 2(1 +$
$B = 4/(1 - d)$;
$u = (1 + y)/(B(1$
$v = u/x = (1 + y$
(Skip a few except

Then (u, v) is a p
a Weierstrass curv
$v^2 = u^3 + (A/B)u$

Easily invert this r
$x = u/v$, $y = (Bu$

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In most cases

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Total cost $1\mathbf{I} + 2\mathbf{M} + 2\mathbf{S}$.

Also handle some exceptions:
$(u_1, v_1) = (u_2, -v_2)$; ∞ as input.
Messy to implement and test.

Birational equivalence

Starting from point (x, y)
on $x^2 + y^2 = 1 + dx^2y^2$:

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Easily invert this map:
$x = u/v$, $y = (Bu - 1)/(B$

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(u_3, v_3) where $(u_3, v_3) =$
$(\lambda^2 - u_1 - u_2, \lambda(u_1 - u_3) - v_1)$.

$u_1 \neq u_2$, "addition" (alert!):
$\lambda = (v_2 - v_1)/(u_2 - u_1)$.
Total cost $1\mathbf{I} + 2\mathbf{M} + 1\mathbf{S}$.

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Attacker can transform Edwards curve to Weierstrass curve and vice versa; $n(x, y) \mapsto n(u, v)$. $\Rightarrow$ Same discrete-log security! Can choose curve representation so that implementation of attack is faster/easier.

System designer can choose curve representation so that protocol runs fastest; no need to worry about security degradation.

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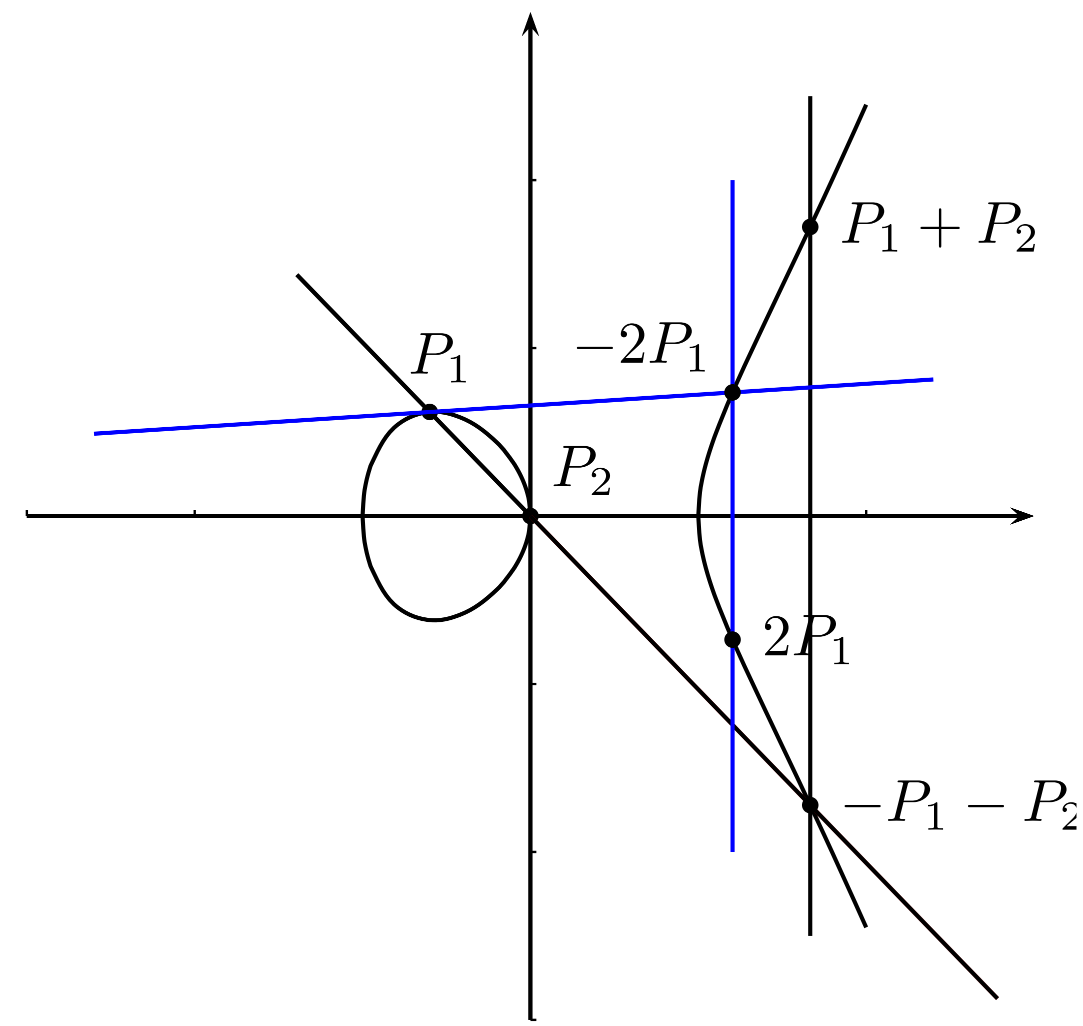
Elliptic-curve groups



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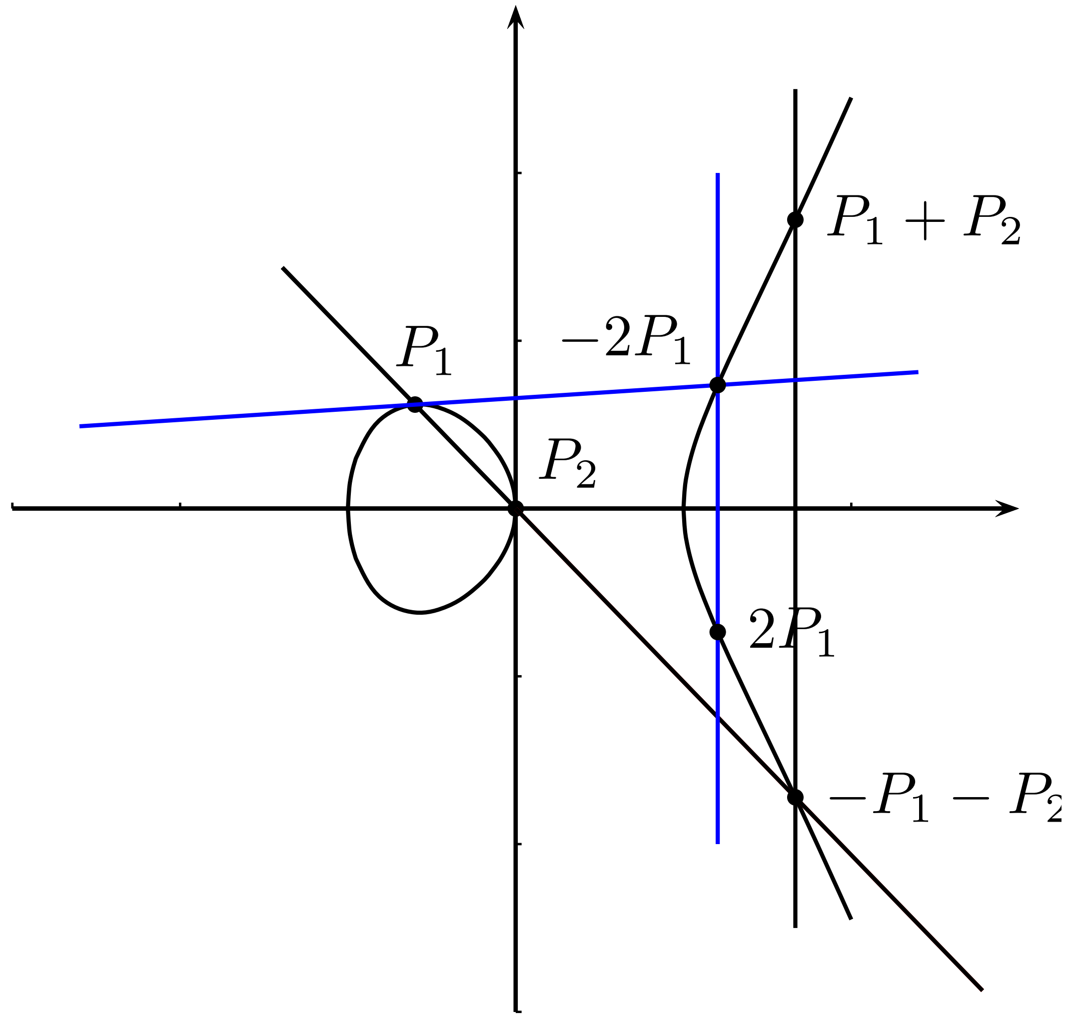

Following algorithms will need a unique representative per point. For that Weierstrass curves are the speed leader.

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$y^2 = 1 + dx^2y^2$:

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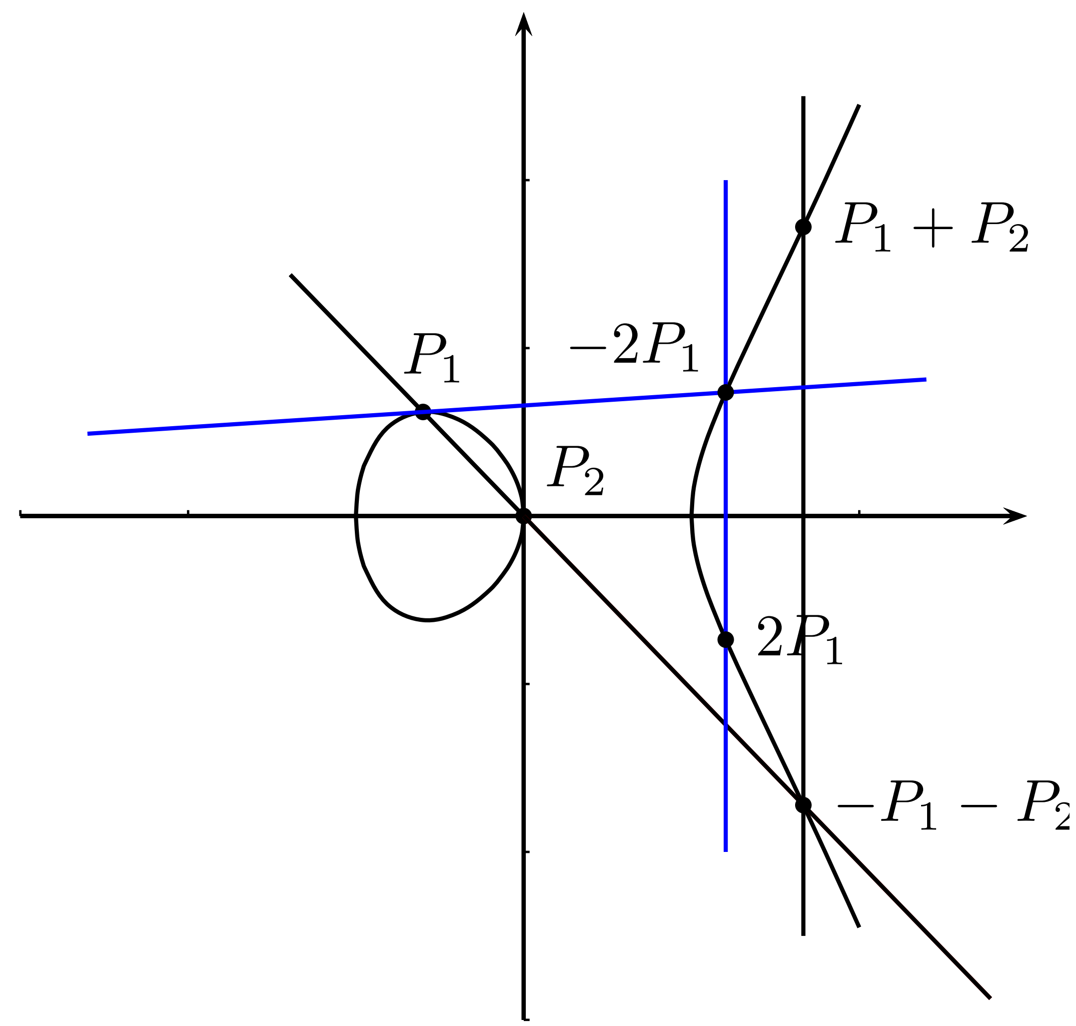
Elliptic-curve groups



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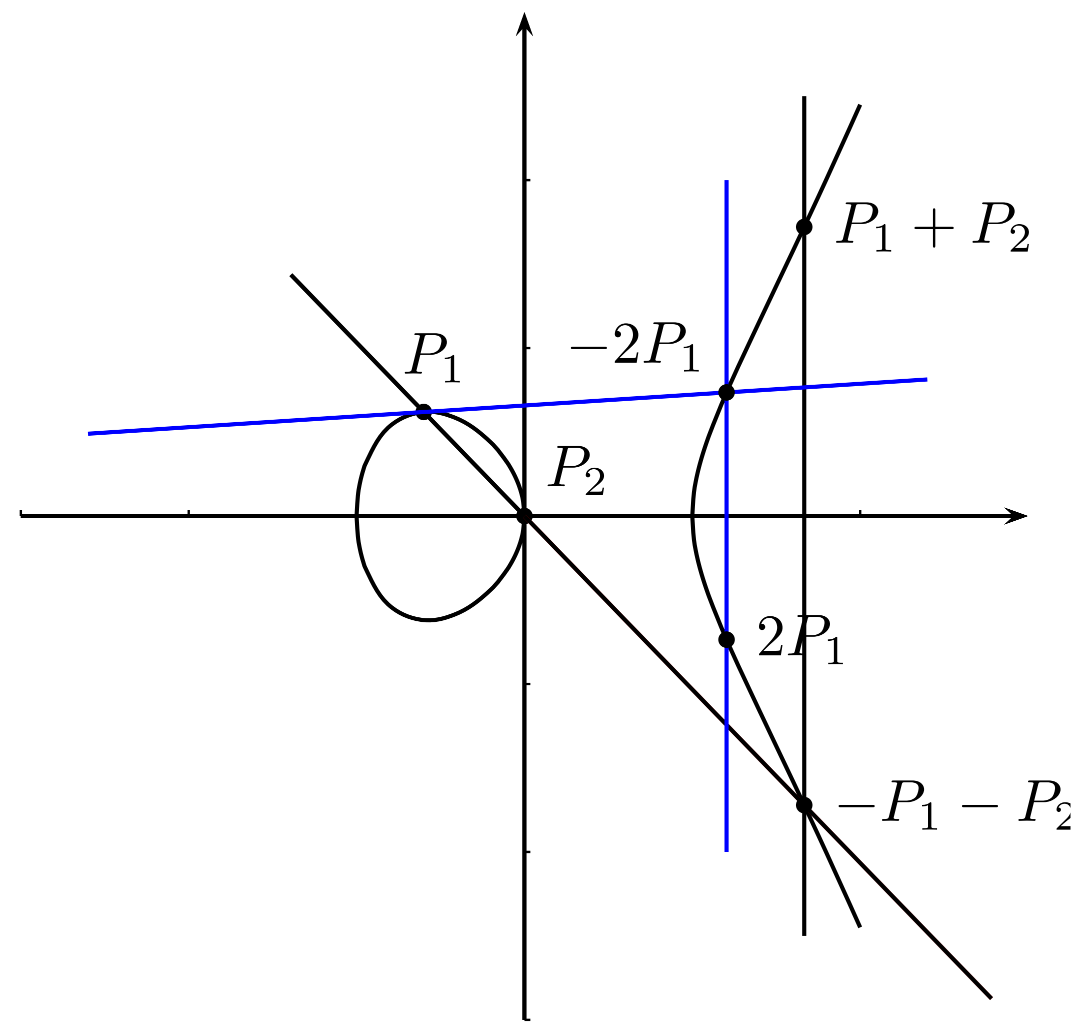

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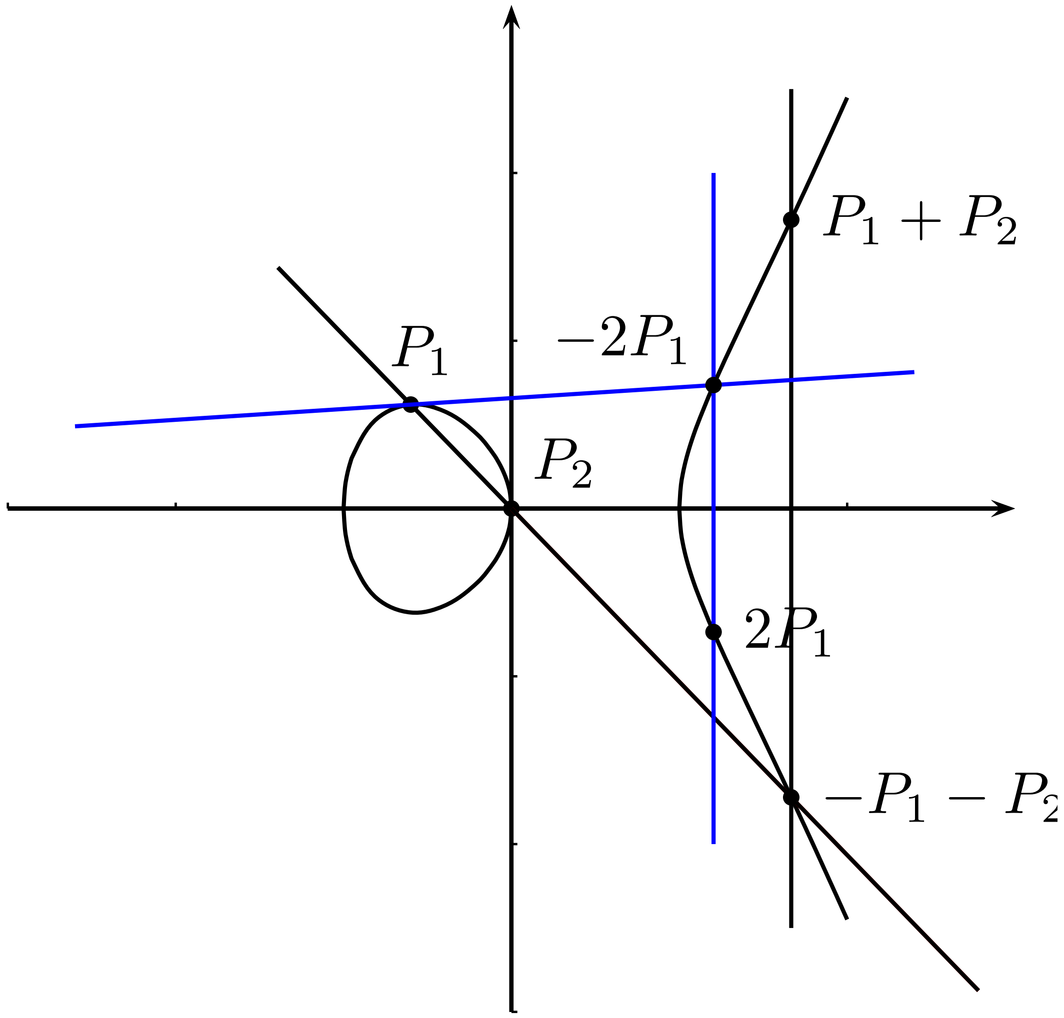

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The discrete-logarithm probl

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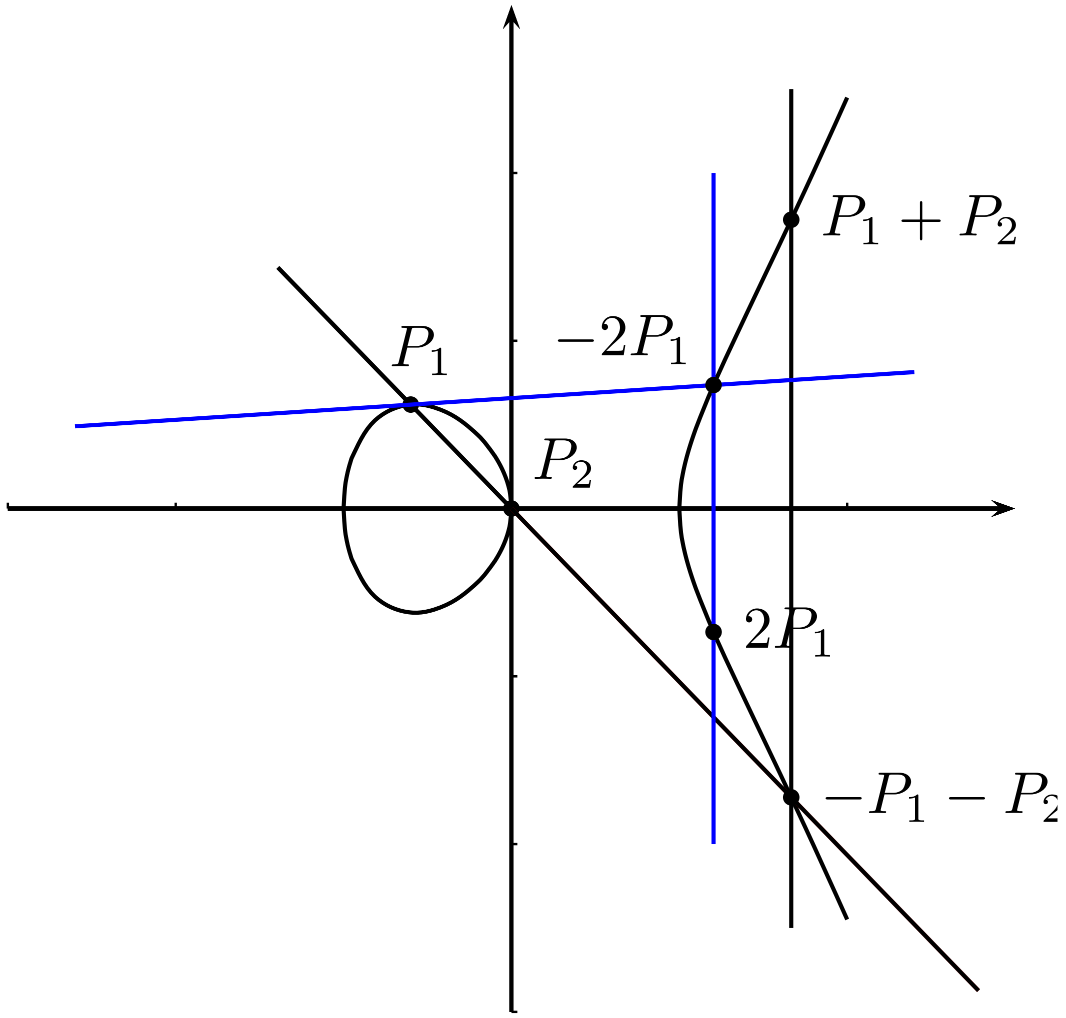

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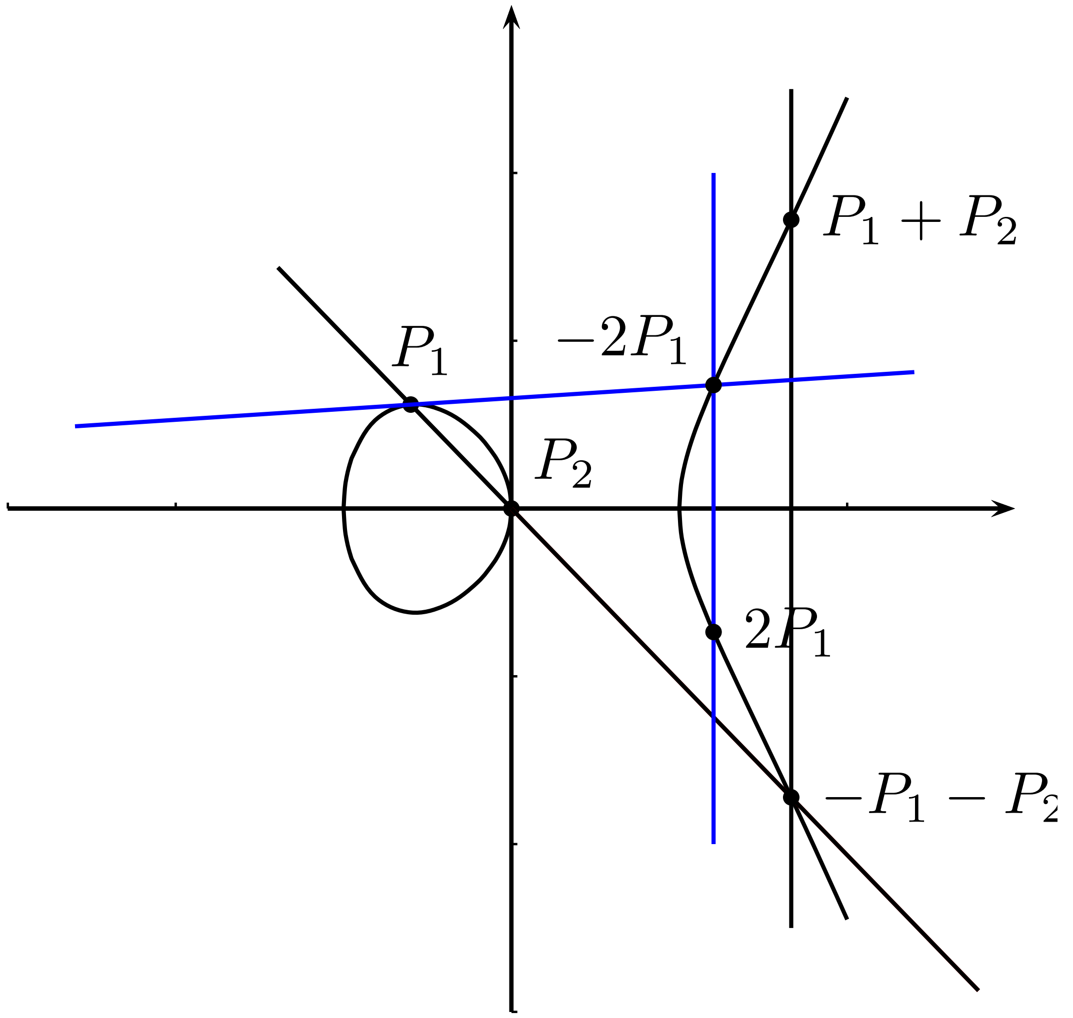

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points and $P = (101384, 614510)$ is a point of order $2 \cdot 53^2 \cdot 89$.

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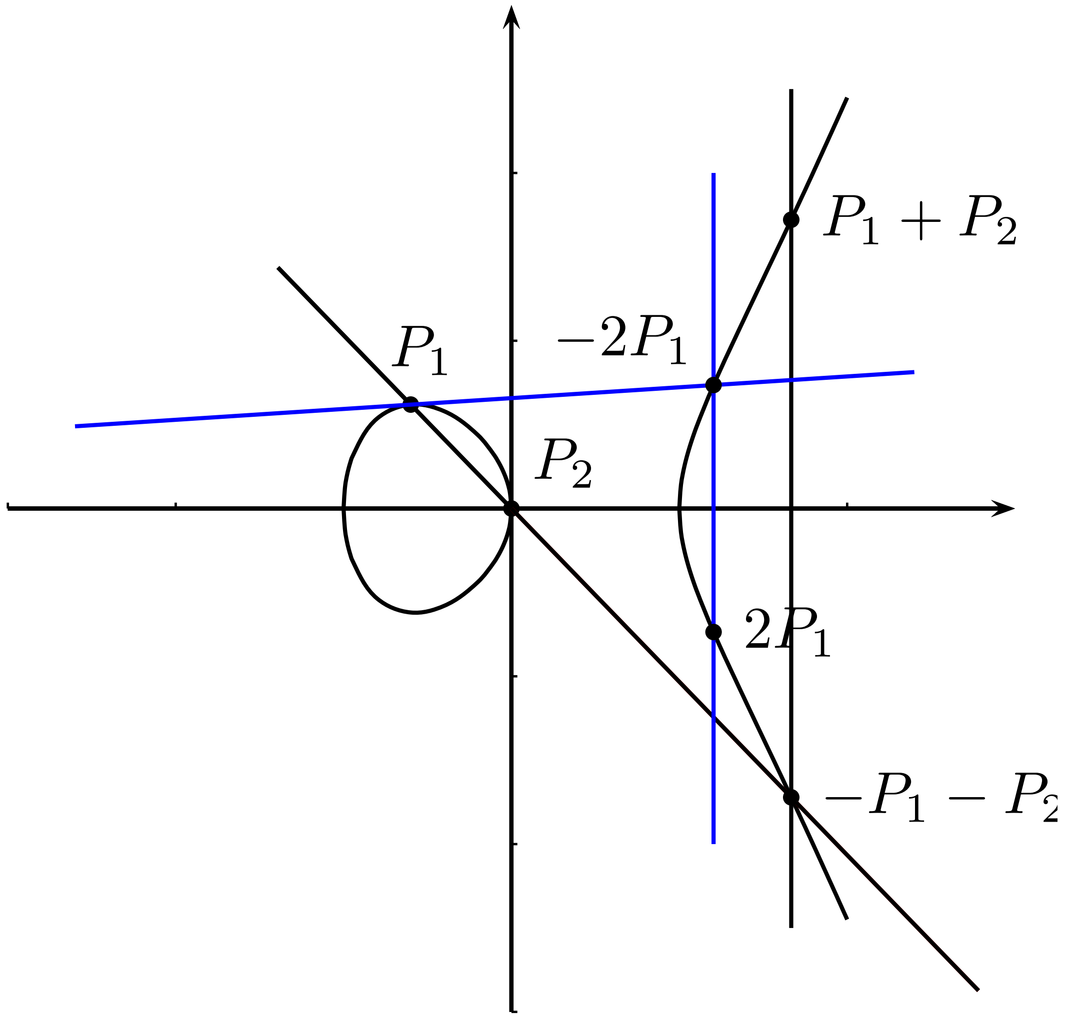

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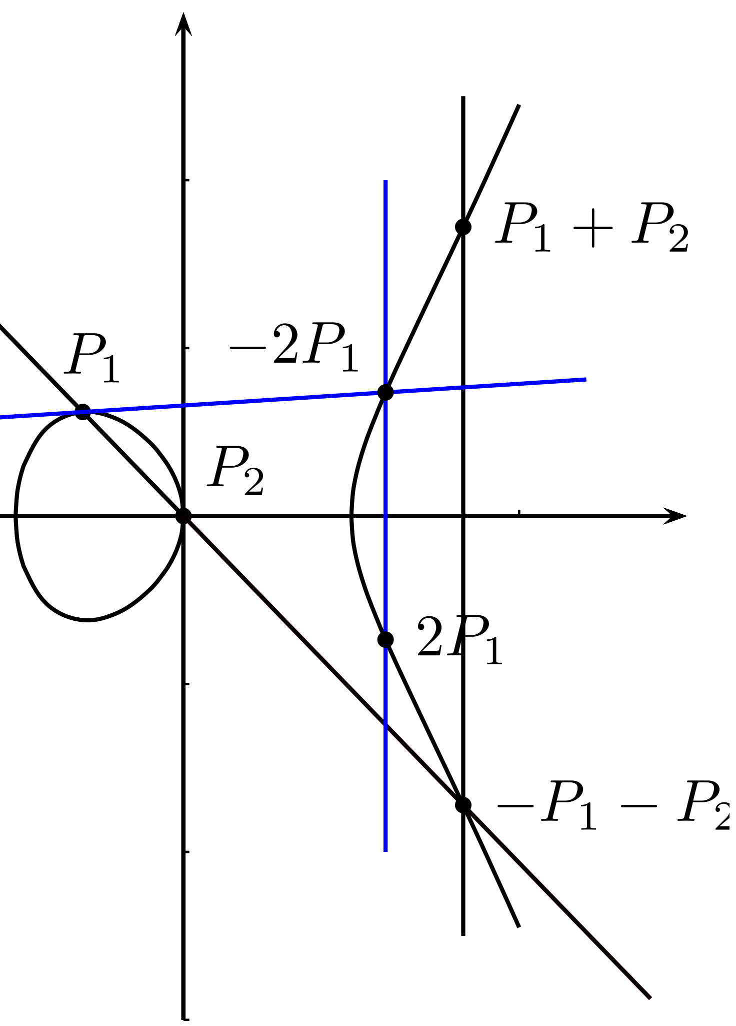
In general, point counting over $\mathbf{F}_p$ runs in time polynomial in $\log p$.

Number of points in $[p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$.

The group is isomorphic to $\mathbf{Z}/n \times \mathbf{Z}/m$, where $n \mid m$ and $n \mid (p - 1)$.

curve groups



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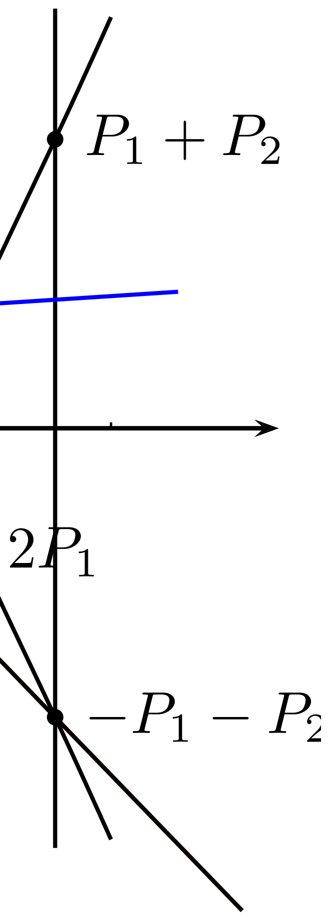

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Can we find an integer $n \in \{1, 2, 3, \ldots, 500001\}$ such that $nP = (670366, 740819)$?

This point was generated as a multiple of P; could also b outside cyclic group.

Could find n by brute force. Is there a faster way?

The discrete-logarithm problem

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Can we find an integer $n \in \{1, 2, 3, \ldots, 500001\}$ such that $nP = (670366, 740819)$?

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Is there a faster way?

Understanding brute force

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$1P = (101384, 614510)$,
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$3P = (77571, 87643)$,
$4P = (650289, 31313)$,
$500001P = -P$.
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Maximum cost of computation: ≤ 500001 additions of P; ≤ 500001 nanoseconds on a CPU that does 1 ADD/nanosecond.

This is negligible work for $p \approx 2^{20}$.

But users can standardize a larger p, making the attack slower.

Attack cost scales linearly:

$\approx 2^{50}$ ADDs for $p \approx 2^{50}$,

$\approx 2^{100}$ ADDs for $p \approx 2^{100}$,

(Not exactly linearly: cost of ADDs grows with p. But this is a minor effect.)

Understanding brute force

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But users can standardize a larger p, making the attack slower.

Attack cost scales linearly: $\approx 2^{50}$ ADDs for $p \approx 2^{50}$, $\approx 2^{100}$ ADDs for $p \approx 2^{100}$, etc.

(Not exactly linearly: cost of ADDs grows with p. But this is a minor effect.)

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choose random r, say 69961;
compute $rP = (593450, 987590)$;
compute $(r + n)P$ as
$(593450, 987590) + (670366, 740819)$;
compute discrete log;
subtract r mod 500002; obtain n.

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Attacker can run many parallel chips.
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Computation can be applied to many targets at once.

Given 100 DL targets n_1P, n_2P, . . . , $n_{100}P$:
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Let ℓ be the order of P.

Given nP:
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Compute $r_1 P + nP$,
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Faster: Choose r_i
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Compute $r_1 P$;
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$2r_1 P + nP$;
$3r_1 P + nP$; etc.
Just 1 ADD for ea

$\approx 100 + \lg \ell + \ell/1$
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Can use random self-reduction to turn a single target into multiple targets.
Let ℓ be the order of P.

Given nP:
Choose random $r_1, r_2, \ldots, r_{100}$.
Compute $r_1P + nP$, $r_2P + nP$, etc.

Solve these 100 DL problems.
Typically $\approx \ell/100$ mults to find *at least one* $r_i + n \bmod \ell$, immediately revealing n.

Also spent some ADDs to compute each r_iP: $\approx \lg p$ ADDs for each i.

Faster: Choose $r_i = ir_1$ with $r_1 \approx \ell/100$.
Compute r_1P;
$r_1P + nP$;
$2r_1P + nP$;
$3r_1P + nP$; etc.
Just 1 ADD for each new i.

$\approx 100 + \lg \ell + \ell/100$ ADDs to find n given nP.

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$2r_1 P + nP$;
$3r_1 P + nP$; etc.
Just 1 ADD for each new i.

$\approx 100 + \lg \ell + \ell/100$ ADDs
to find n given nP.

Faster: Increase 10
Only $\approx 2\sqrt{\ell}$ ADD
to solve one DL pr
"Shanks baby-step
discrete-logarithm

Example: p = 10
500002, $P = (101$
$Q = nP = (67036$
Compute $708P$=(
Then compute 707
$708P + Q = (3428$
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$3r_1 P + nP$; etc.
Just 1 ADD for each new i.

$\approx 100 + \lg \ell + \ell/100$ ADDs
to find n given nP.

Faster: Increase 100 to $\approx \sqrt{}$
Only $\approx 2\sqrt{\ell}$ ADDs
to solve one DL problem!
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discrete-logarithm algorithm

Example: $p = 1000003, \ell$
500002, $P = (101384, 61451$
$Q = nP = (670366, 740819)$
Compute $708P$=(393230, 42
Then compute 707 targets:
$708P + Q = (342867, 15381$
$2 \cdot 708P + nP = (430321, 99$
$3 \cdot 708P + nP = (423151, 63$
$\ldots, 706 \cdot 708P + nP =$
(534170, 450849).

Also spent some ADDs to compute each $r_i P$: $\approx \lg p$ ADDs for each i.

Faster: Choose $r_i = i r_1$ with $r_1 \approx \ell/100$.
Compute $r_1 P$;
$r_1 P + nP$;
$2r_1 P + nP$;
$3r_1 P + nP$; etc.
Just 1 ADD for each new i.

$\approx 100 + \lg \ell + \ell/100$ ADDs to find n given nP.

Faster: Increase 100 to $\approx \sqrt{\ell}$.
Only $\approx 2\sqrt{\ell}$ ADDs to solve one DL problem!
"Shanks baby-step-giant-step discrete-logarithm algorithm."

Example: $p = 1000003, \ell = 500002$, $P = (101384, 614510)$, $Q = nP = (670366, 740819)$.
Compute $708P$=$(393230, 421116)$.
Then compute 707 targets:
$708P + Q = (342867, 153817)$,
$2 \cdot 708P + nP = (430321, 994742)$,
$3 \cdot 708P + nP = (423151, 635197)$,
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$\ldots$, $706 \cdot 708P + nP = (534170, 450849)$.

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Compute $708P = (393230, 421116)$.
Then compute 707 targets:
$708P + Q = (342867, 153817)$,
$2 \cdot 708P + nP = (430321, 994742)$,
$3 \cdot 708P + nP = (423151, 635197)$,
$\ldots$, $706 \cdot 708P + nP =$
$(534170, 450849)$.

Build a sorted tab
$600 \cdot 708P + Q = ($
$27 \cdot 708P + Q = ($
$219 \cdot 708P + Q = ($
...
$242 \cdot 708P + Q = ($
...
$317 \cdot 708P + Q = ($

Look up P, $2P$, 3

$620P = (950652,$
$596 \cdot 708P + Q = ($
in the table of tar
so $620 = 596 \cdot 708$
deduce $n = 7865$

Faster: Increase 100 to $\approx \sqrt{\ell}$.
Only $\approx 2\sqrt{\ell}$ ADDs
to solve one DL problem!
"Shanks baby-step-giant-step
discrete-logarithm algorithm."

Example: $p = 1000003, \ell = 500002$, $P = (101384, 614510)$, $Q = nP = (670366, 740819)$.
Compute $708P{=}(393230, 421116)$.
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Build a sorted table of targe
$600{\cdot}708P{+}Q = (799978, 92$
$27 \cdot 708P + Q = (785344, 83$
$219{\cdot}708P{+}Q = (425475, 79$
...
$242{\cdot}708P{+}Q = (262804, 34$
...
$317{\cdot}708P{+}Q = (599785, 18$

Look up P, $2P$, $3P$, etc. in

$620P = (950652, 688508)$; f
$596 \cdot 708P + Q = (950652, 68$
in the table of targets;
so $620 = 596{\cdot}708{+}n \bmod 5$
deduce $n = 78654$.

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Build a sorted table of targets:
$600 \cdot 708P + Q = (799978, 929249)$,
$27 \cdot 708P + Q = (785344, 831127)$,
$219 \cdot 708P + Q = (425475, 793466)$,
...
$242 \cdot 708P + Q = (262804, 347755)$,
...
$317 \cdot 708P + Q = (599785, 189116)$.

Look up P, $2P$, $3P$, etc. in table.

$620P = (950652, 688508)$; find
$596 \cdot 708P + Q = (950652, 688508)$
in the table of targets;
so $620 = 596 \cdot 708 + n \bmod 500002$;
deduce $n = 78654$.

Increase 100 to $\approx \sqrt{\ell}$.
$2\sqrt{\ell}$ ADDs
one DL problem!
baby-step-giant-step
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$P = (101384, 614510)$,
$= (670366, 740819)$.
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mpute 707 targets:
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$620P = (950652, 688508)$; find
$596 \cdot 708P + Q = (950652, 688508)$
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so $620 = 596 \cdot 708 + n \bmod 500002$;
deduce $n = 78654$.

Factors

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Given Q

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393230, 421116).
7 targets:
867, 153817),
430321, 994742),
423151, 635197),
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Build a sorted table of targets:
$600 \cdot 708P + Q = (799978, 929249)$,
$27 \cdot 708P + Q = (785344, 831127)$,
$219 \cdot 708P + Q = (425475, 793466)$,
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$620P = (950652, 688508)$; find
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Factors of the gro

P has order $2 \cdot 53^2$

Given $Q = nP$, fin

$R = (53^2 \cdot 89)P$ h
$S = (53^2 \cdot 89)Q$ is
Compute $n_1 = \log$

$R = (2 \cdot 53 \cdot 89)P$
and
$S = (2 \cdot 53 \cdot 89)Q$
Compute
$n_2 = \log_R S \equiv n$ m
This is a DLP in a
of size 53.

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Build a sorted table of targets:

$600 \cdot 708P + Q = (799978, 929249)$,

$27 \cdot 708P + Q = (785344, 831127)$,

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Look up P, $2P$, $3P$, etc. in table.

$620P = (950652, 688508)$; find $596 \cdot 708P + Q = (950652, 688508)$ in the table of targets; so $620 = 596 \cdot 708 + n \bmod 500002$; deduce $n = 78654$.

Factors of the group order

P has order $2 \cdot 53^2 \cdot 89$.

Given $Q = nP$, find $n = \log$

$R = (53^2 \cdot 89)P$ has order 2

$S = (53^2 \cdot 89)Q$ is multiple

Compute $n_1 = \log_R S \equiv n$

$R = (2 \cdot 53 \cdot 89)P$ has orde

and

$S = (2 \cdot 53 \cdot 89)Q$ is multipl

Compute

$n_2 = \log_R S \equiv n \bmod 53$.

This is a DLP in a group of size 53.

Build a sorted table of targets:

$600 \cdot 708P + Q = (799978, 929249)$,

$27 \cdot 708P + Q = (785344, 831127)$,

$219 \cdot 708P + Q = (425475, 793466)$,

...

$242 \cdot 708P + Q = (262804, 347755)$,

...

$317 \cdot 708P + Q = (599785, 189116)$.

Look up P, $2P$, $3P$, etc. in table.

$620P = (950652, 688508)$; find $596 \cdot 708P + Q = (950652, 688508)$ in the table of targets; so $620 = 596 \cdot 708 + n \bmod 500002$; deduce $n = 78654$.

<u>Factors of the group order</u>

P has order $2 \cdot 53^2 \cdot 89$.

Given $Q = nP$, find $n = \log_P Q$:

$R = (53^2 \cdot 89)P$ has order 2, and $S = (53^2 \cdot 89)Q$ is multiple of R. Compute $n_1 = \log_R S \equiv n \bmod 2$.

$R = (2 \cdot 53 \cdot 89)P$ has order 53, and $S = (2 \cdot 53 \cdot 89)Q$ is multiple of R. Compute $n_2 = \log_R S \equiv n \bmod 53$. This is a DLP in a group of size 53.

sorted table of targets:

$P+Q = (799978, 929249)$,

$P+Q = (785344, 831127)$,

$P+Q = (425475, 793466)$,

$P+Q = (262804, 347755)$,

$P+Q = (599785, 189116)$.

P, $2P$, $3P$, etc. in table.

$(950652, 688508)$; find

$P+Q = (950652, 688508)$

able of targets;

$= 596{\cdot}708+n$ mod 500002;

$n = 78654$.

Factors of the group order

P has order $2 \cdot 53^2 \cdot 89$.

Given $Q = nP$, find $n = \log_P Q$:

$R = (53^2 \cdot 89)P$ has order 2, and $S = (53^2 \cdot 89)Q$ is multiple of R.
Compute $n_1 = \log_R S \equiv n \bmod 2$.

$R = (2 \cdot 53 \cdot 89)P$ has order 53, and
$S = (2 \cdot 53 \cdot 89)Q$ is multiple of R.
Compute
$n_2 = \log_R S \equiv n \bmod 53$.
This is a DLP in a group of size 53.

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$n_3 = \log$

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$R = (2$

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$n \equiv n_1$

$n \equiv n_2$

$n \equiv n_4$

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425475, 793466),

262804, 347755),

599785, 189116).

P, etc. in table.

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Factors of the group order

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$n_2 = \log_R S \equiv n \bmod 53$.
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of size 53.

$T = (2 \cdot 89)(Q -$
a multiple of R, i.
Compute
$n_3 = \log_R T \equiv n$
Now $n_2 + 53n_3 \equiv$

$R = (2 \cdot 53^2)P$ ha
$S = (2 \cdot 53^2)Q$ is
Compute
$n_4 = \log_R S \equiv n$

Use Chinese Rema
$n \equiv n_1 \bmod 2$,
$n \equiv n_2 + 53n_3$ m
$n \equiv n_4 \bmod 89$,
to determine n m

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<u>Factors of the group order</u>

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$n_2 = \log_R S \equiv n \bmod 53$.
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of size 53.

$T = (2 \cdot 89)(Q - n_2 P)$ is als
a multiple of R, i.e., has ord
Compute
$n_3 = \log_R T \equiv n \bmod 53$.
Now $n_2 + 53n_3 \equiv n \bmod 53$

$R = (2 \cdot 53^2)P$ has order 89
$S = (2 \cdot 53^2)Q$ is multiple o
Compute
$n_4 = \log_R S \equiv n \bmod 89$.

Use Chinese Remainder The
$n \equiv n_1 \bmod 2$,
$n \equiv n_2 + 53n_3 \bmod 53^2$,
$n \equiv n_4 \bmod 89$,
to determine n modulo $2 \cdot 5$

Factors of the group order

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$S = (2 \cdot 53 \cdot 89)Q$ is multiple of R.
Compute
$n_2 = \log_R S \equiv n \bmod 53$.
This is a DLP in a group
of size 53.

$T = (2 \cdot 89)(Q - n_2 P)$ is also
a multiple of R, i.e., has order 53.
Compute
$n_3 = \log_R T \equiv n \bmod 53$.
Now $n_2 + 53n_3 \equiv n \bmod 53^2$.

$R = (2 \cdot 53^2)P$ has order 89, and
$S = (2 \cdot 53^2)Q$ is multiple of R.
Compute
$n_4 = \log_R S \equiv n \bmod 89$.

Use Chinese Remainder Theorem
$n \equiv n_1 \bmod 2$,
$n \equiv n_2 + 53n_3 \bmod 53^2$,
$n \equiv n_4 \bmod 89$,
to determine n modulo $2 \cdot 53^2 \cdot 89$.

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rder $2 \cdot 53^2 \cdot 89$.

$= nP$, find $n = \log_P Q$:

$^2 \cdot 89)P$ has order 2, and

$^2 \cdot 89)Q$ is multiple of R.

e $n_1 = \log_R S \equiv n \bmod 2$.

$\cdot 53 \cdot 89)P$ has order 53,

$53 \cdot 89)Q$ is multiple of R.

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$g_R S \equiv n \bmod 53$.

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$T = (2 \cdot 89)(Q - n_2 P)$ is also a multiple of R, i.e., has order 53.

Compute

$n_3 = \log_R T \equiv n \bmod 53$.

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$R = (2 \cdot 53^2)P$ has order 89, and $S = (2 \cdot 53^2)Q$ is multiple of R.

Compute

$n_4 = \log_R S \equiv n \bmod 89$.

Use Chinese Remainder Theorem

$n \equiv n_1 \bmod 2$,

$n \equiv n_2 + 53n_3 \bmod 53^2$,

$n \equiv n_4 \bmod 89$,

to determine n modulo $2 \cdot 53^2 \cdot 89$.

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$T = (2 \cdot 89)(Q - n_2 P)$ is also a multiple of R, i.e., has order 53. Compute $n_3 = \log_R T \equiv n \bmod 53$. Now $n_2 + 53n_3 \equiv n \bmod 53^2$.

$R = (2 \cdot 53^2)P$ has order 89, and $S = (2 \cdot 53^2)Q$ is multiple of R. Compute $n_4 = \log_R S \equiv n \bmod 89$.

Use Chinese Remainder Theorem
$n \equiv n_1 \bmod 2$,
$n \equiv n_2 + 53n_3 \bmod 53^2$,
$n \equiv n_4 \bmod 89$,
to determine n modulo $2 \cdot 53^2 \cdot 89$.

This "Pohlig-Helln
converts an order-
an order-a DL, an
and a few scalar m

Here $(53^2 \cdot 89)P$
$(53^2 \cdot 89)Q = \infty$,

$(2 \cdot 53 \cdot 89)P = (5$
$(2 \cdot 53 \cdot 89)Q = (7$
A search quickly fi
$(2 \cdot 89)(Q - 2P) =$
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$T = (2 \cdot 89)(Q - n_2 P)$ is also a multiple of R, i.e., has order 53.
Compute
$n_3 = \log_R T \equiv n \bmod 53$.
Now $n_2 + 53 n_3 \equiv n \bmod 53^2$.

$R = (2 \cdot 53^2)P$ has order 89, and
$S = (2 \cdot 53^2)Q$ is multiple of R.
Compute
$n_4 = \log_R S \equiv n \bmod 89$.

Use Chinese Remainder Theorem
$n \equiv n_1 \bmod 2$,
$n \equiv n_2 + 53 n_3 \bmod 53^2$,
$n \equiv n_4 \bmod 89$,
to determine n modulo $2 \cdot 53^2 \cdot 89$.

This "Pohlig-Hellman metho
converts an order-ab DL int
an order-a DL, an order-b D
and a few scalar multiplicati

Here $(53^2 \cdot 89)P = (1, 0)$ a
$(53^2 \cdot 89)Q = \infty$, thus $n_1 =$

$(2 \cdot 53 \cdot 89)P = (539296, 48$
$(2 \cdot 53 \cdot 89)Q = (782288, 57$
A search quickly finds $n_2 =$
$(2 \cdot 89)(Q - 2P) = \infty$, thus
and $n_2 + 53 n_3 = 2$.

$T = (2 \cdot 89)(Q - n_2 P)$ is also a multiple of R, i.e., has order 53. Compute $n_3 = \log_R T \equiv n \bmod 53$. Now $n_2 + 53n_3 \equiv n \bmod 53^2$.

$R = (2 \cdot 53^2)P$ has order 89, and $S = (2 \cdot 53^2)Q$ is multiple of R. Compute $n_4 = \log_R S \equiv n \bmod 89$.

Use Chinese Remainder Theorem
$n \equiv n_1 \bmod 2$,
$n \equiv n_2 + 53n_3 \bmod 53^2$,
$n \equiv n_4 \bmod 89$,
to determine n modulo $2 \cdot 53^2 \cdot 89$.

This "Pohlig-Hellman method" converts an order-ab DL into an order-a DL, an order-b DL, and a few scalar multiplications.

Here $(53^2 \cdot 89)P = (1, 0)$ and $(53^2 \cdot 89)Q = \infty$, thus $n_1 = 0$.

$(2 \cdot 53 \cdot 89)P = (539296, 488875)$,
$(2 \cdot 53 \cdot 89)Q = (782288, 572333)$.
A search quickly finds $n_2 = 2$.
$(2 \cdot 89)(Q - 2P) = \infty$, thus $n_3 = 0$ and $n_2 + 53n_3 = 2$.

$89)(Q - n_2 P)$ is also
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$_R T \equiv n$ mod 53.
$+ 53n_3 \equiv n$ mod 53^2.

$53^2)P$ has order 89, and
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$(2 \cdot 53^2)P = (8775$
$(2 \cdot 53^2)Q = (822$
Compute $n_4 = 67$
e.g. using BSGS.

Use Chinese Rema
$n \equiv 0$ mod 2,
$n \equiv 2$ mod 53^2,
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A search quickly finds $n_2 = 2$.
$(2 \cdot 89)(Q - 2P) = \infty$, thus $n_3 = 0$ and $n_2 + 53n_3 = 2$.

$(2 \cdot 53^2)P = (877560, 94784$
$(2 \cdot 53^2)Q = (822491, 1182$
Compute $n_4 = 67$, e.g. using BSGS.

Use Chinese Remainder The
$n \equiv 0 \bmod 2$,
$n \equiv 2 \bmod 53^2$,
$n \equiv 67 \bmod 89$,
to determine $n = 78654$.

Pohlig-Hellman method redu
security of discrete logarithm
problem in group generated
to security of largest prime o
subgroup.

This "Pohlig-Hellman method" converts an order-ab DL into an order-a DL, an order-b DL, and a few scalar multiplications.

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A search quickly finds $n_2 = 2$.
$(2 \cdot 89)(Q - 2P) = \infty$, thus $n_3 = 0$ and $n_2 + 53n_3 = 2$.

$(2 \cdot 53^2)P = (877560, 947848)$ and $(2 \cdot 53^2)Q = (822491, 118220)$.
Compute $n_4 = 67$, e.g. using BSGS.

Use Chinese Remainder Theorem
$n \equiv 0 \bmod 2$,
$n \equiv 2 \bmod 53^2$,
$n \equiv 67 \bmod 89$,
to determine $n = 78654$.

Pohlig-Hellman method reduces security of discrete logarithm problem in group generated by P to security of largest prime order subgroup.

ohlig-Hellman method"
an order-ab DL into
-a DL, an order-b DL,
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$3^2 \cdot 89)P = (1,0)$ and
$)Q = \infty$, thus $n_1 = 0$.

$89)P = (539296, 488875)$,
$89)Q = (782288, 572333)$.
quickly finds $n_2 = 2$.
$Q - 2P) = \infty$, thus $n_3 = 0$
$+ 53n_3 = 2$.

$(2 \cdot 53^2)P = (877560, 947848)$ and
$(2 \cdot 53^2)Q = (822491, 118220)$.
Compute $n_4 = 67$,
e.g. using BSGS.

Use Chinese Remainder Theorem
$n \equiv 0 \bmod 2$,
$n \equiv 2 \bmod 53^2$,
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Pohlig-Hellman method reduces security of discrete logarithm problem in group generated by P to security of largest prime order subgroup.

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Compute $n_4 = 67$,
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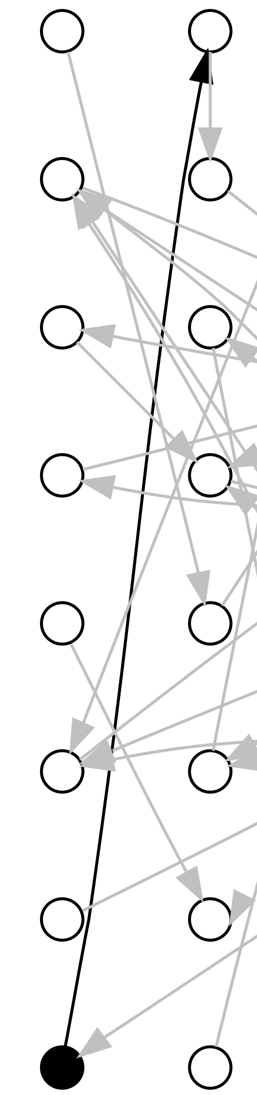

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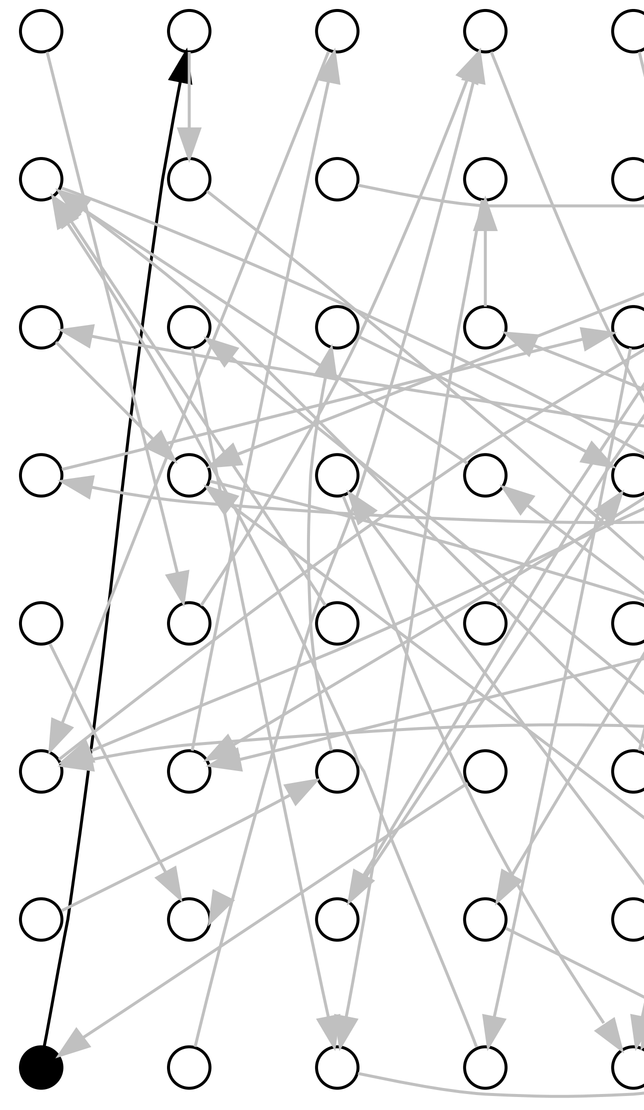

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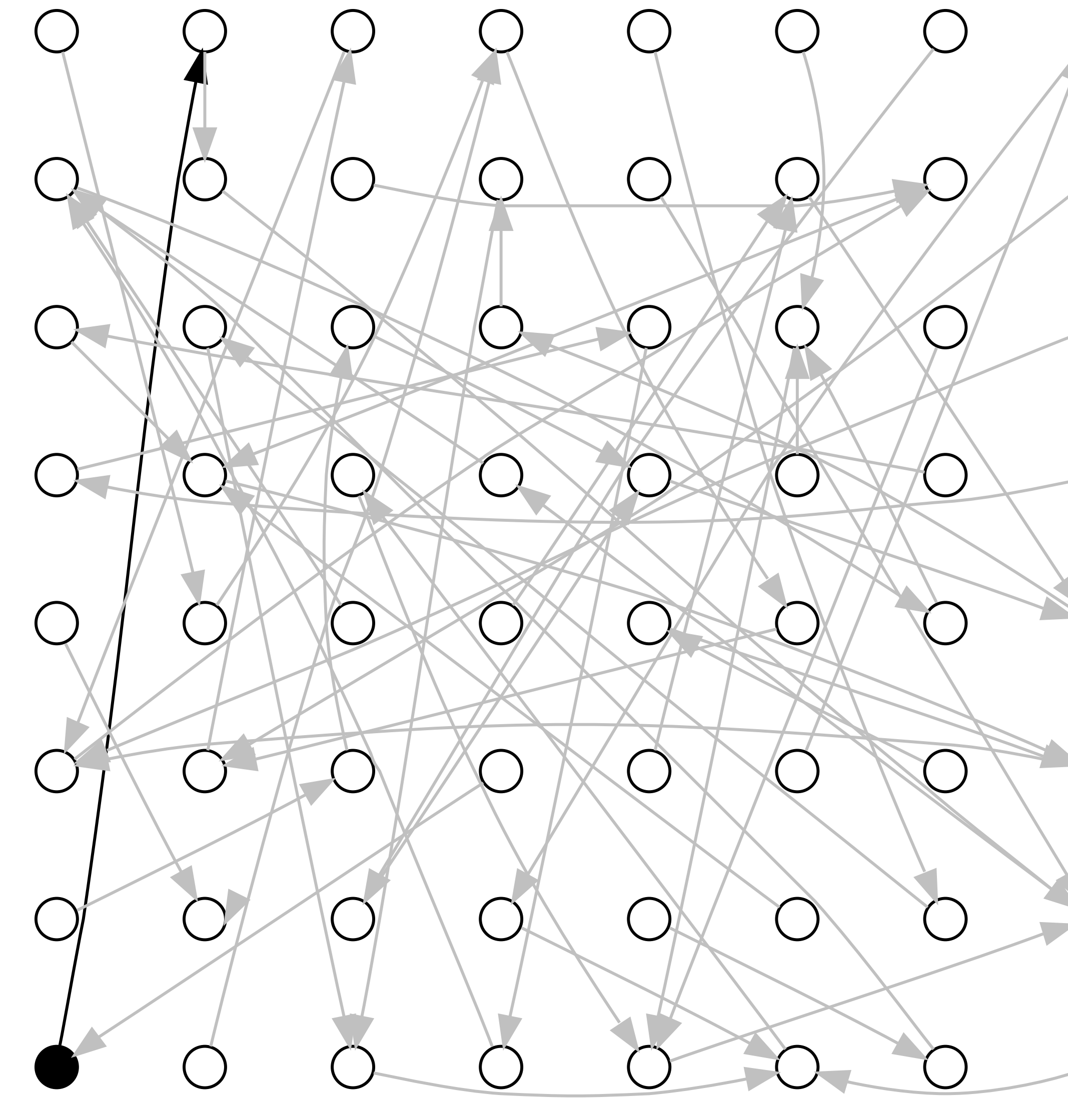

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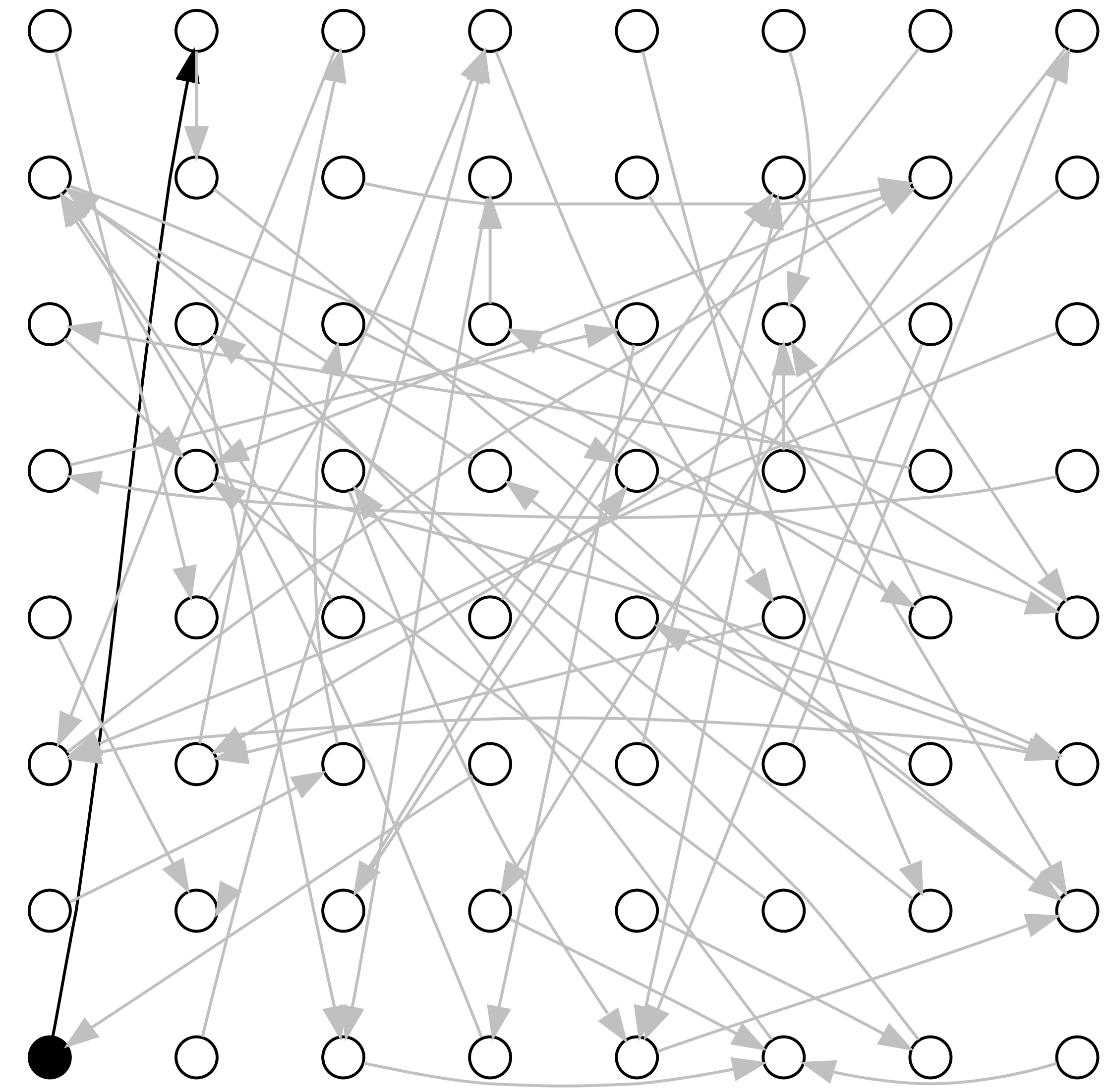

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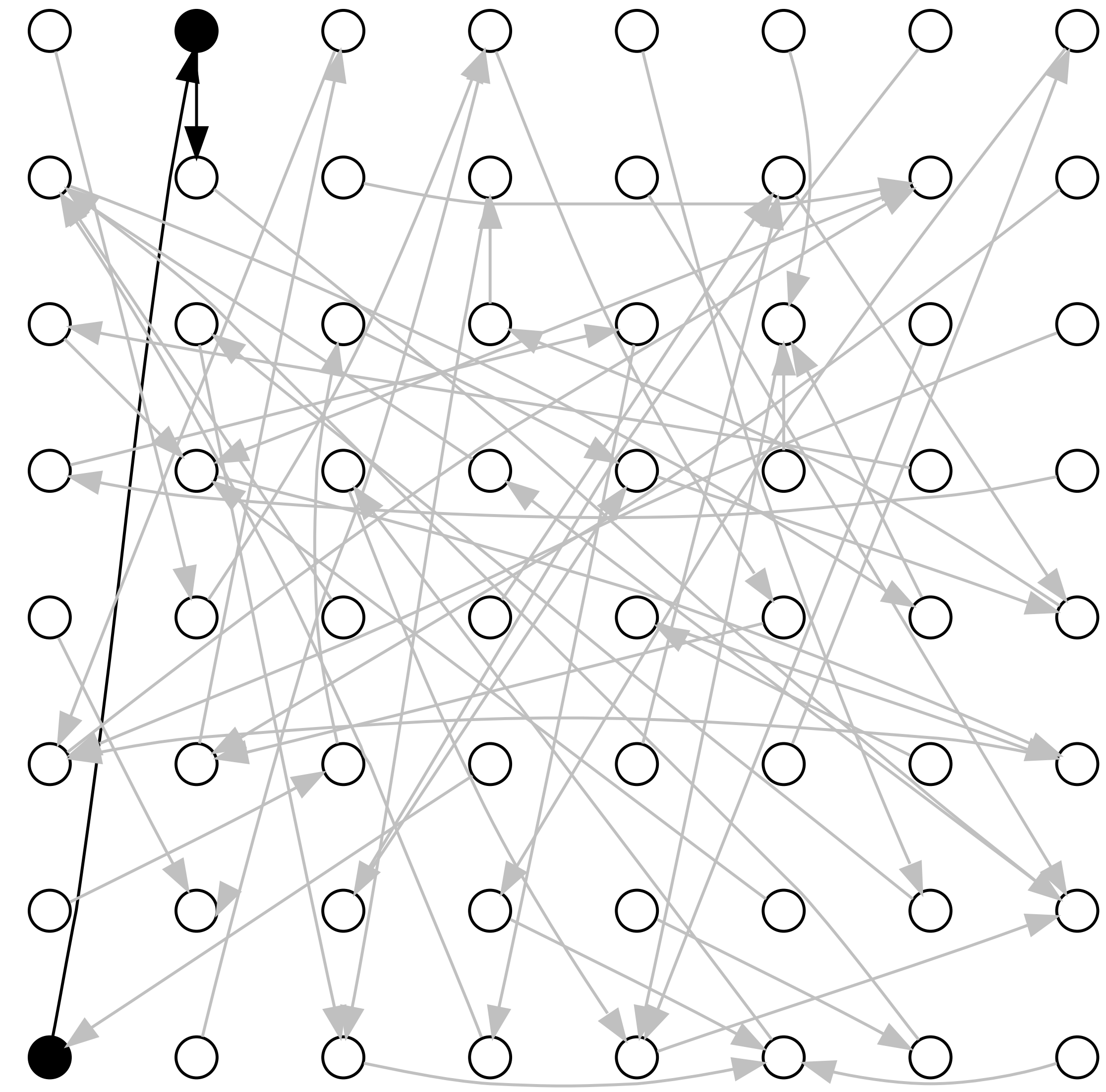

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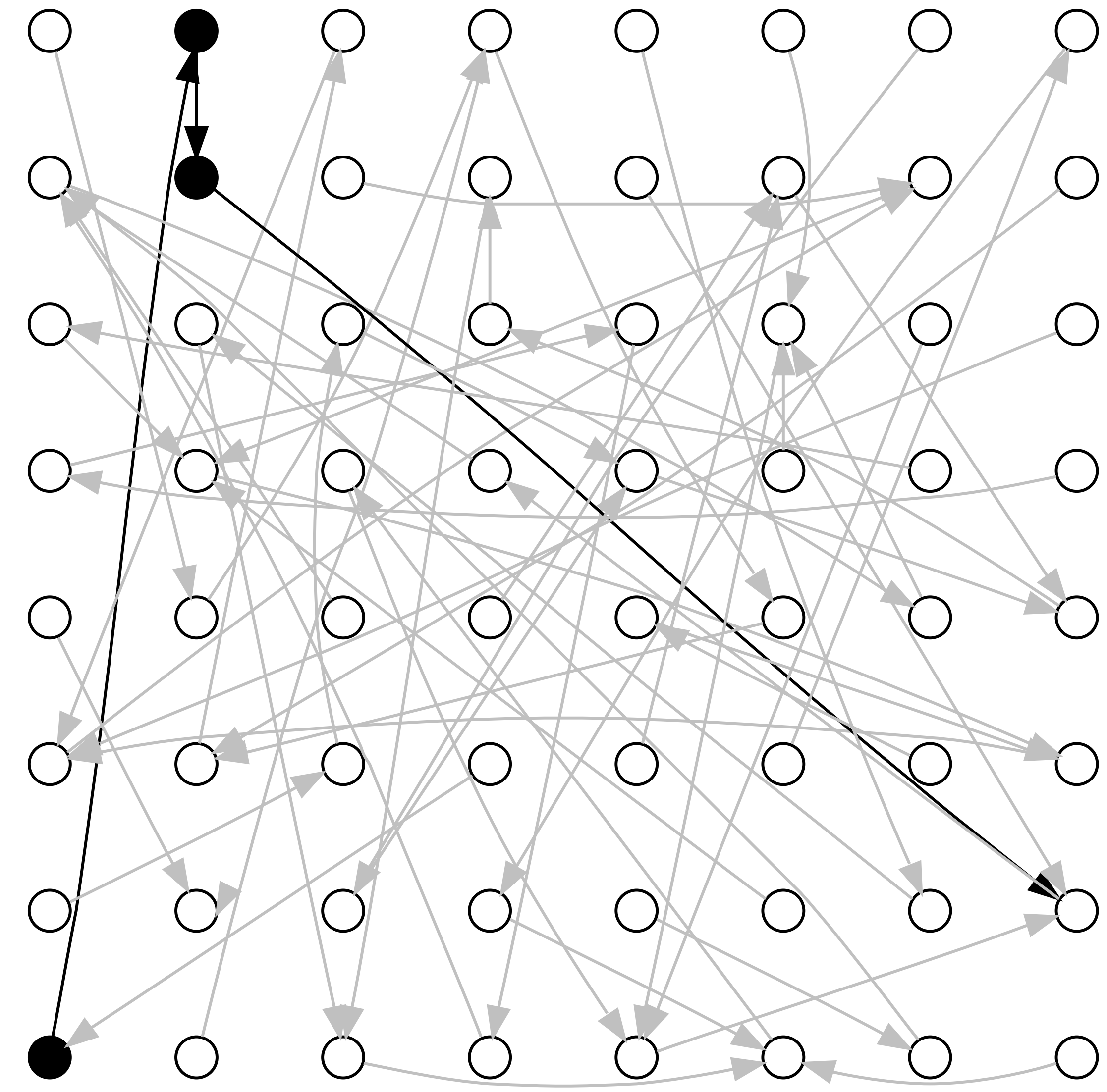

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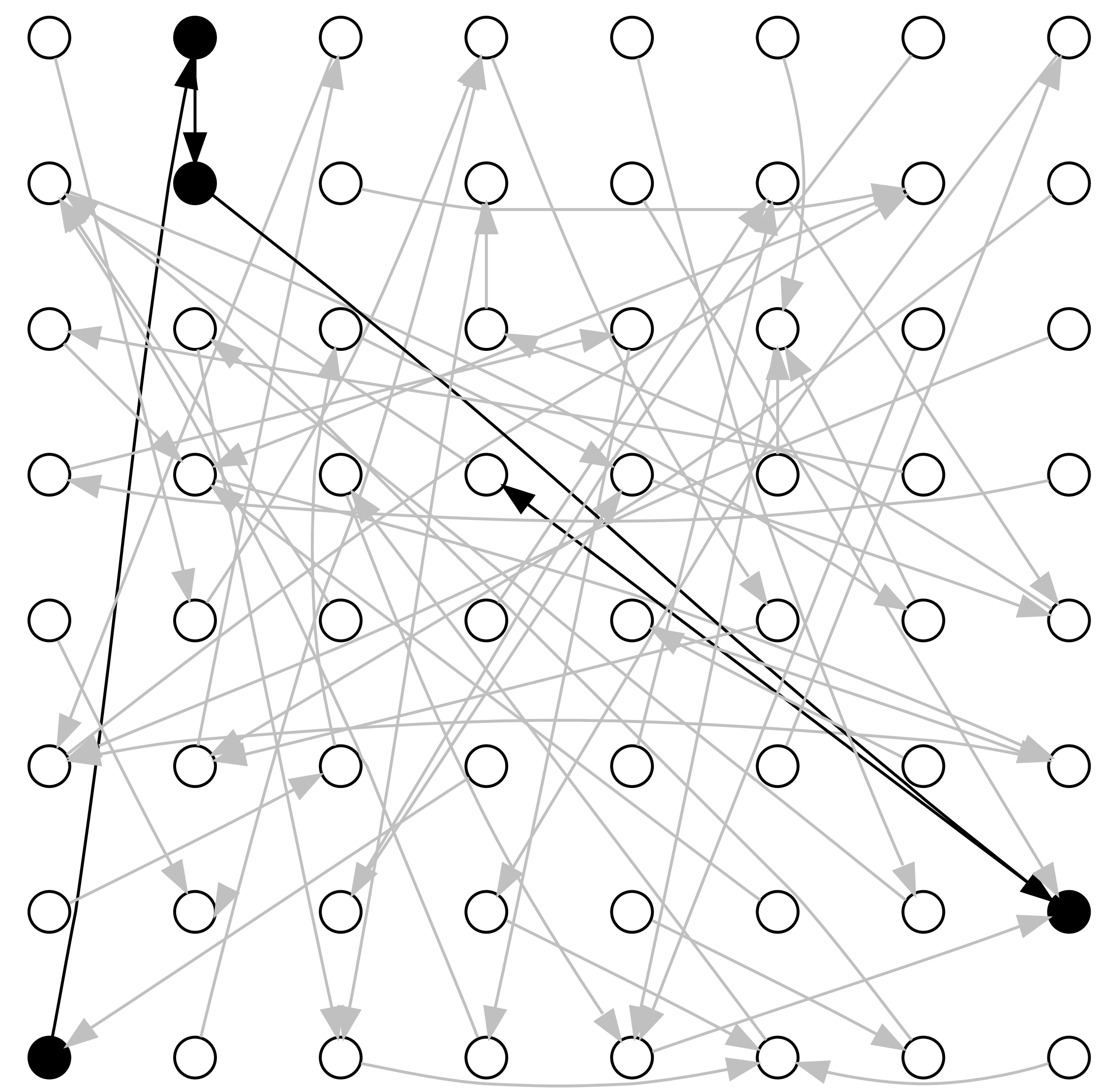

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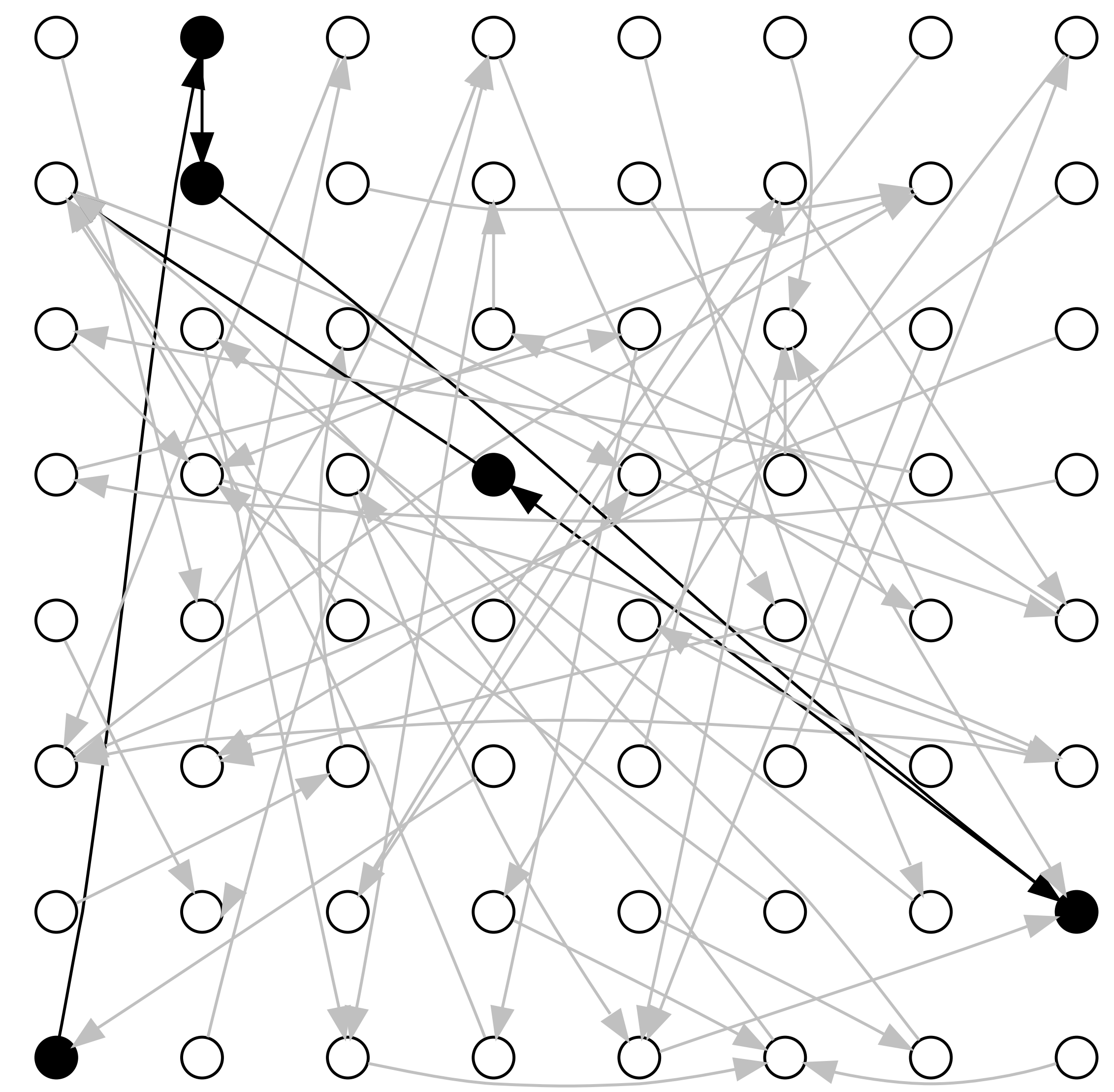

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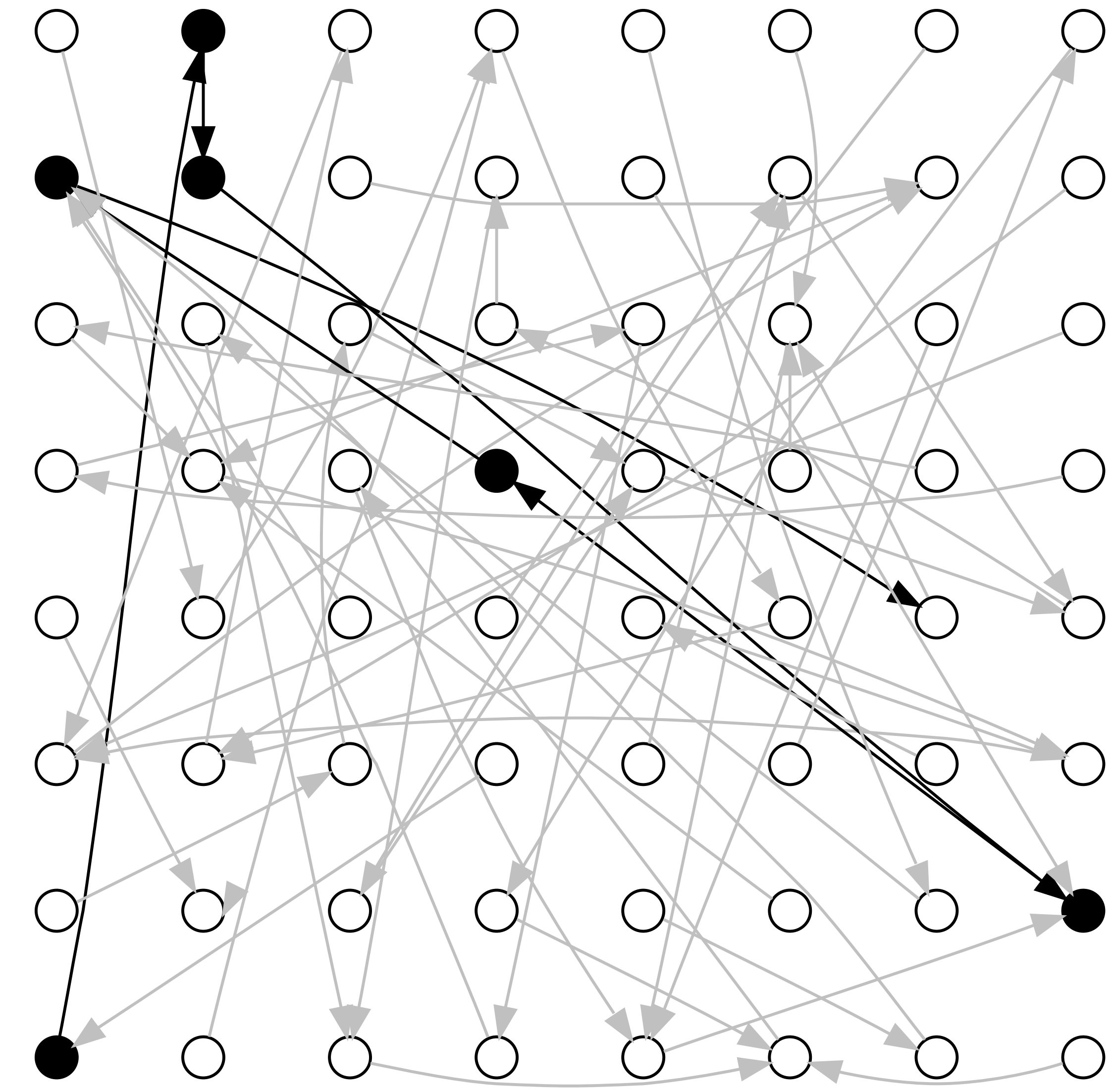

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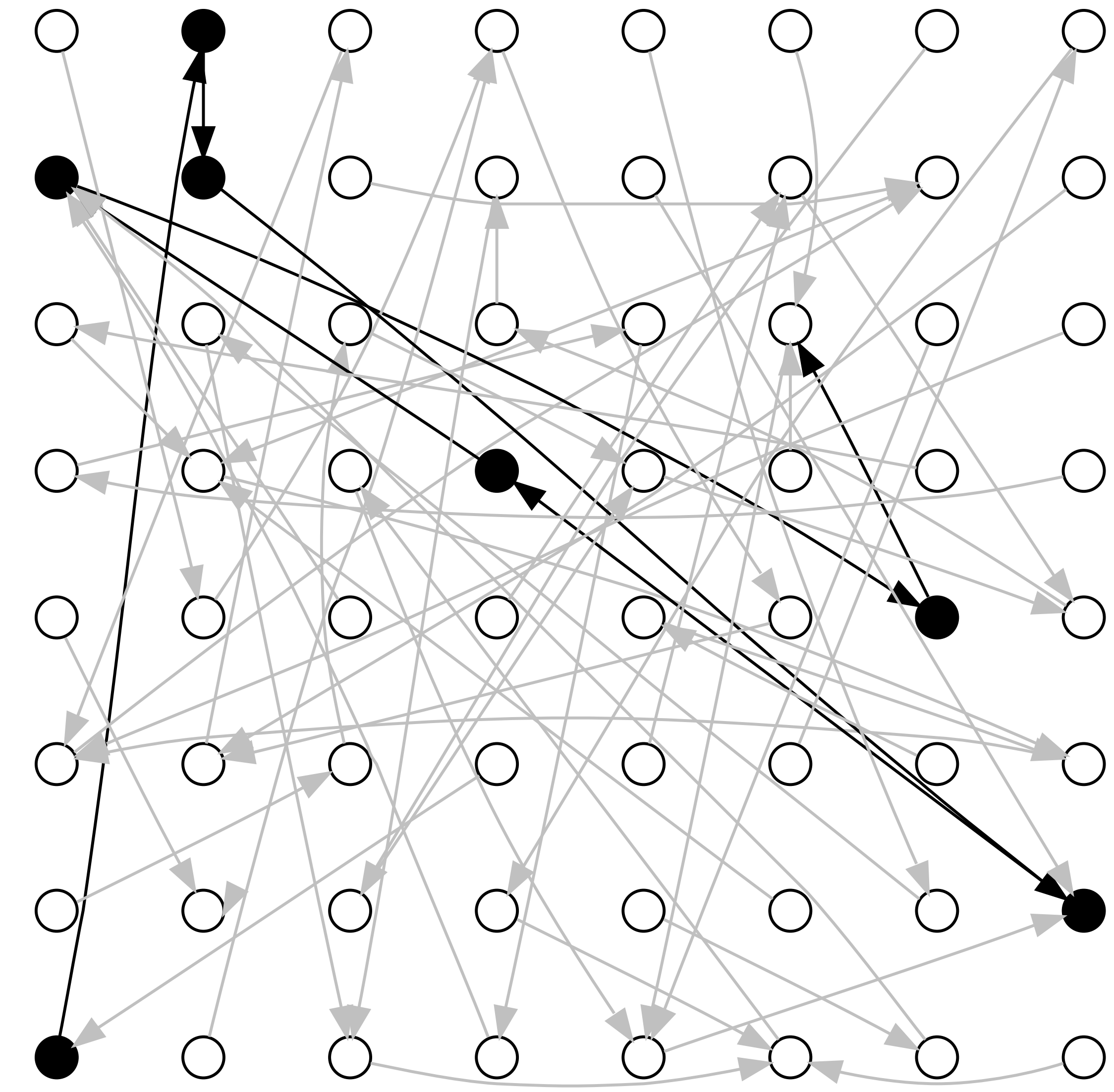

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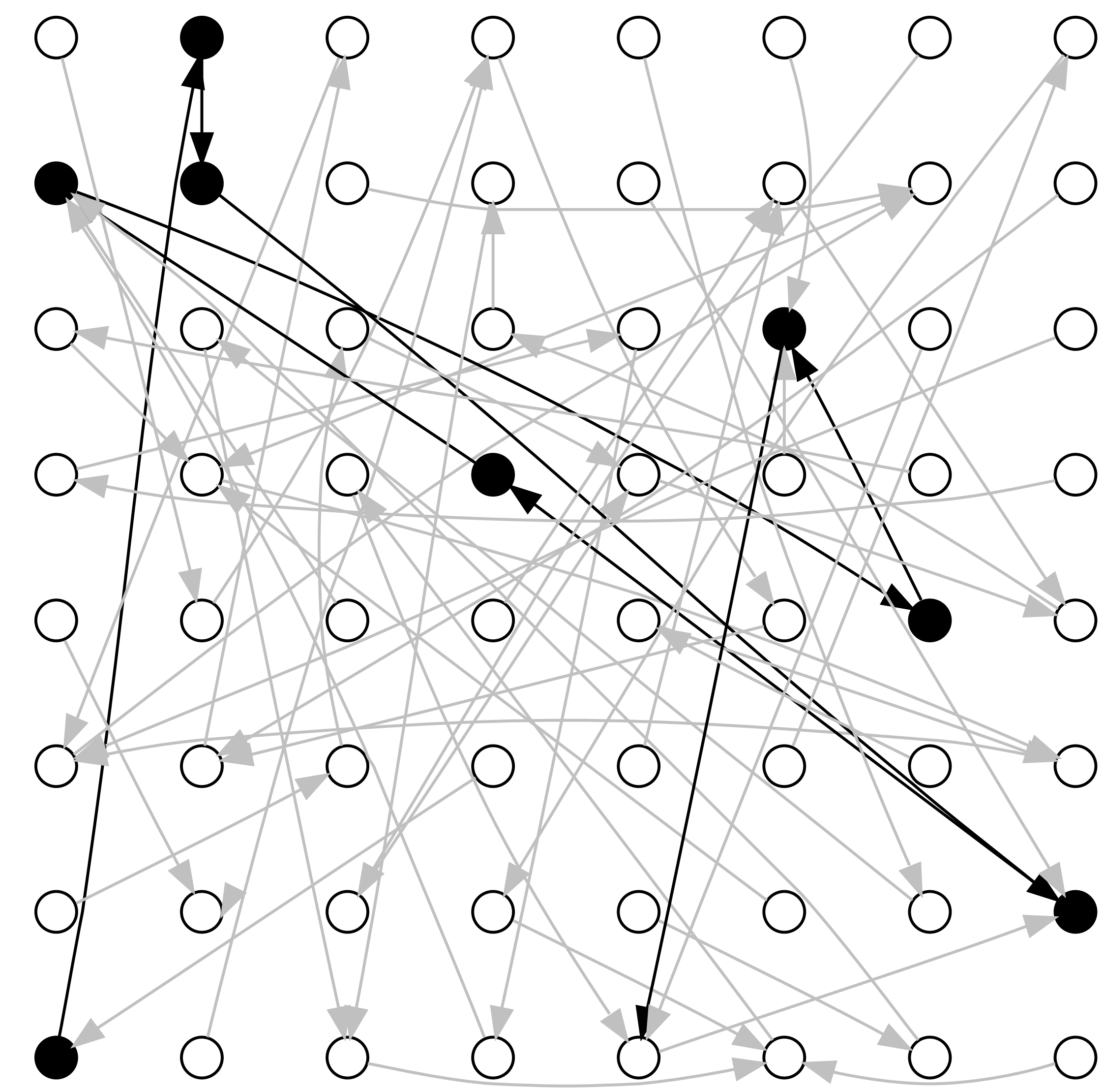

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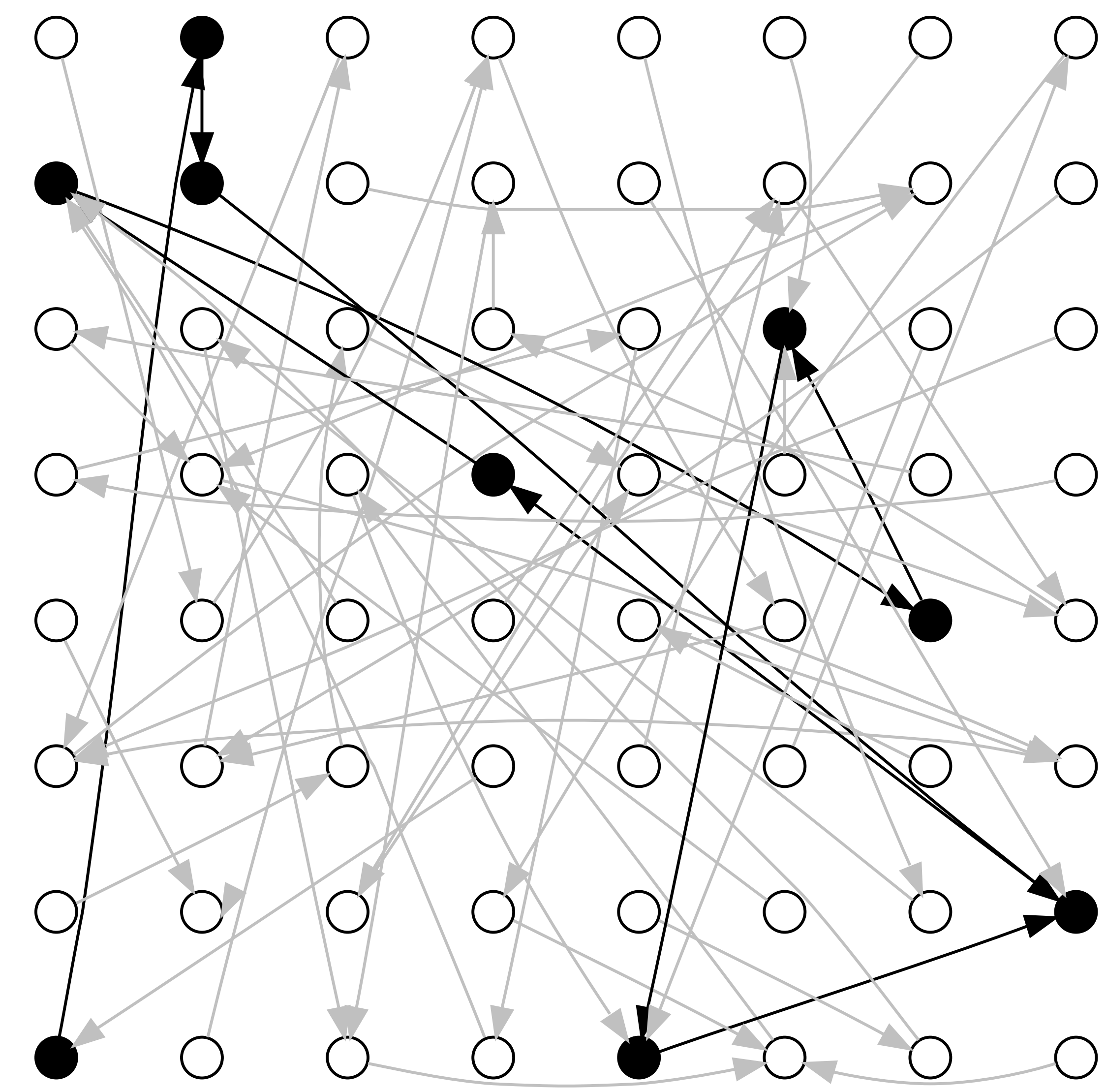

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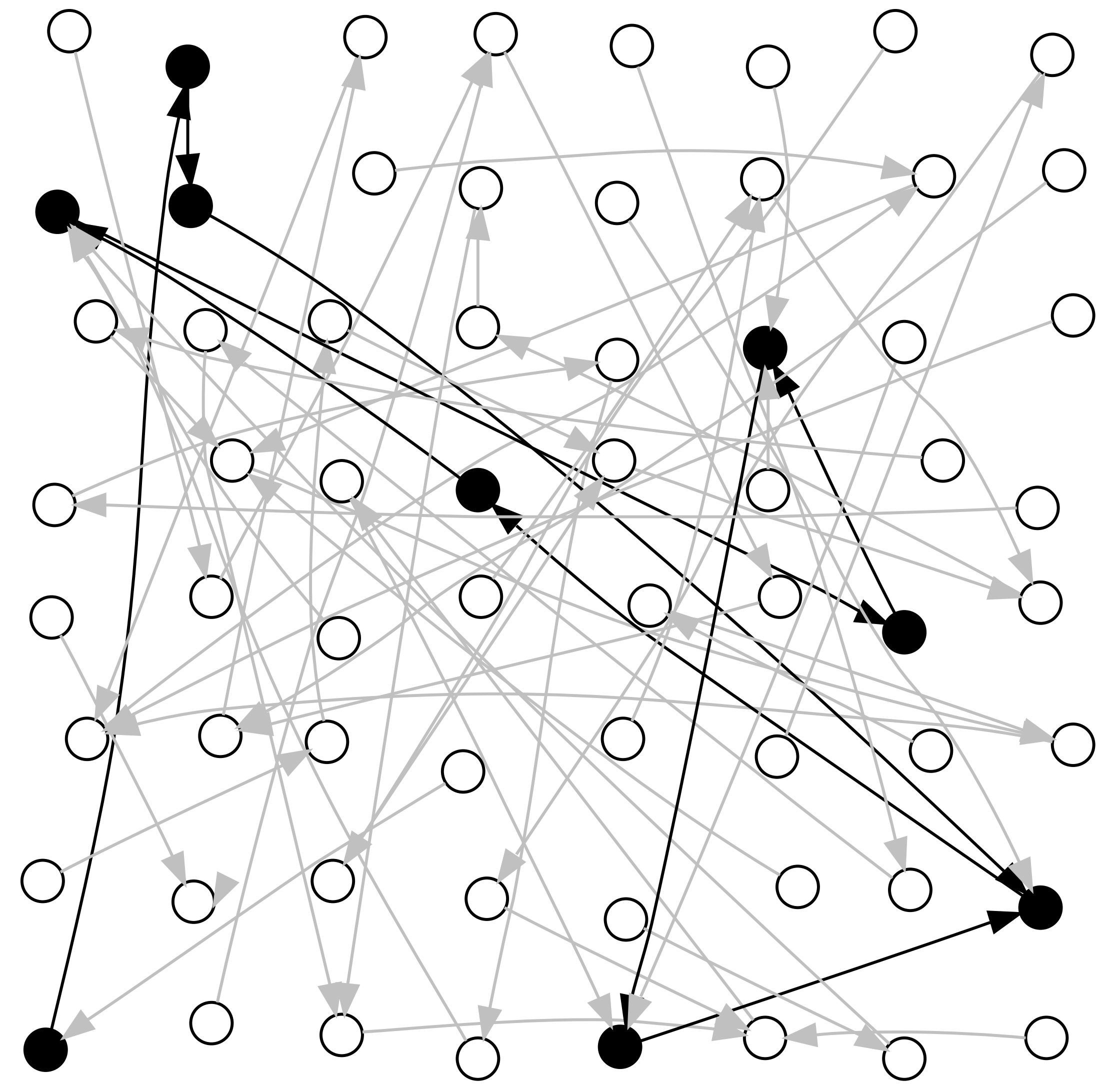

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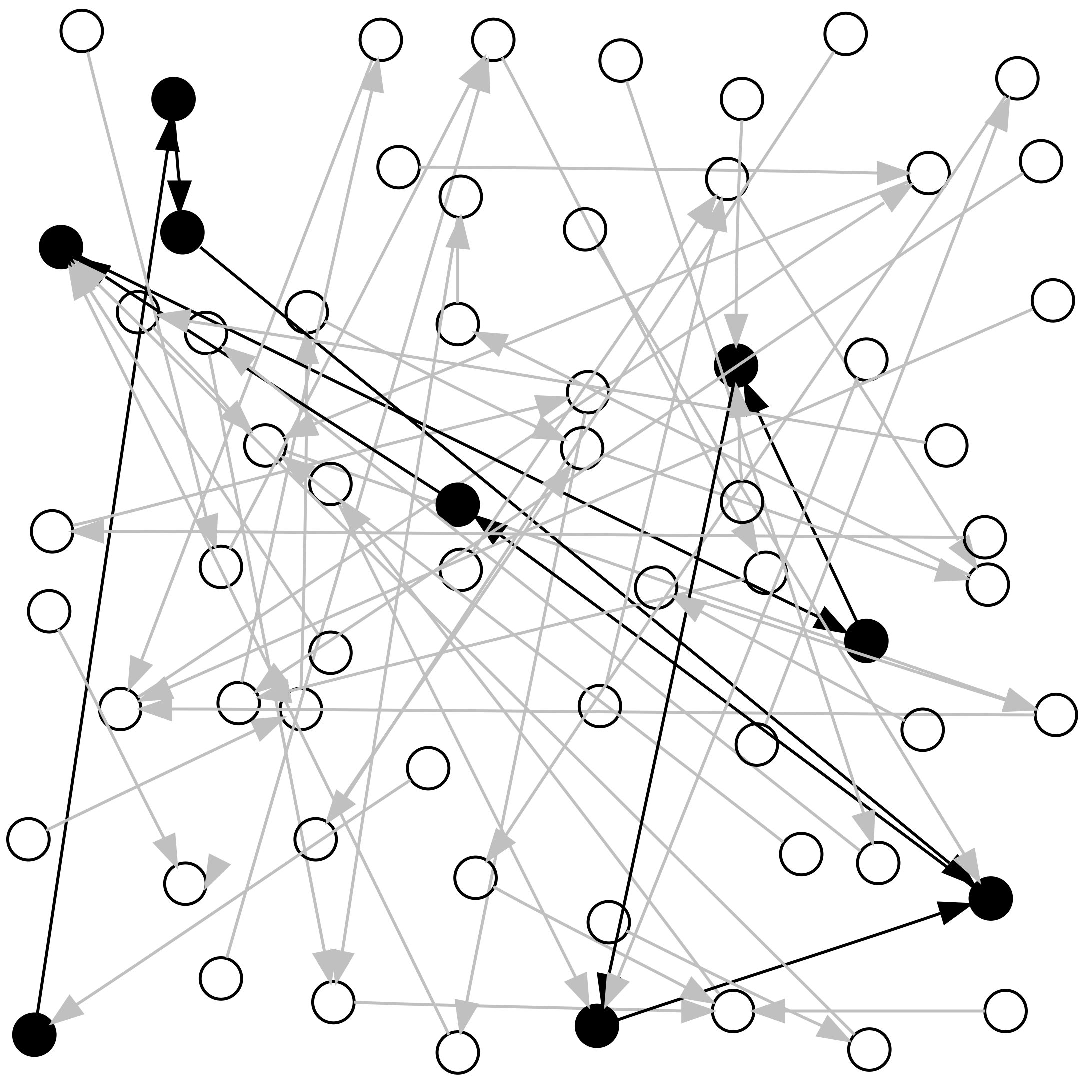

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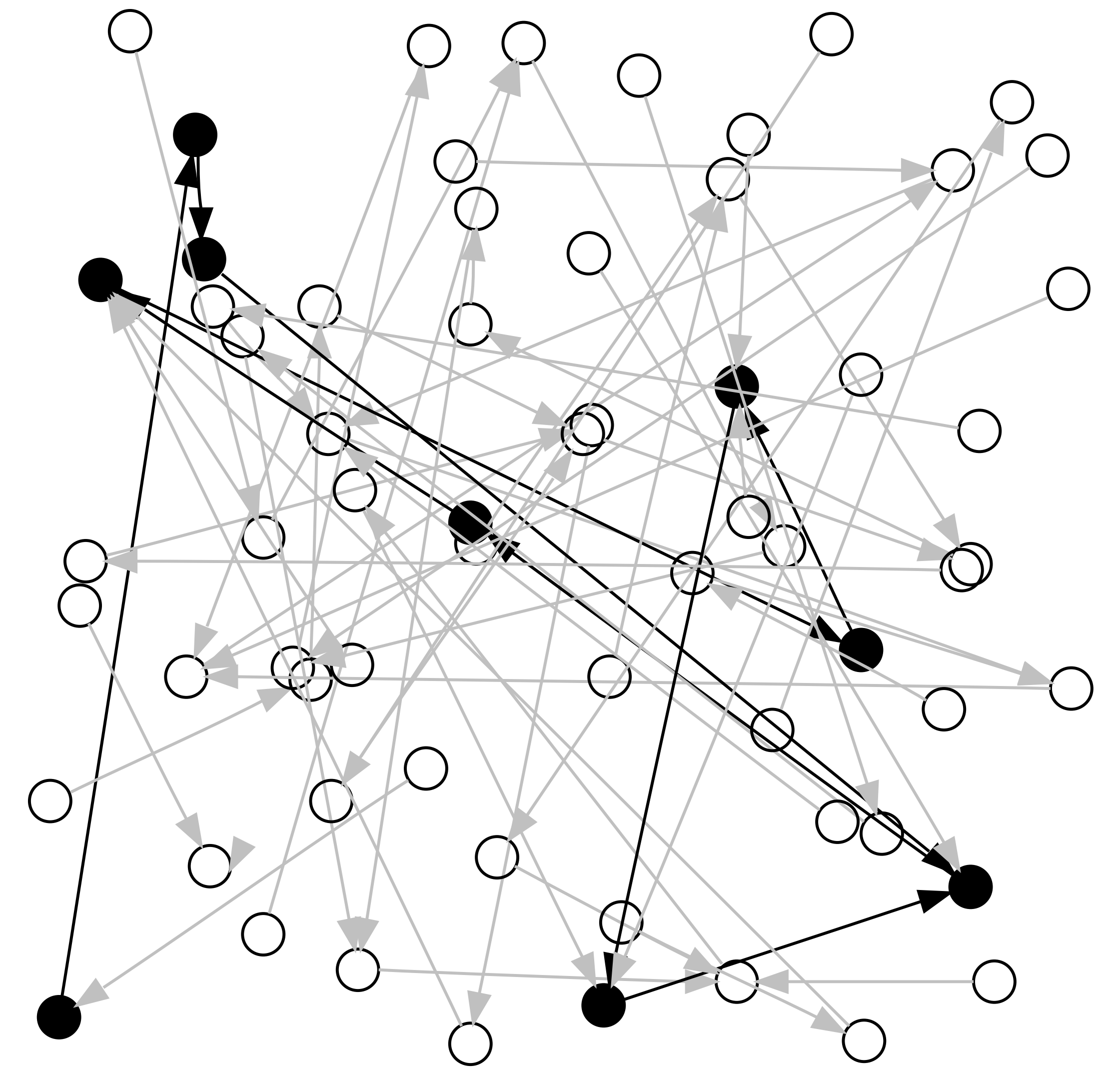

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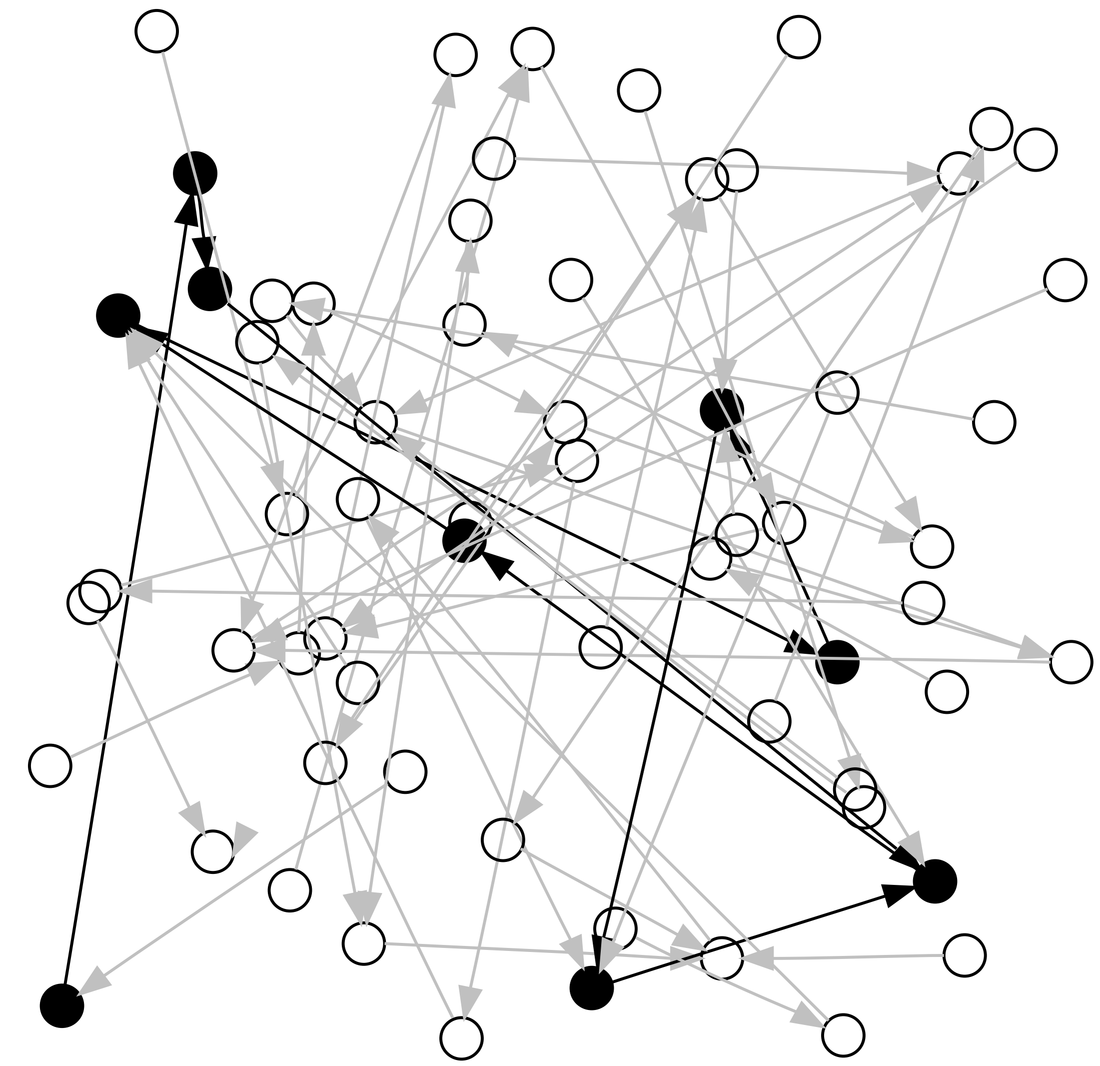

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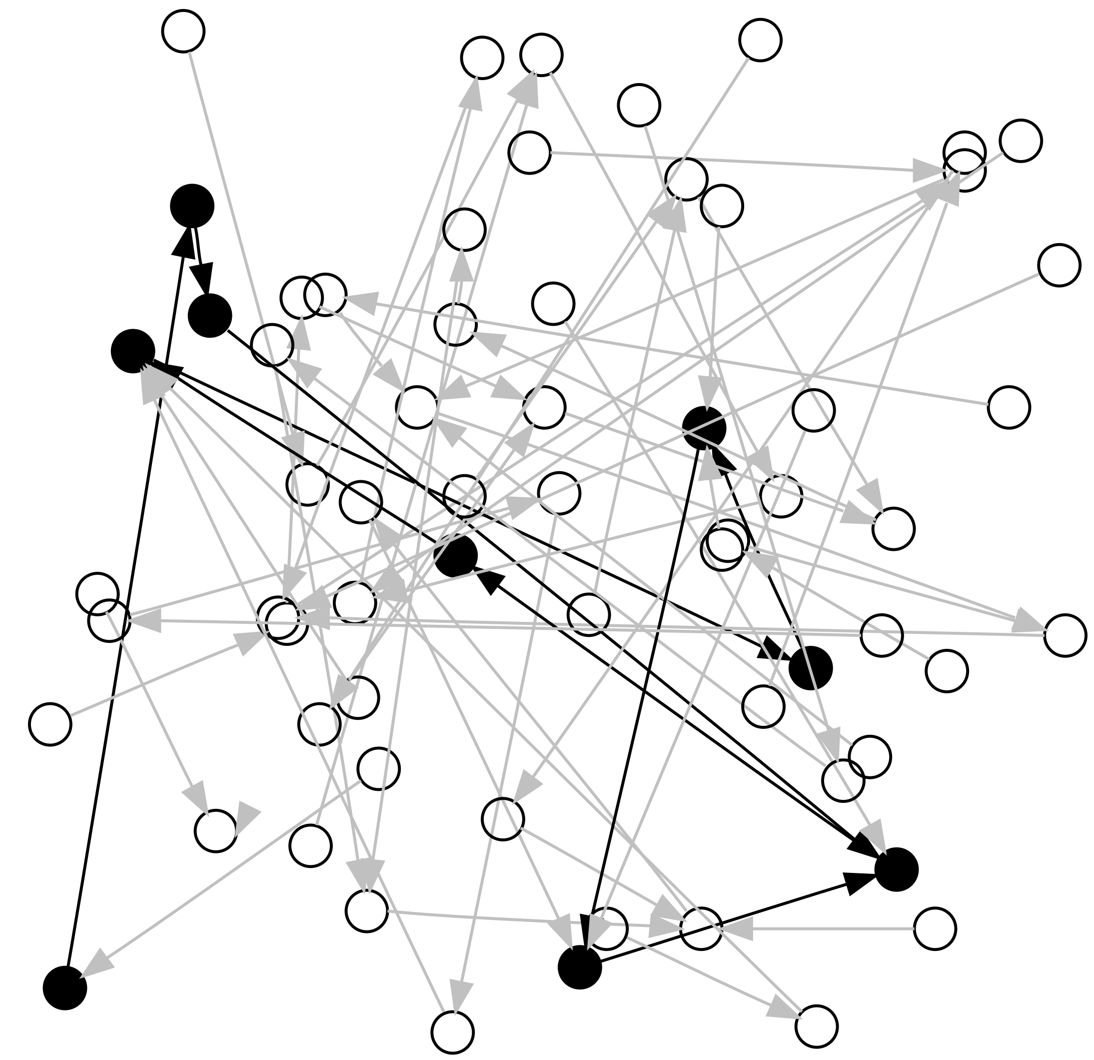

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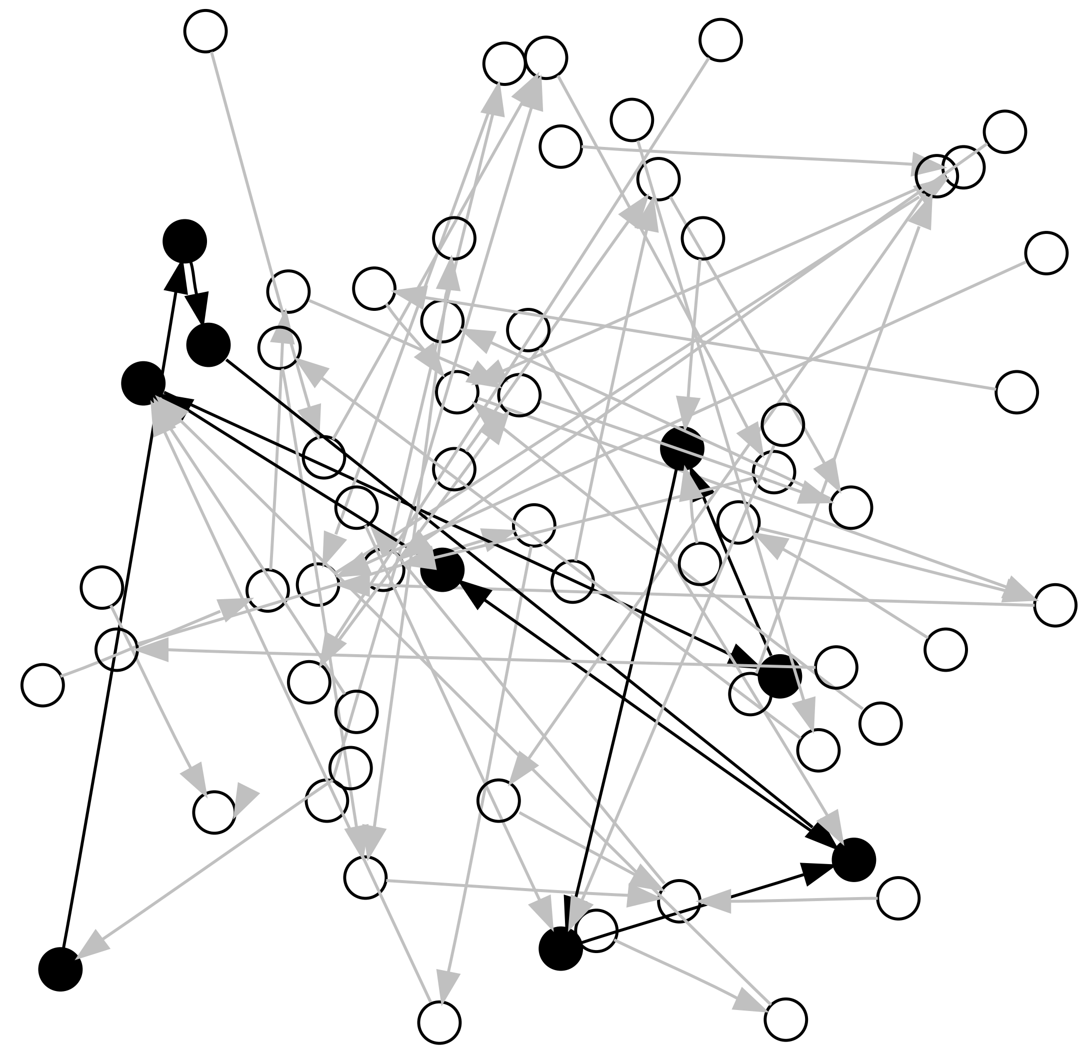

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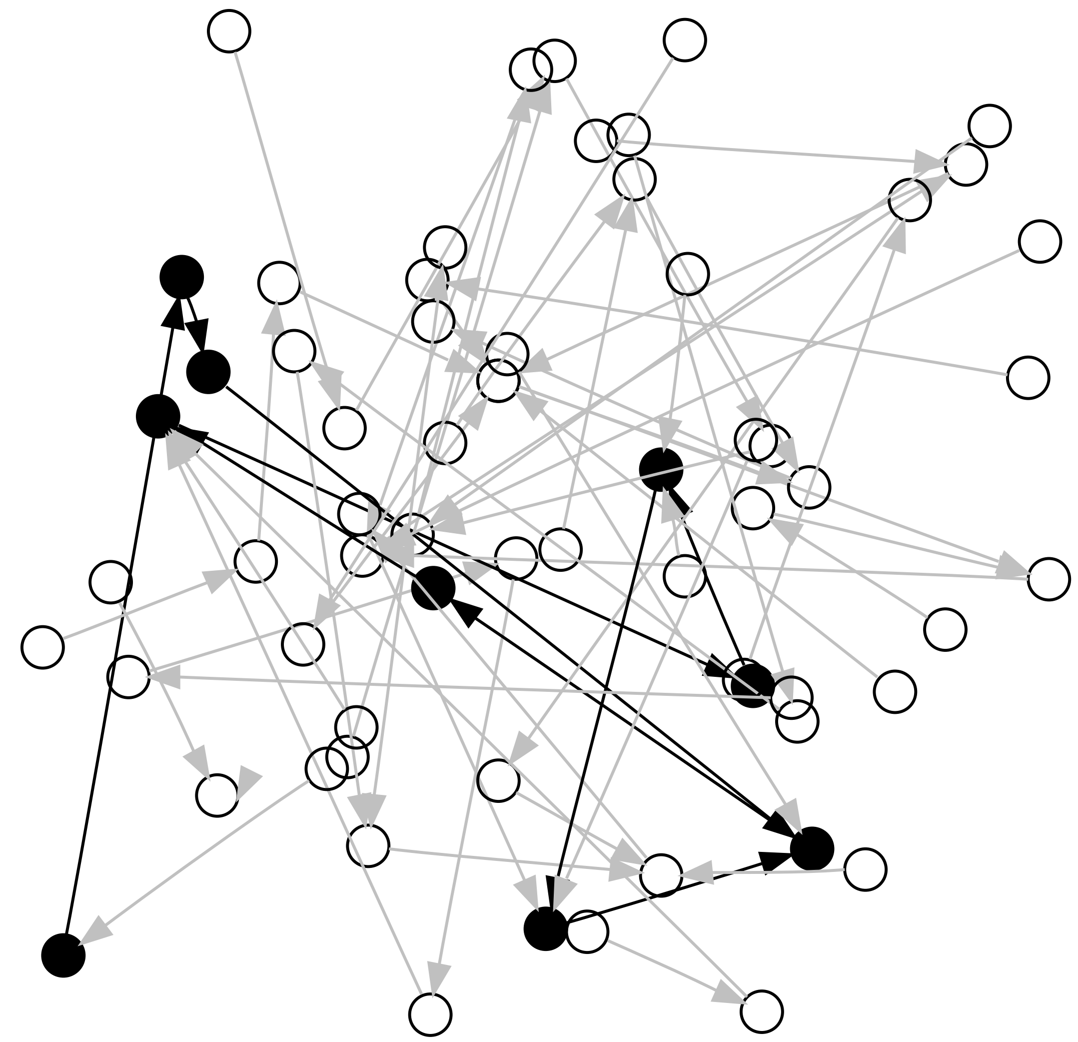

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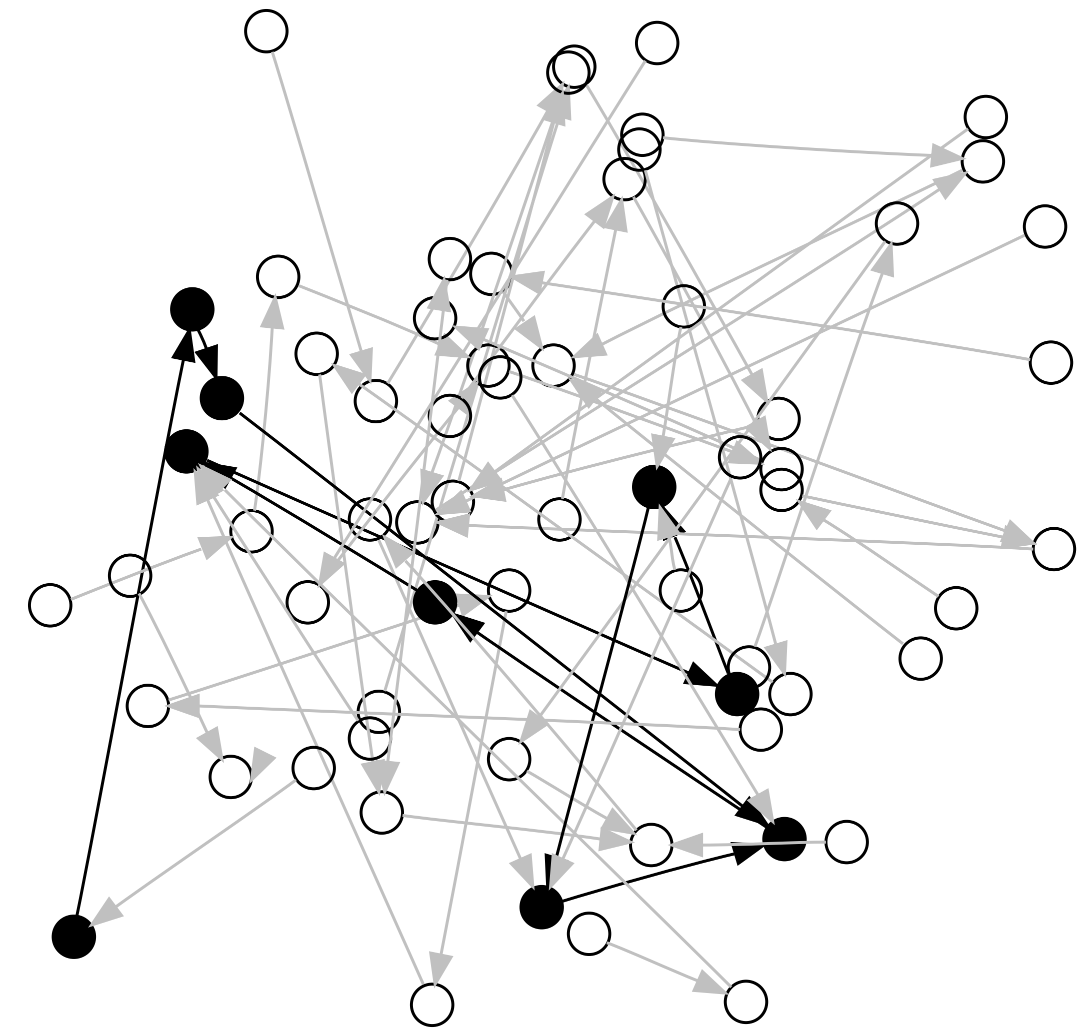

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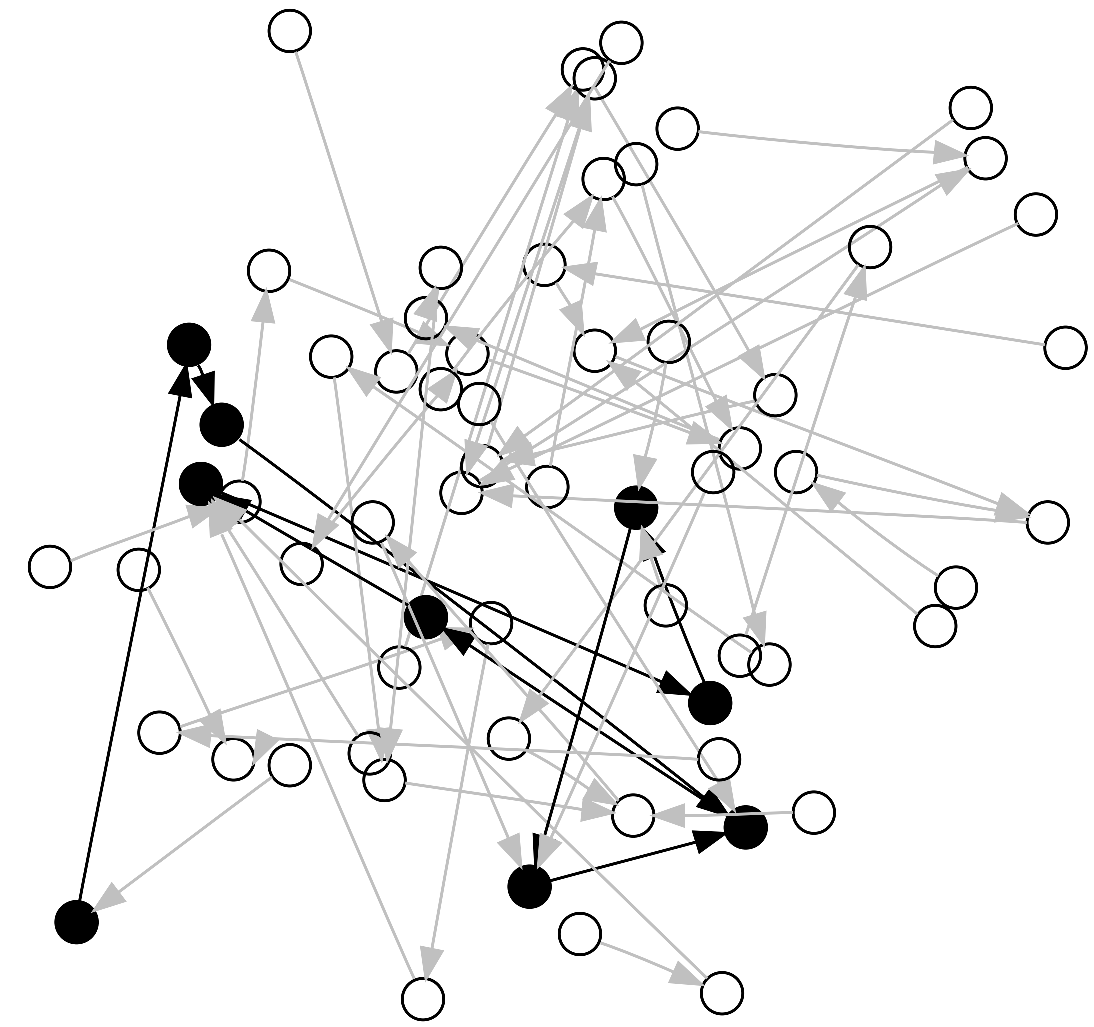

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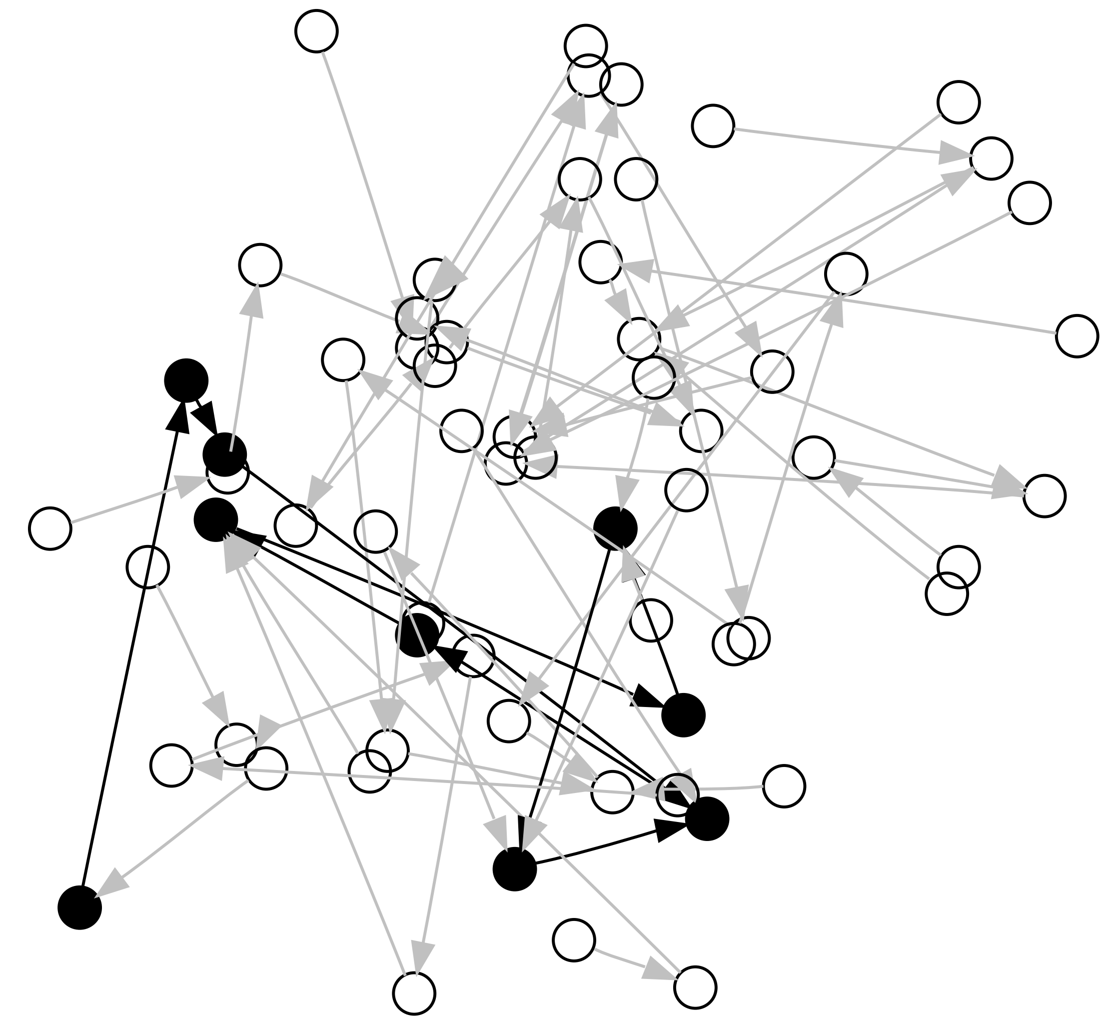

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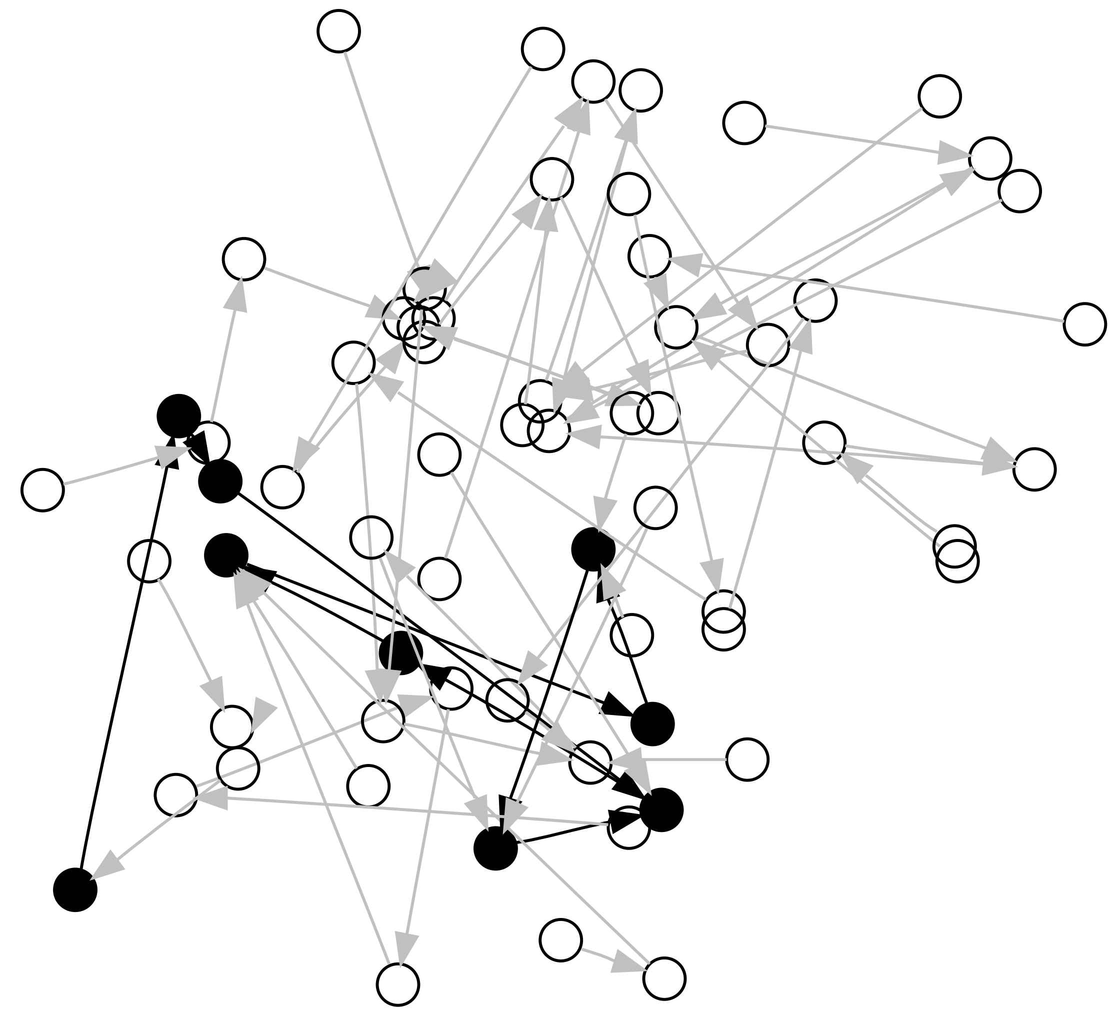

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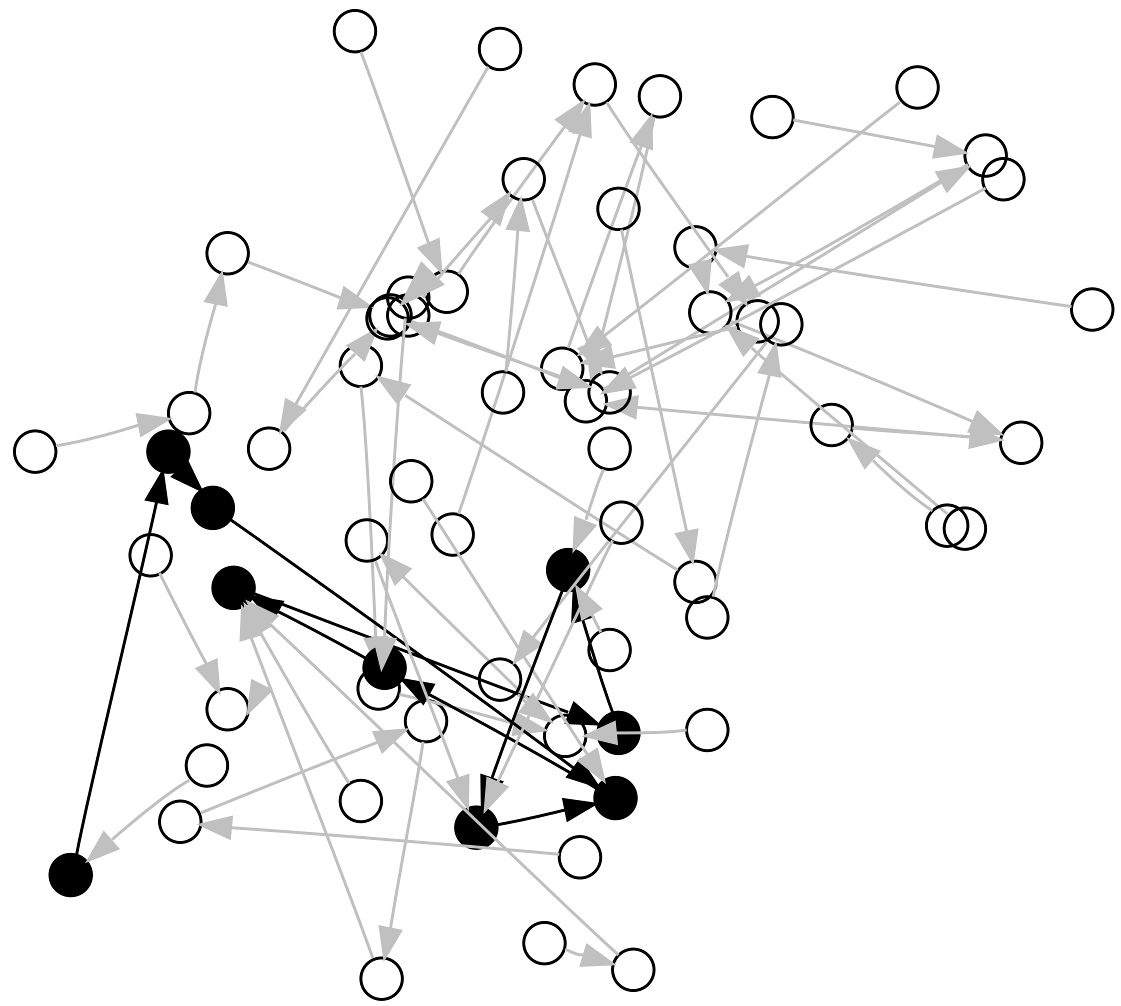

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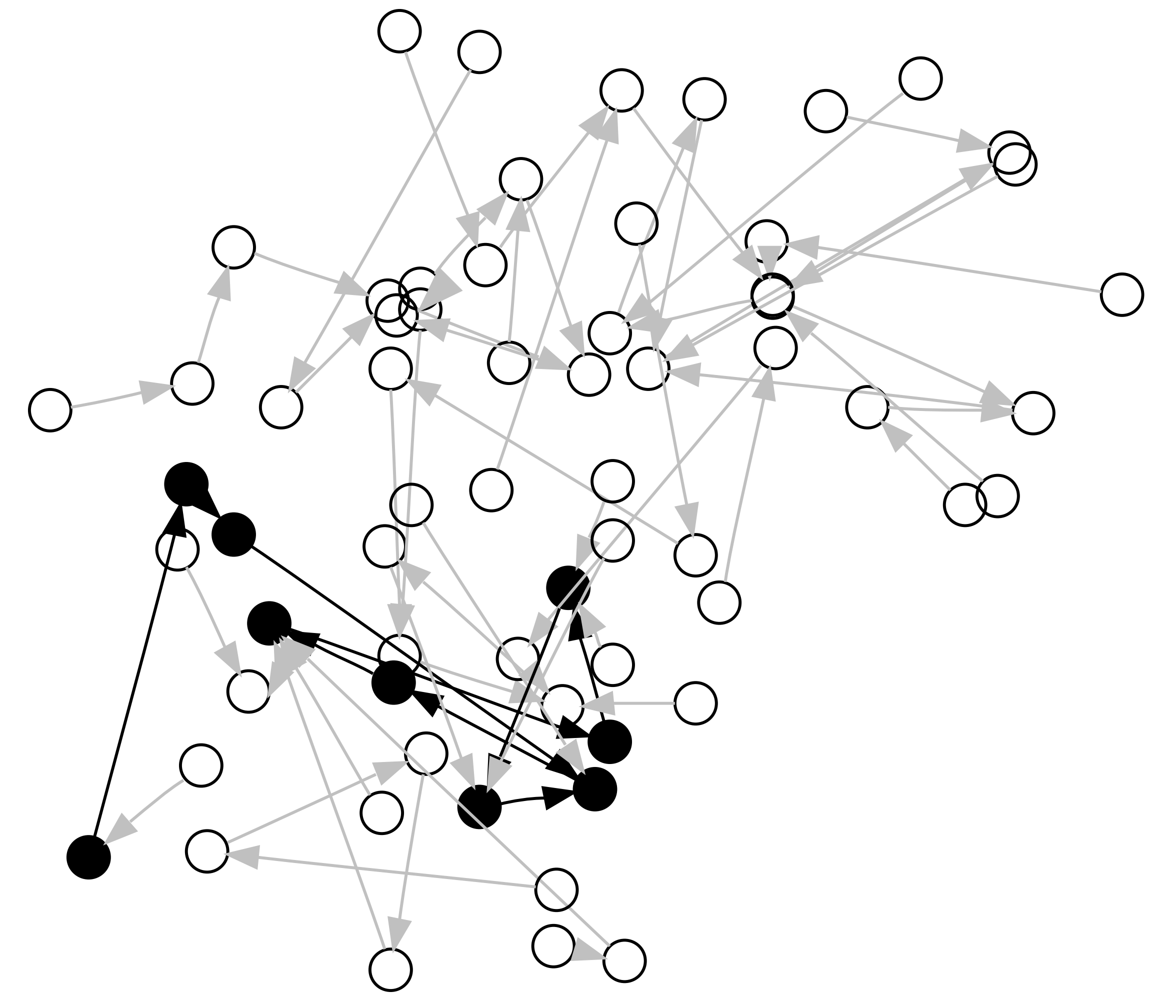

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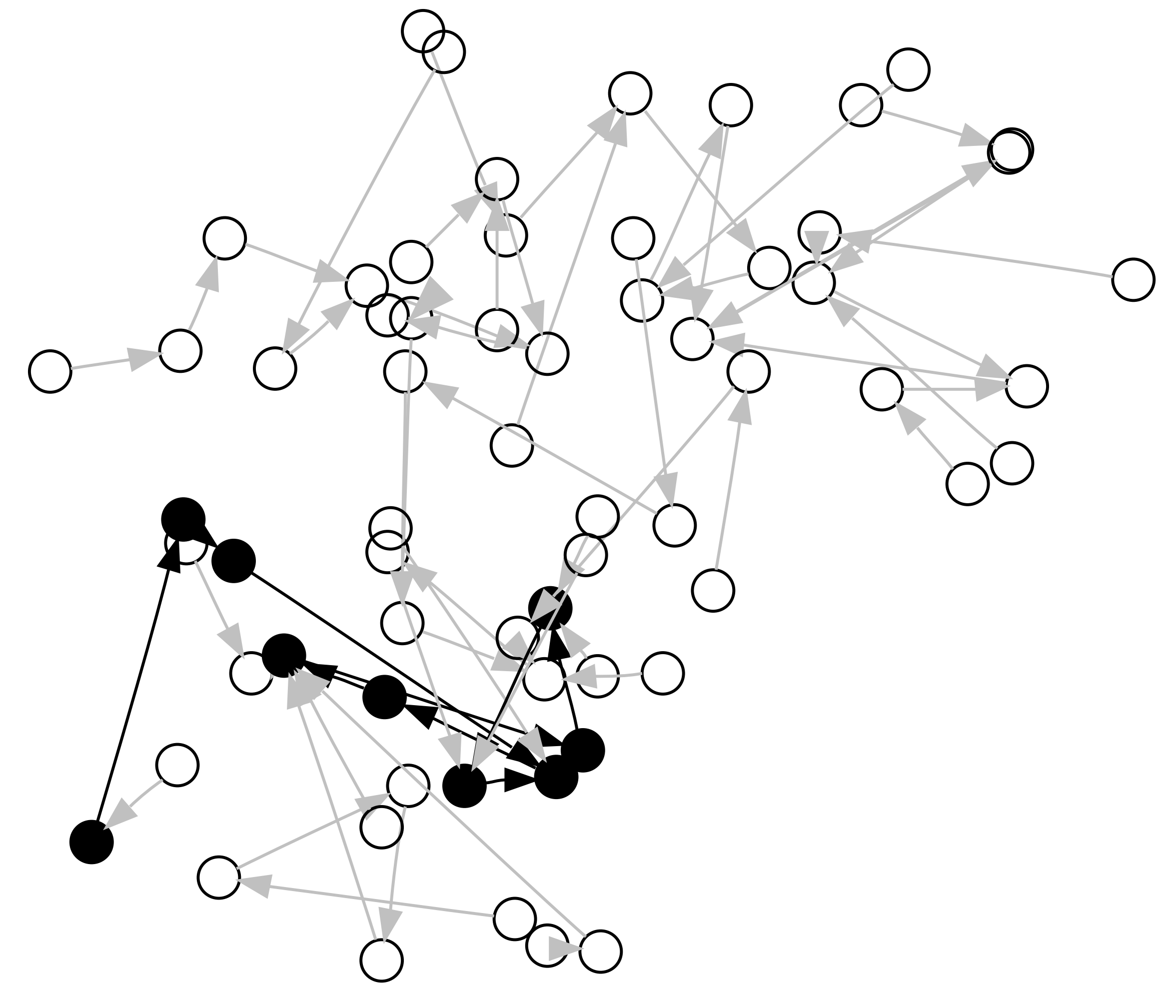

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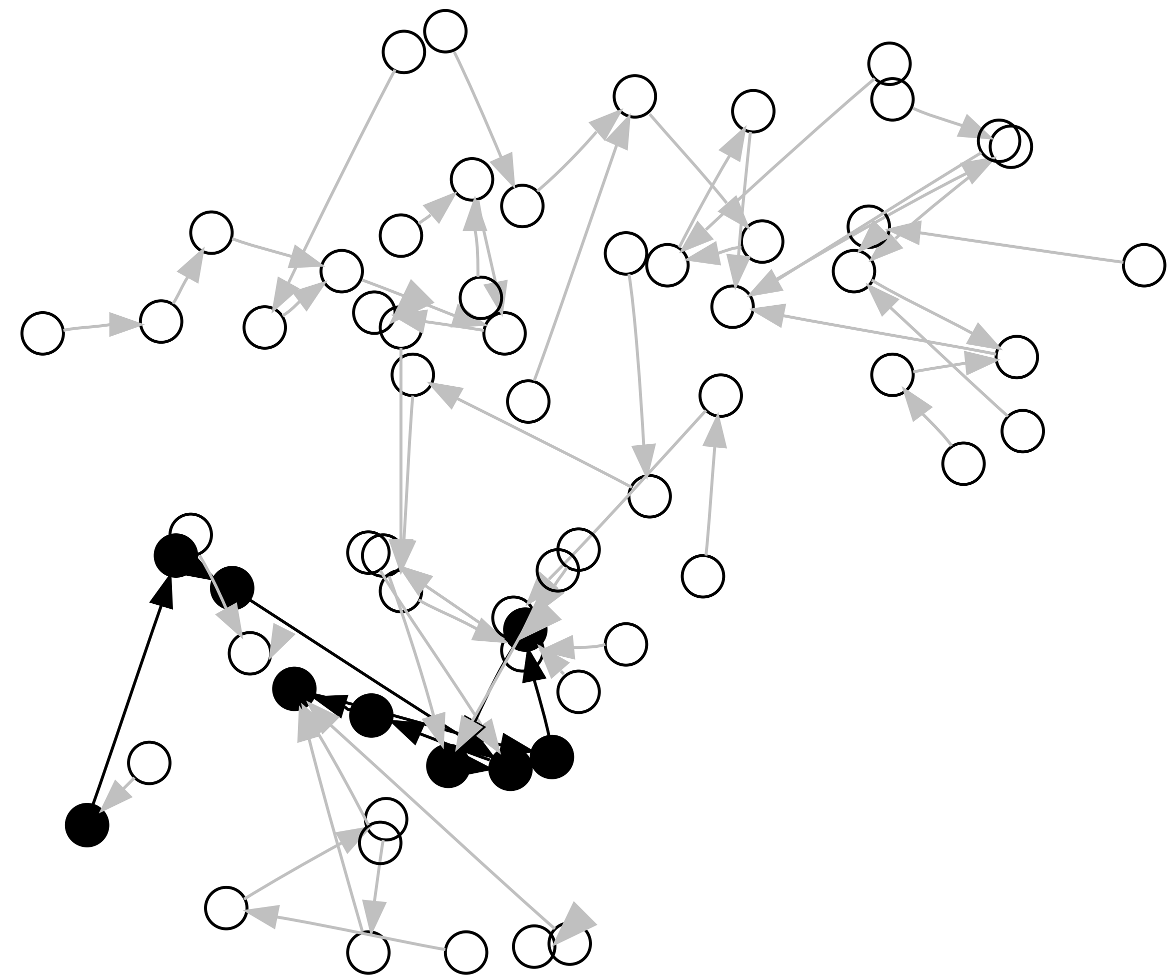

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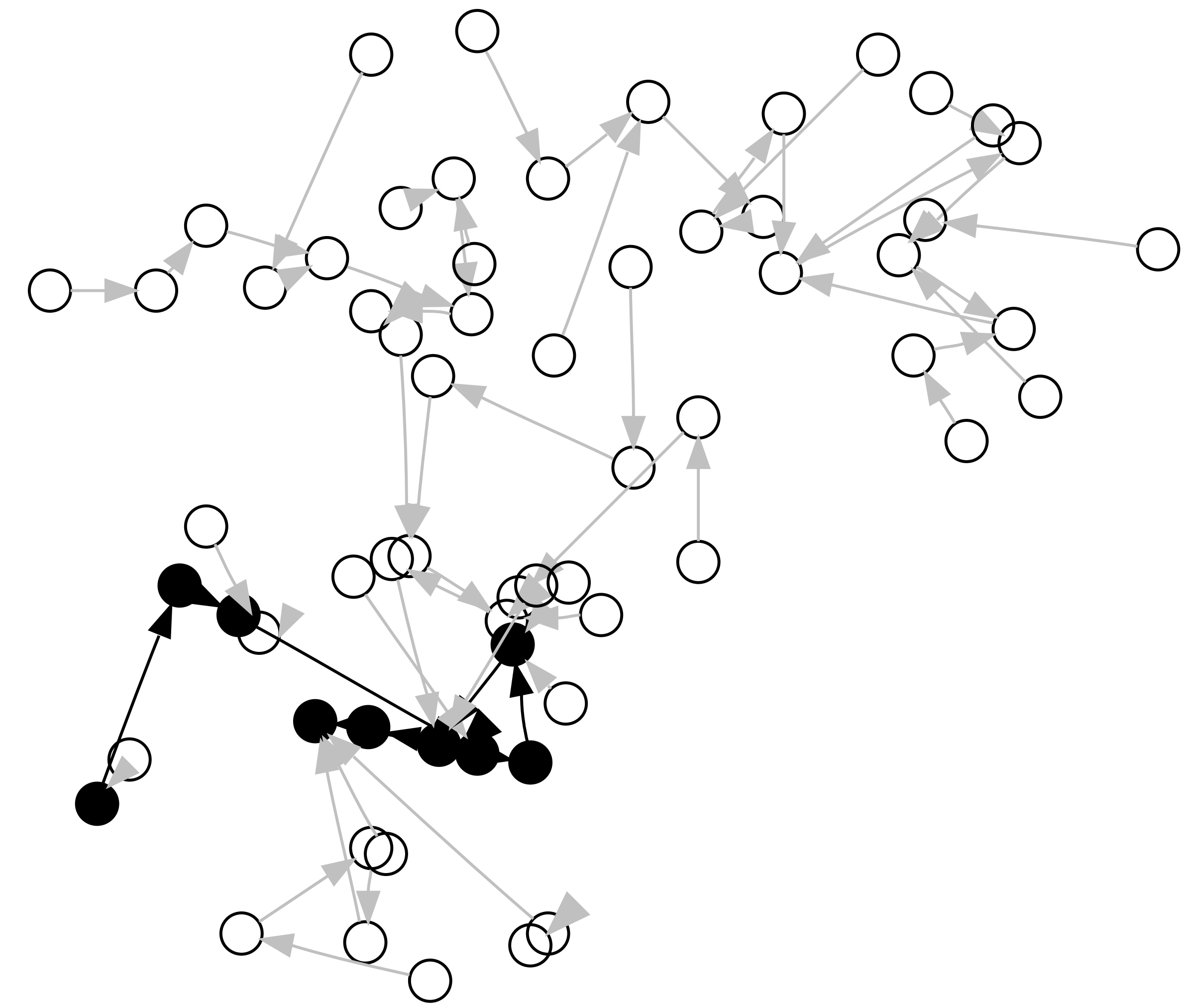

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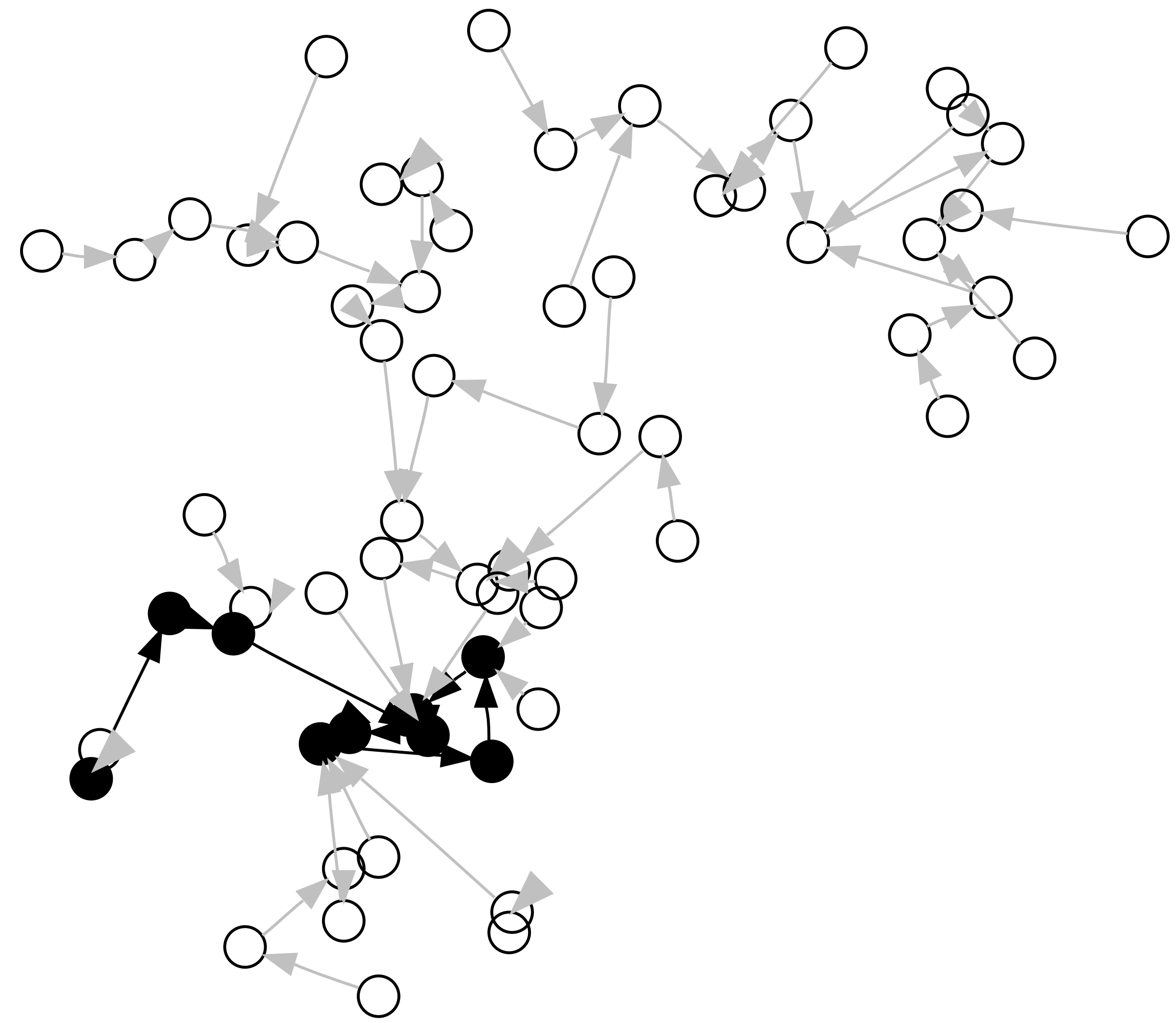

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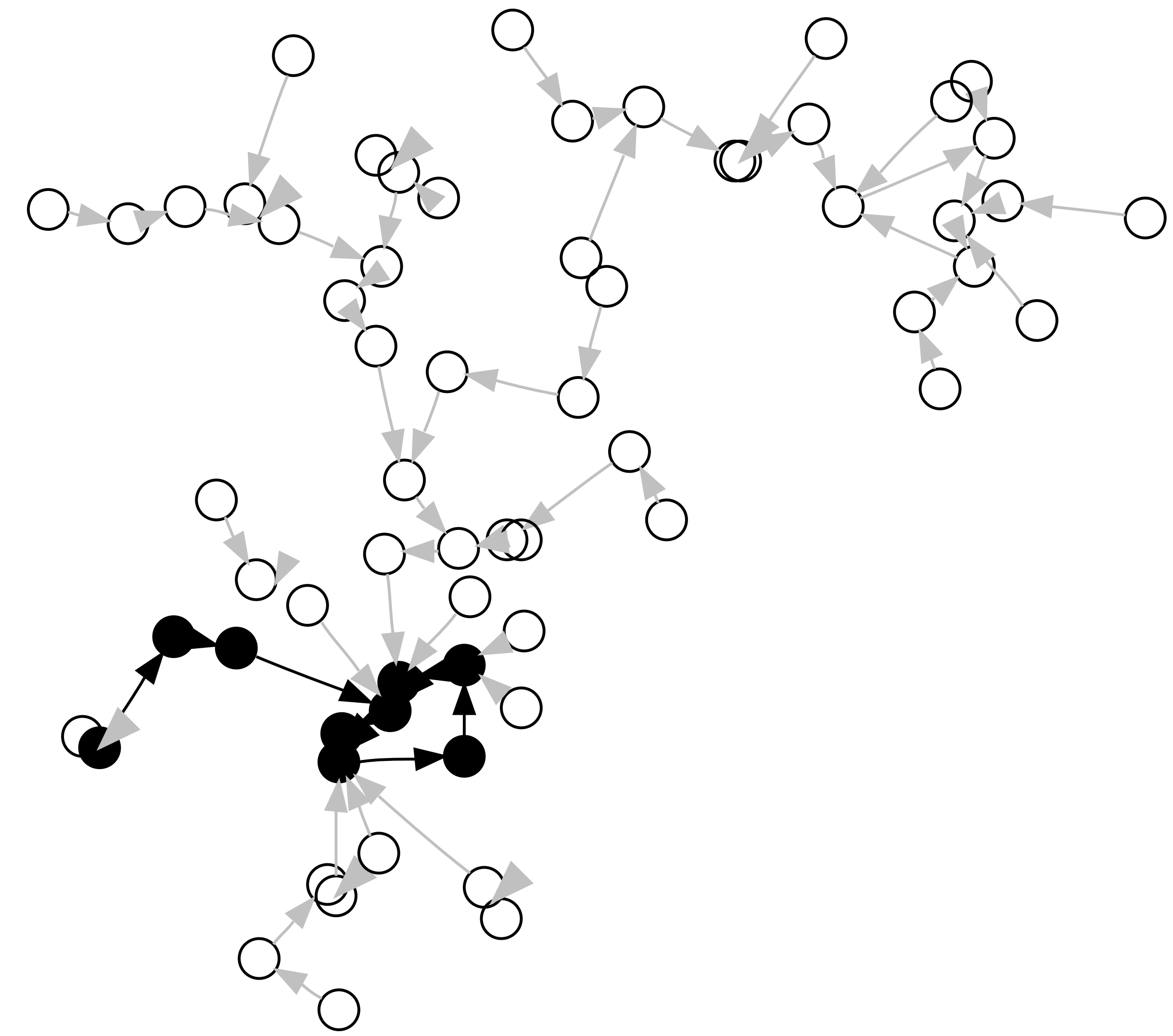

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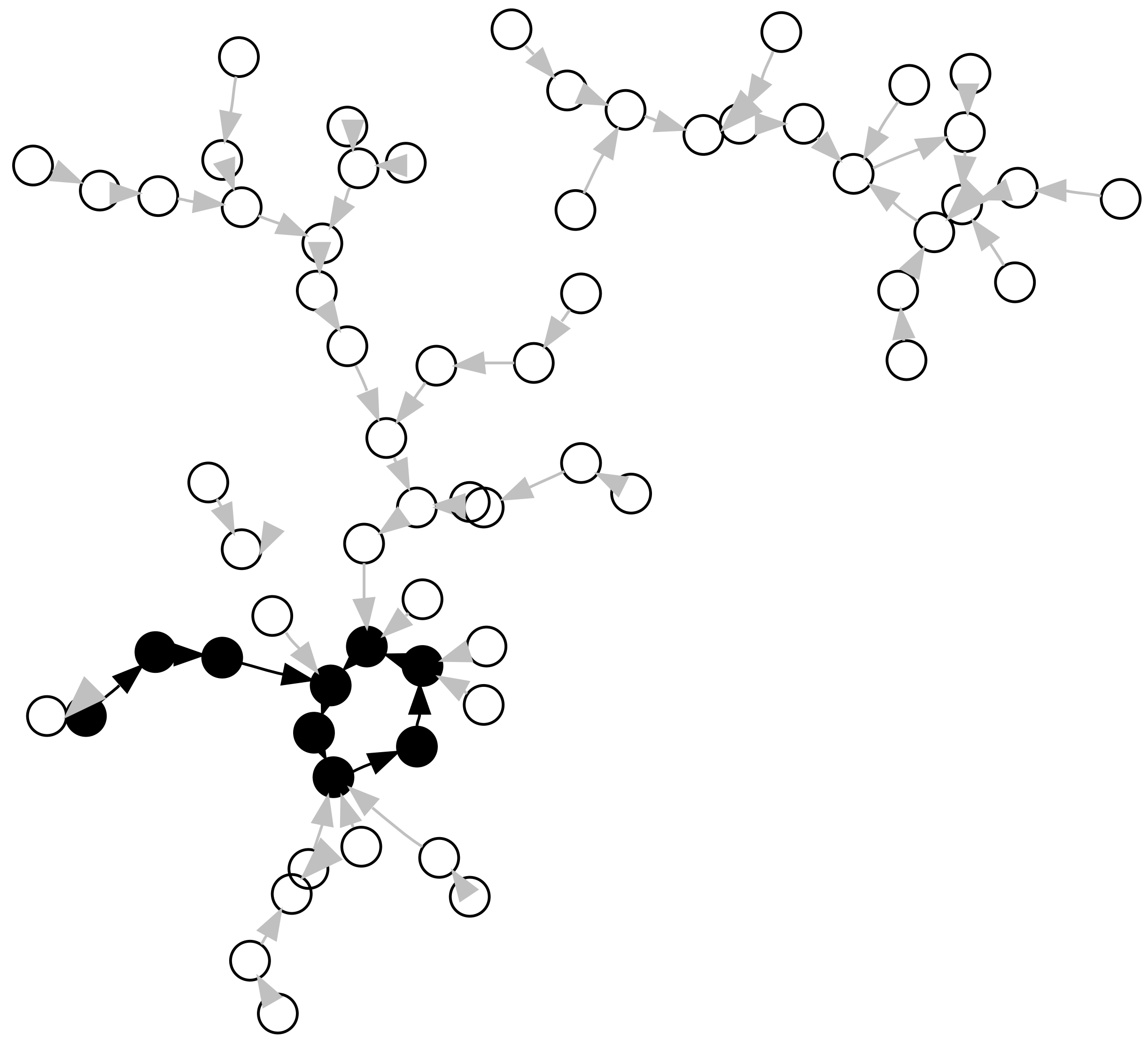

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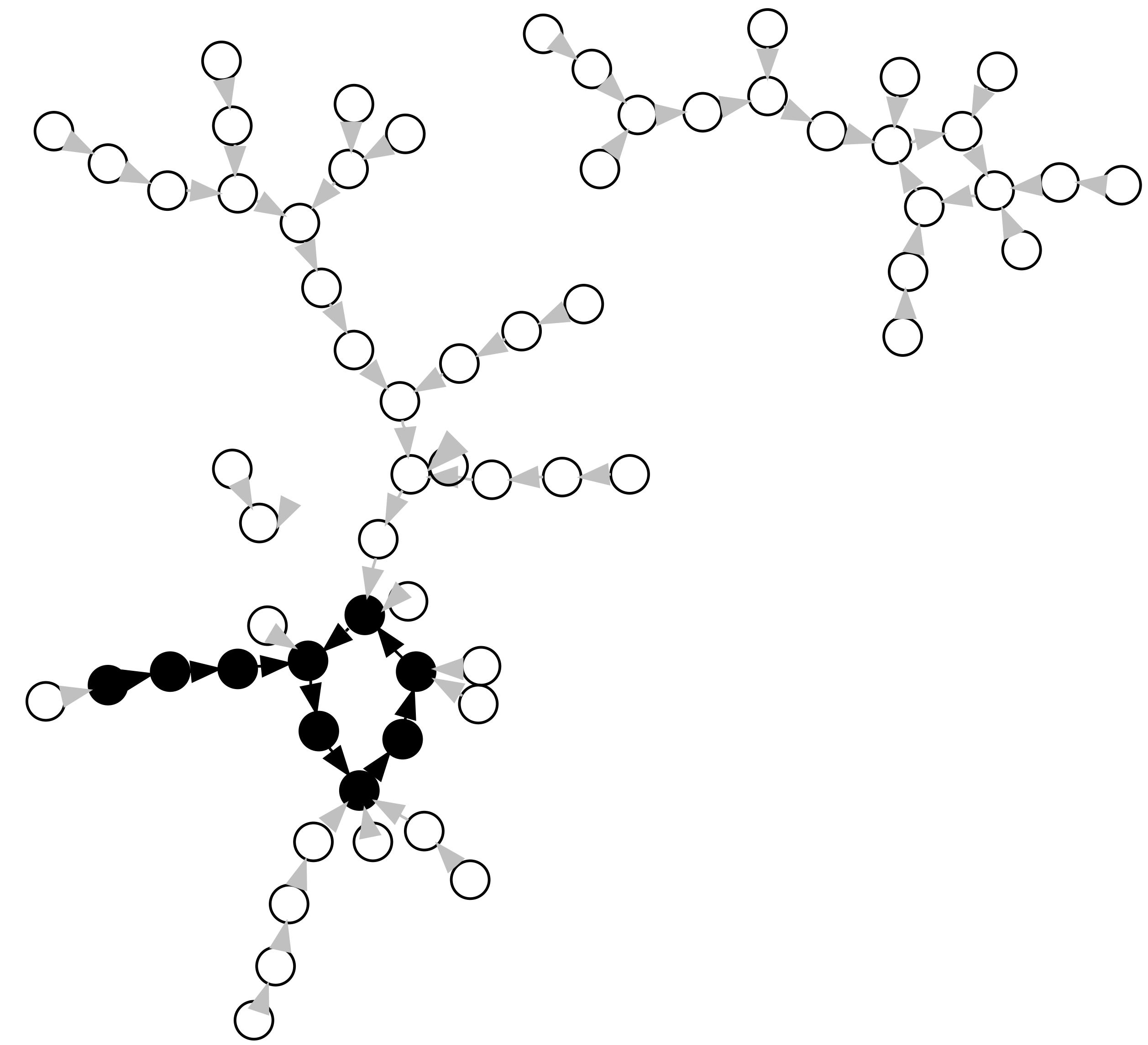

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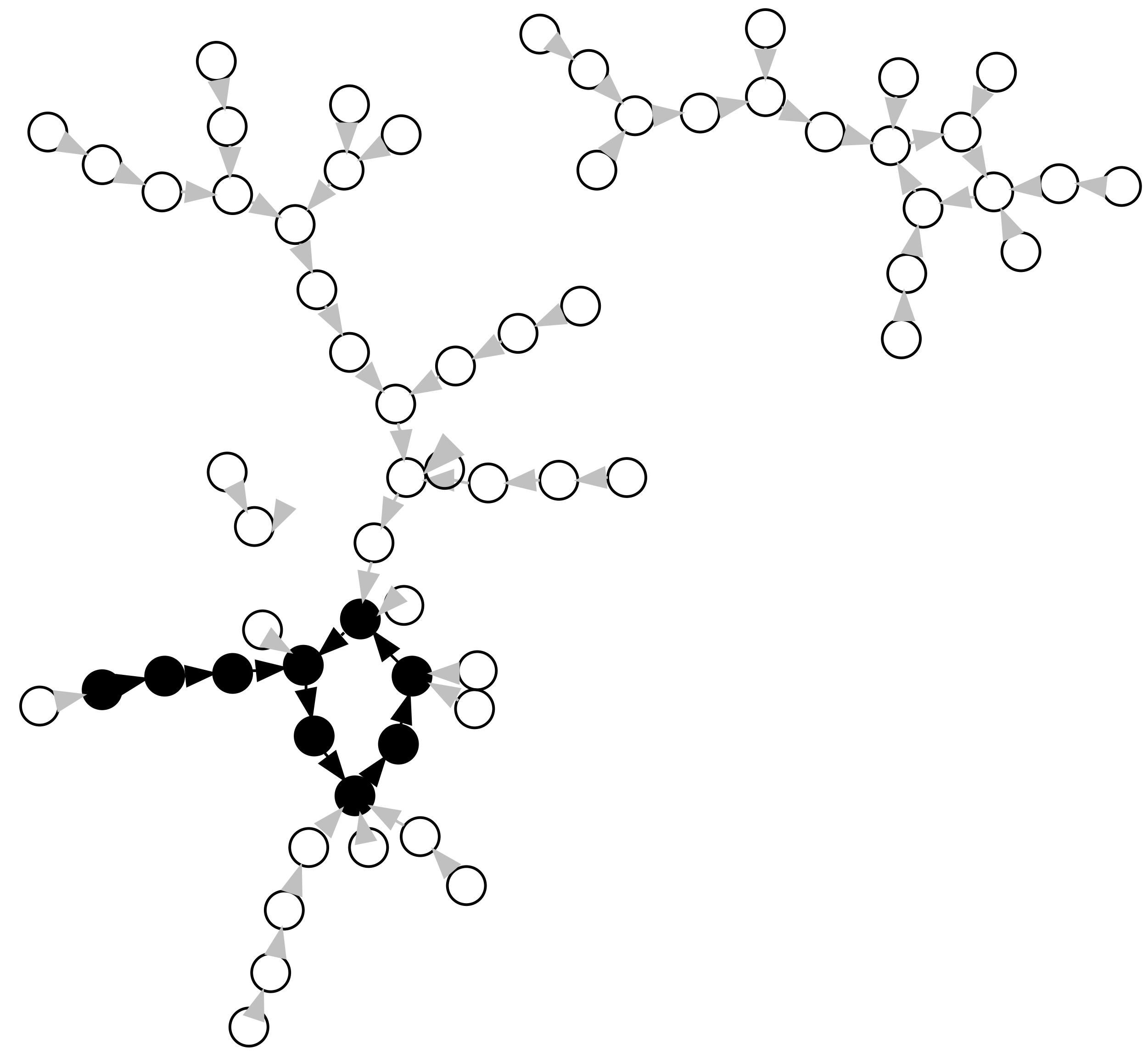

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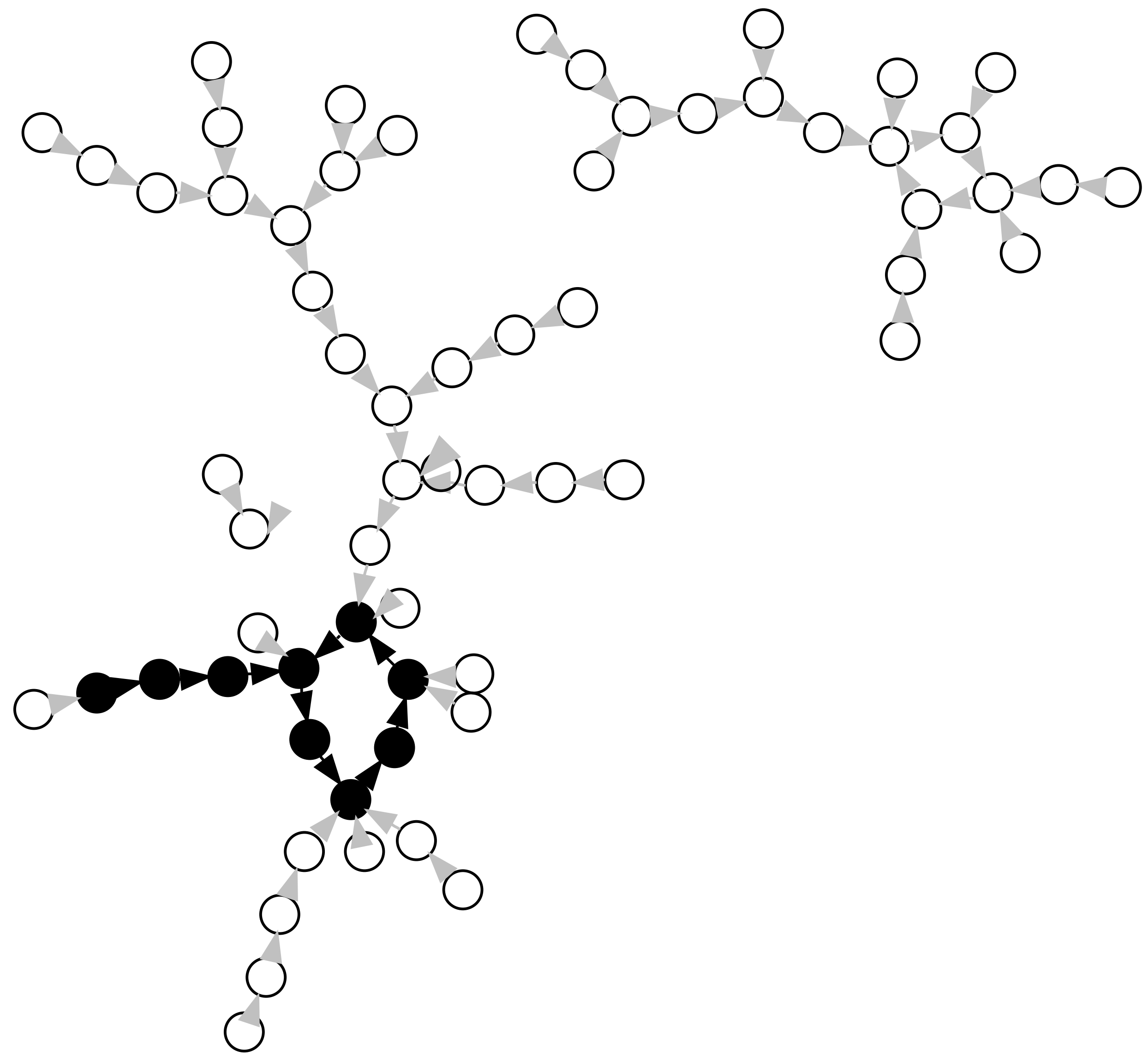

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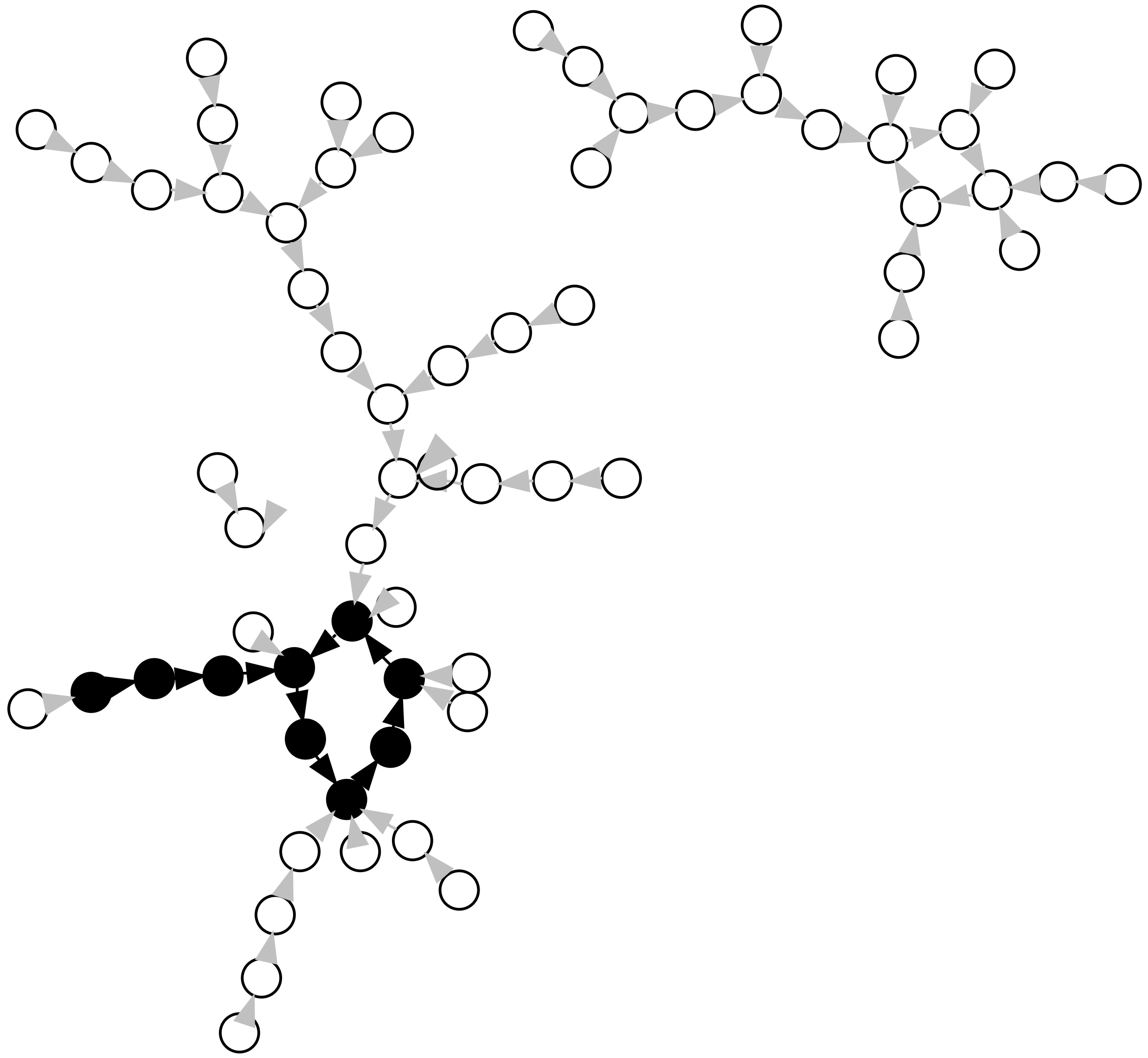

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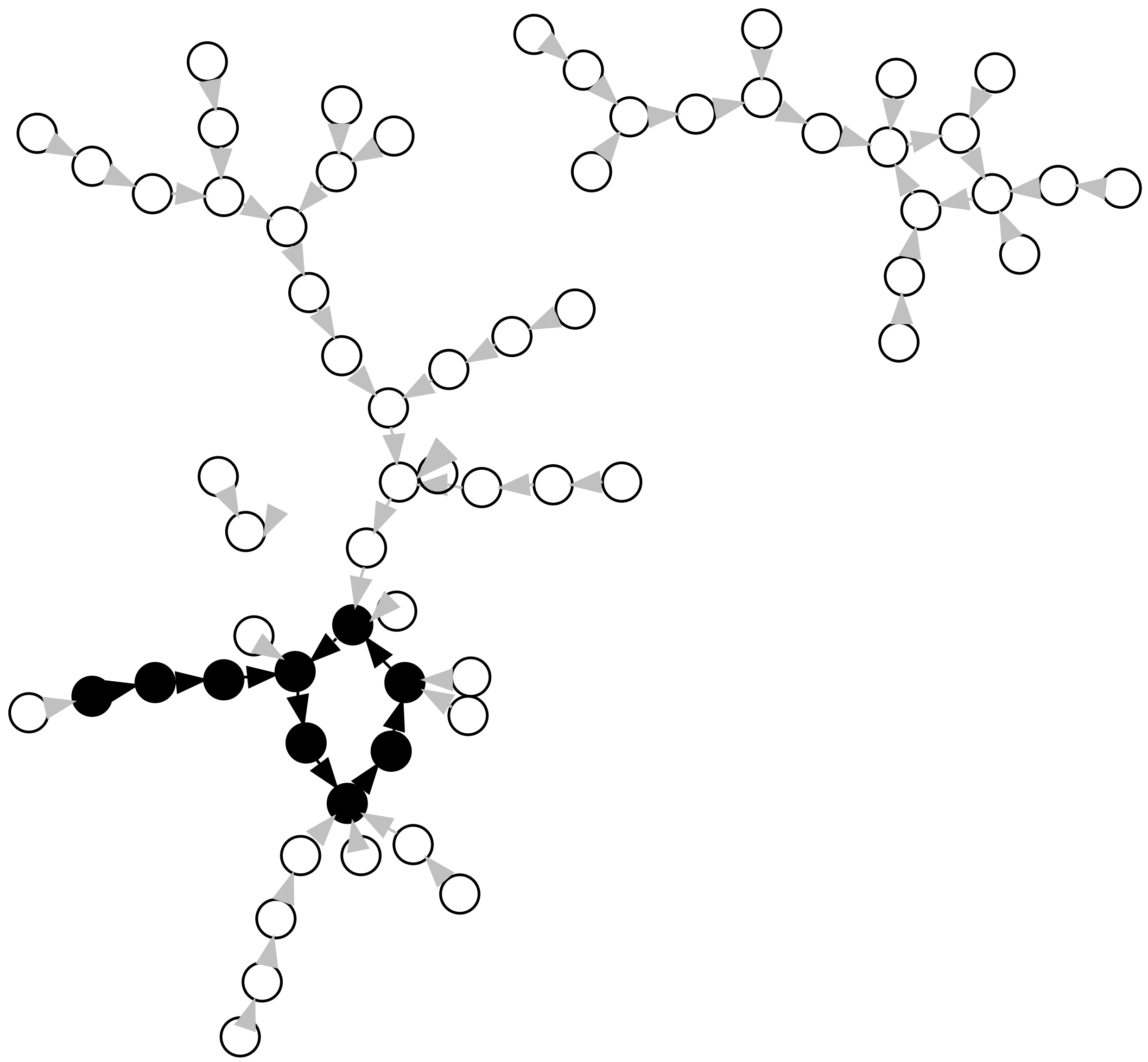

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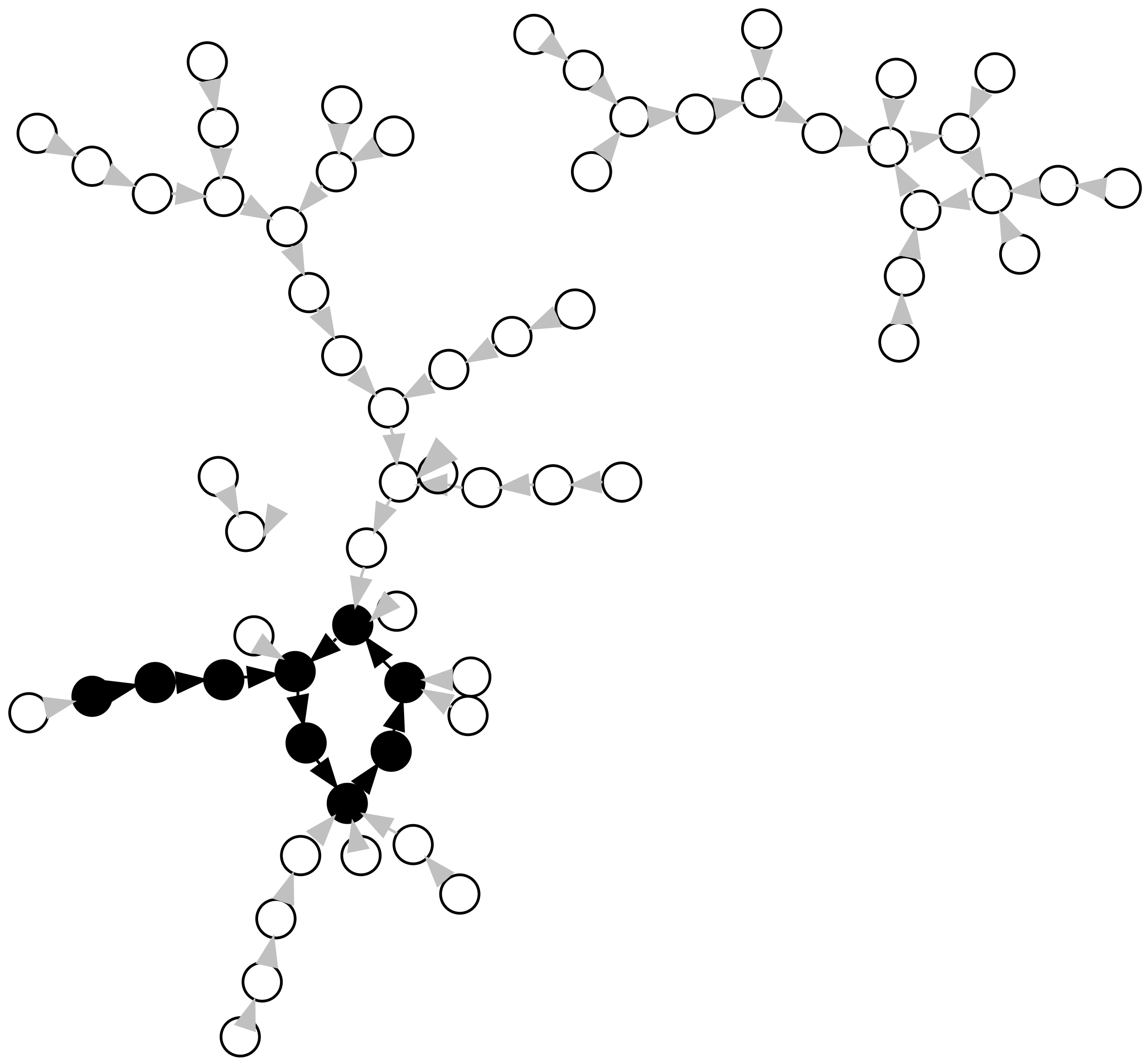

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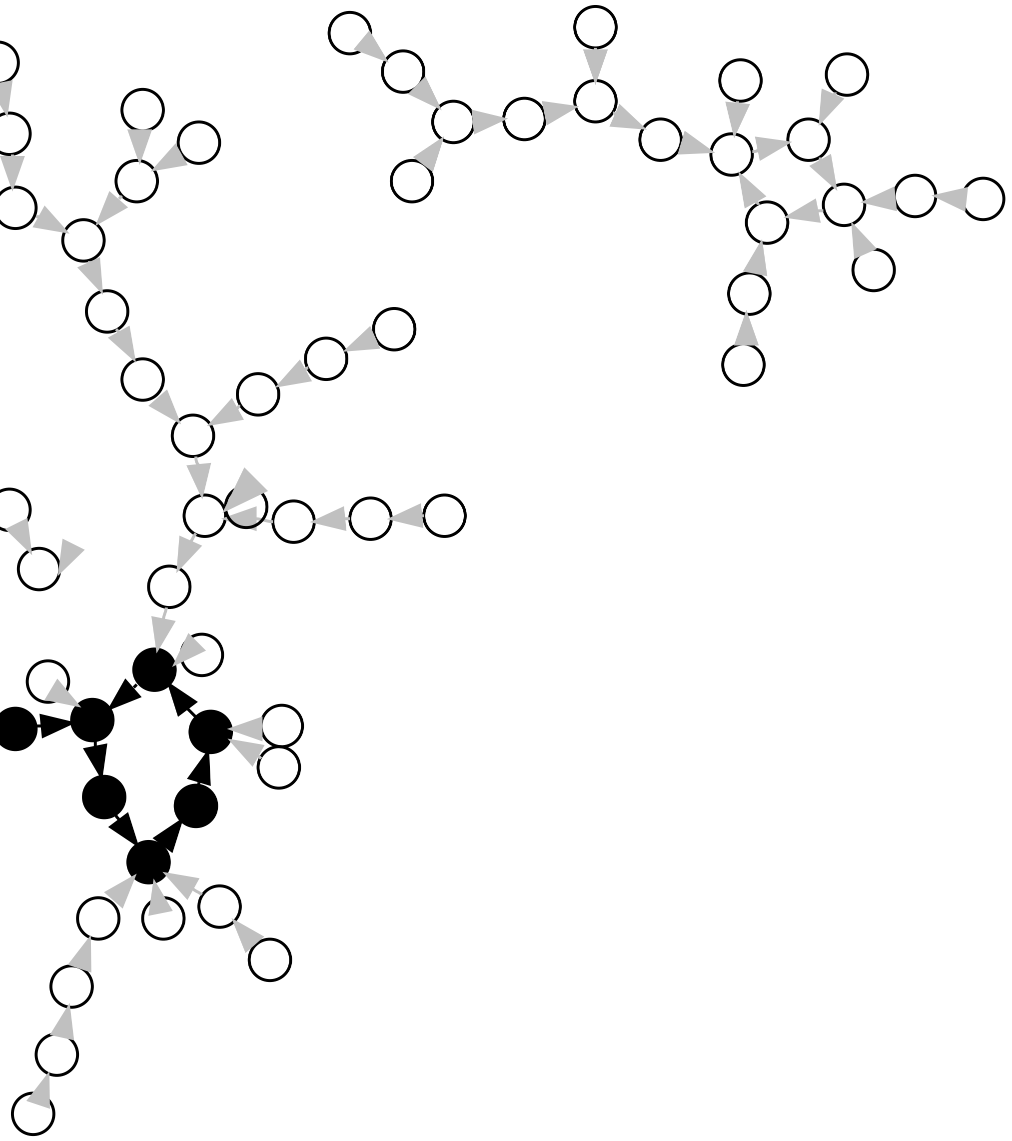

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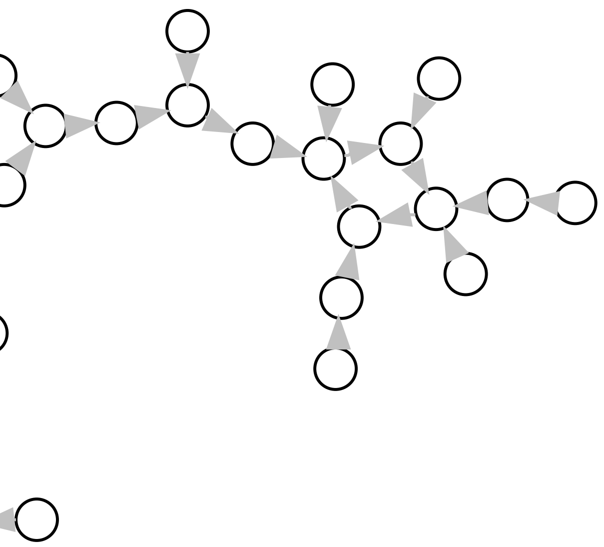

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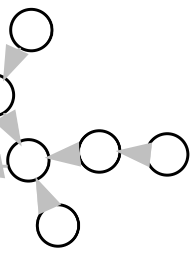

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In general, Pollard's rho method can be combined with any easily computed group automorphism of small order.

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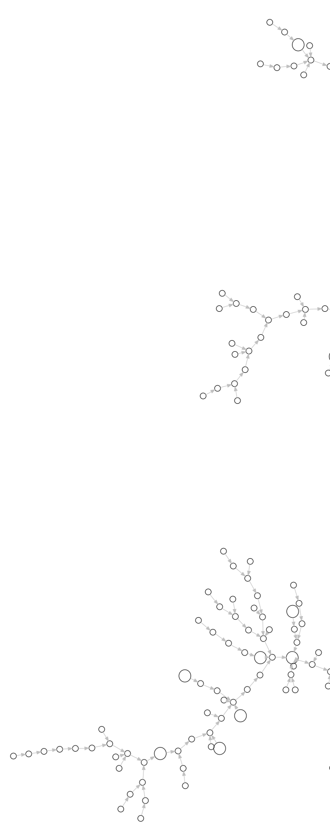

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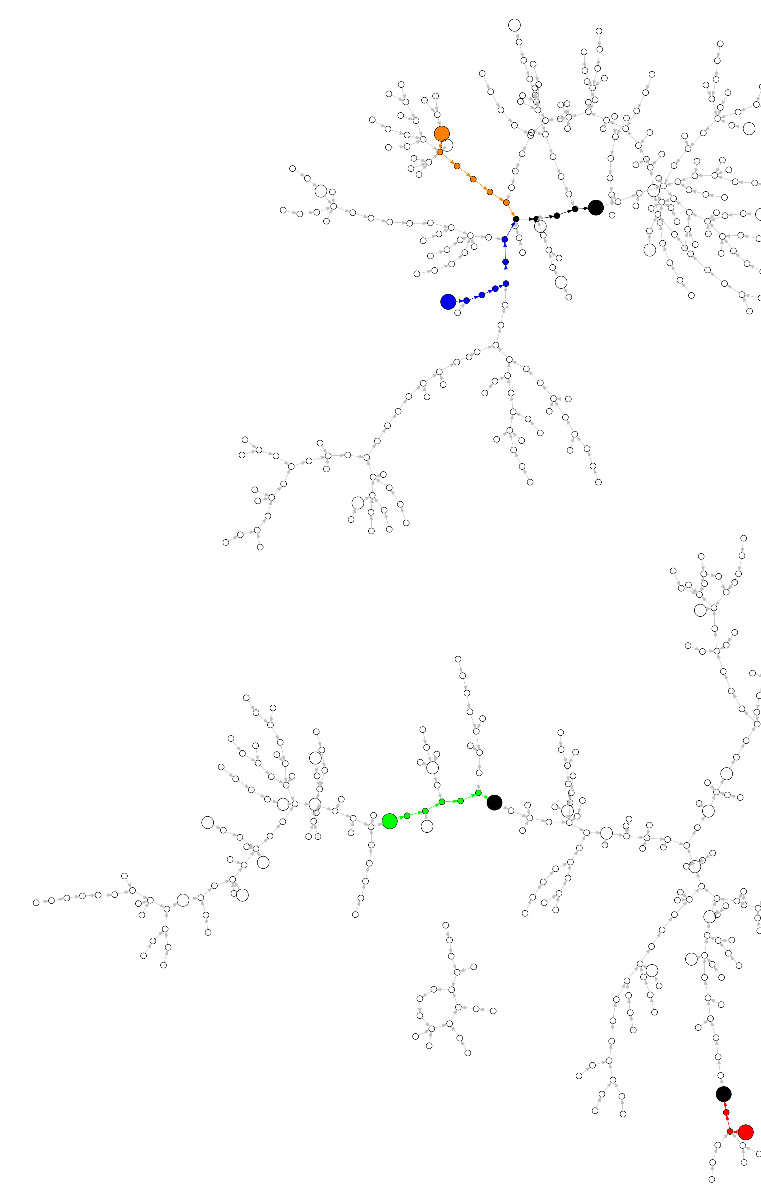

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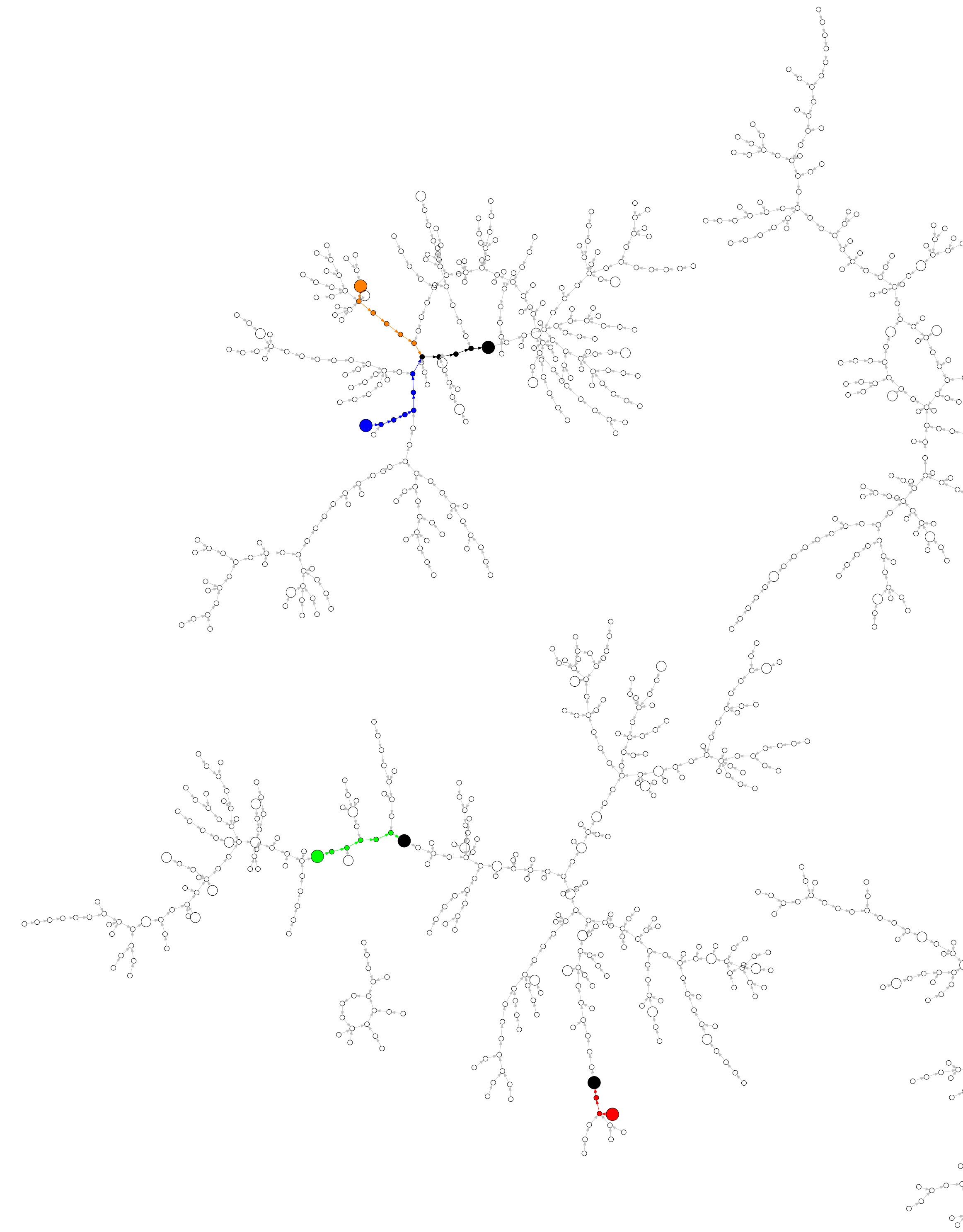

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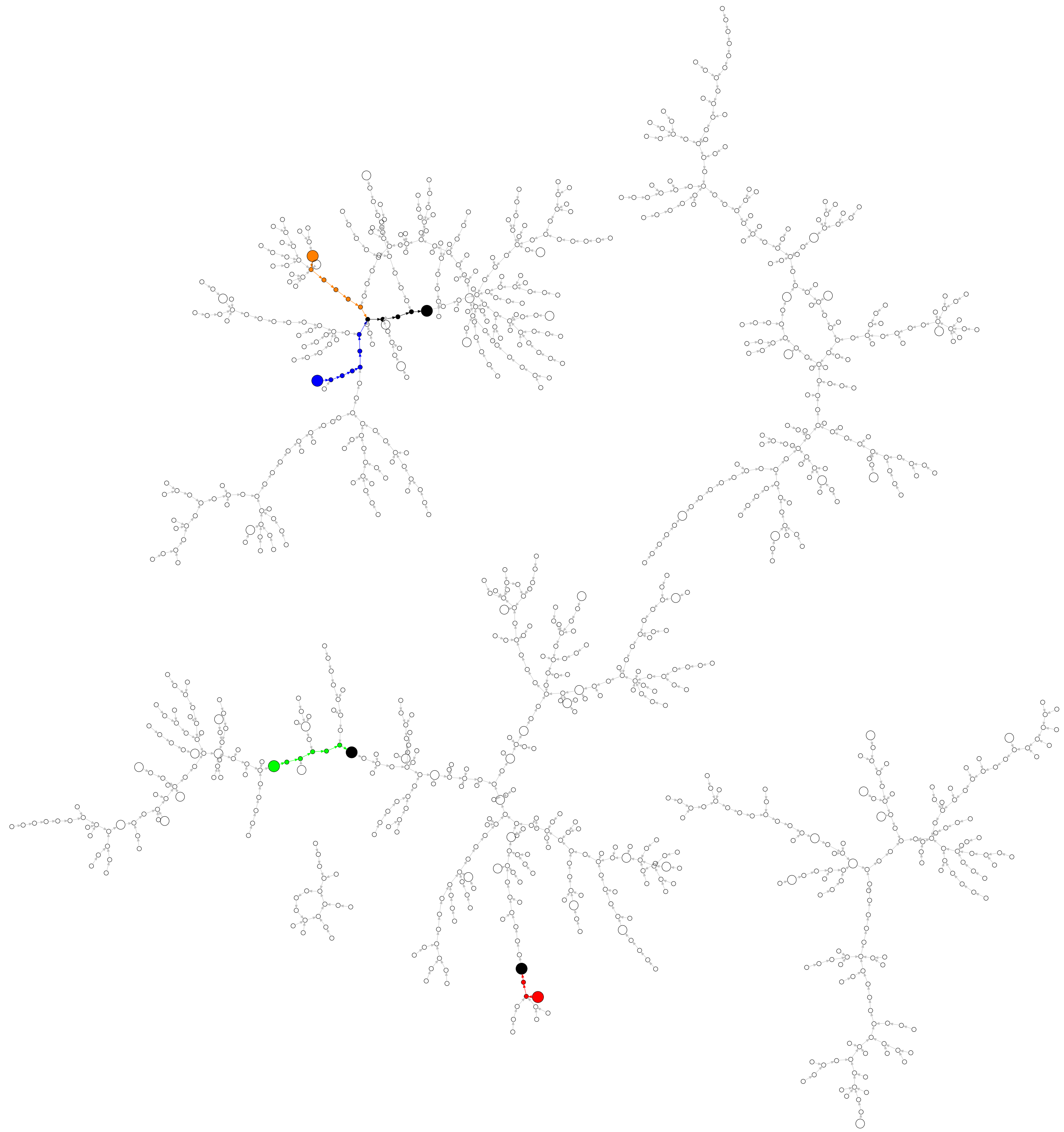

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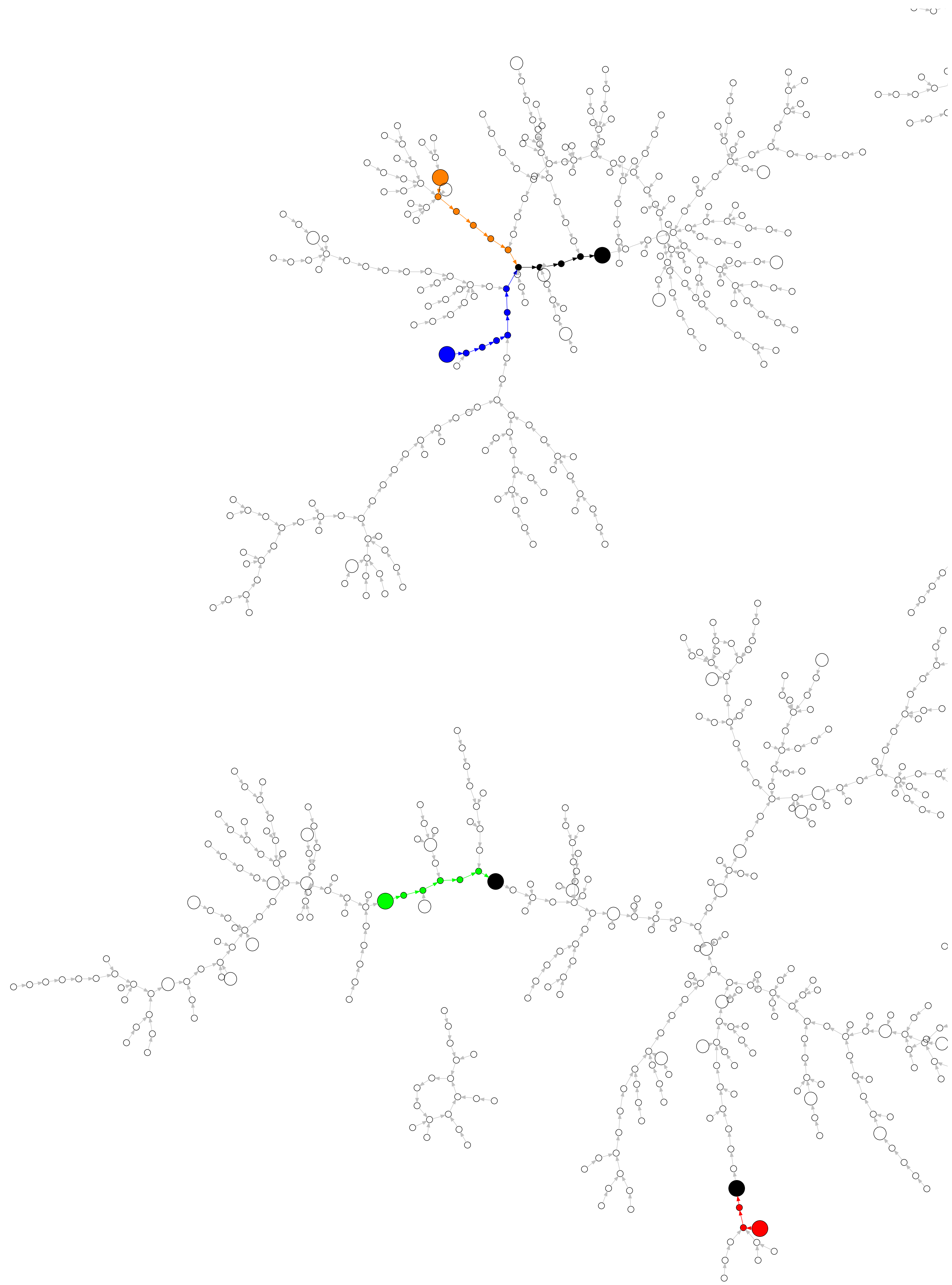

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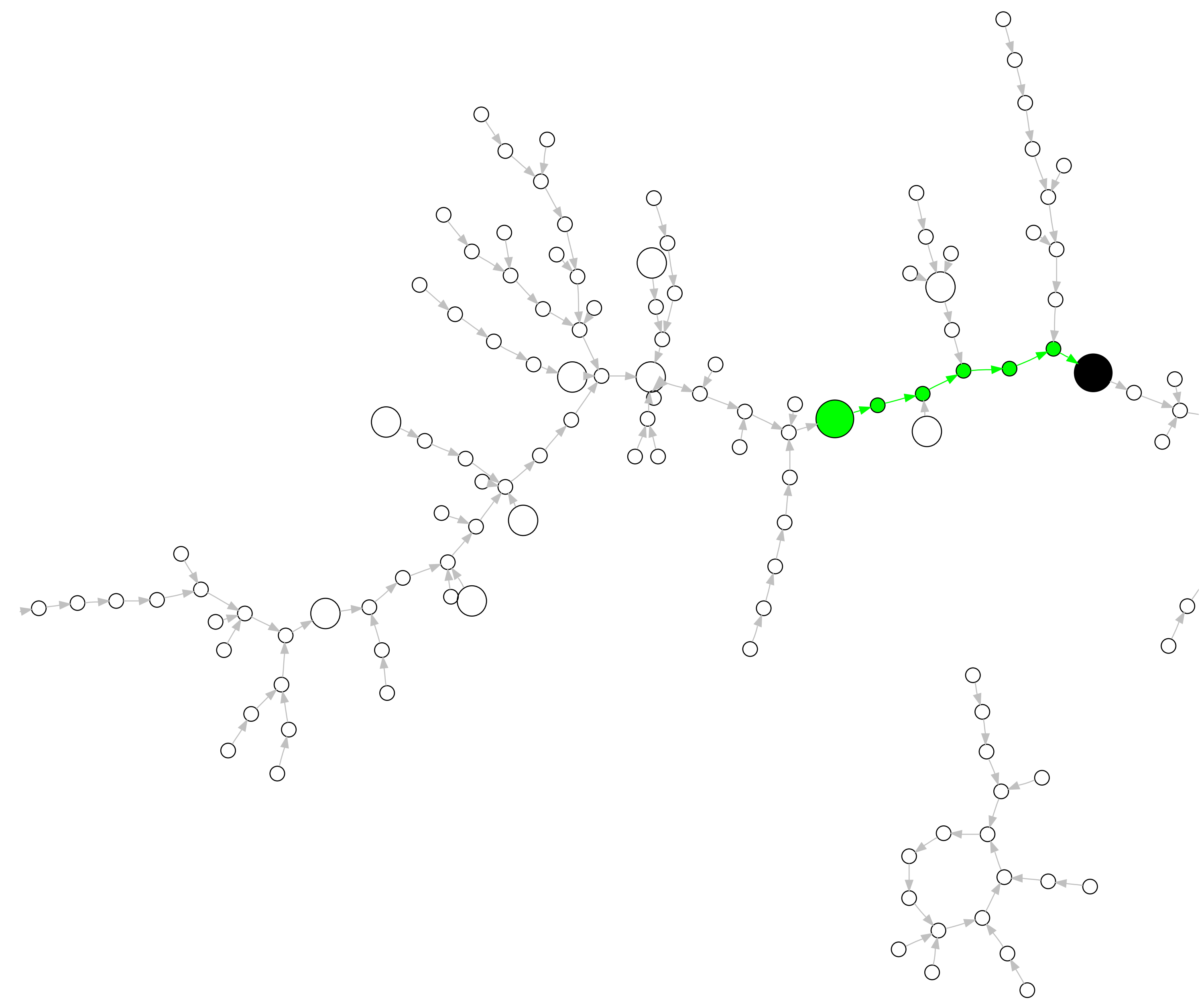

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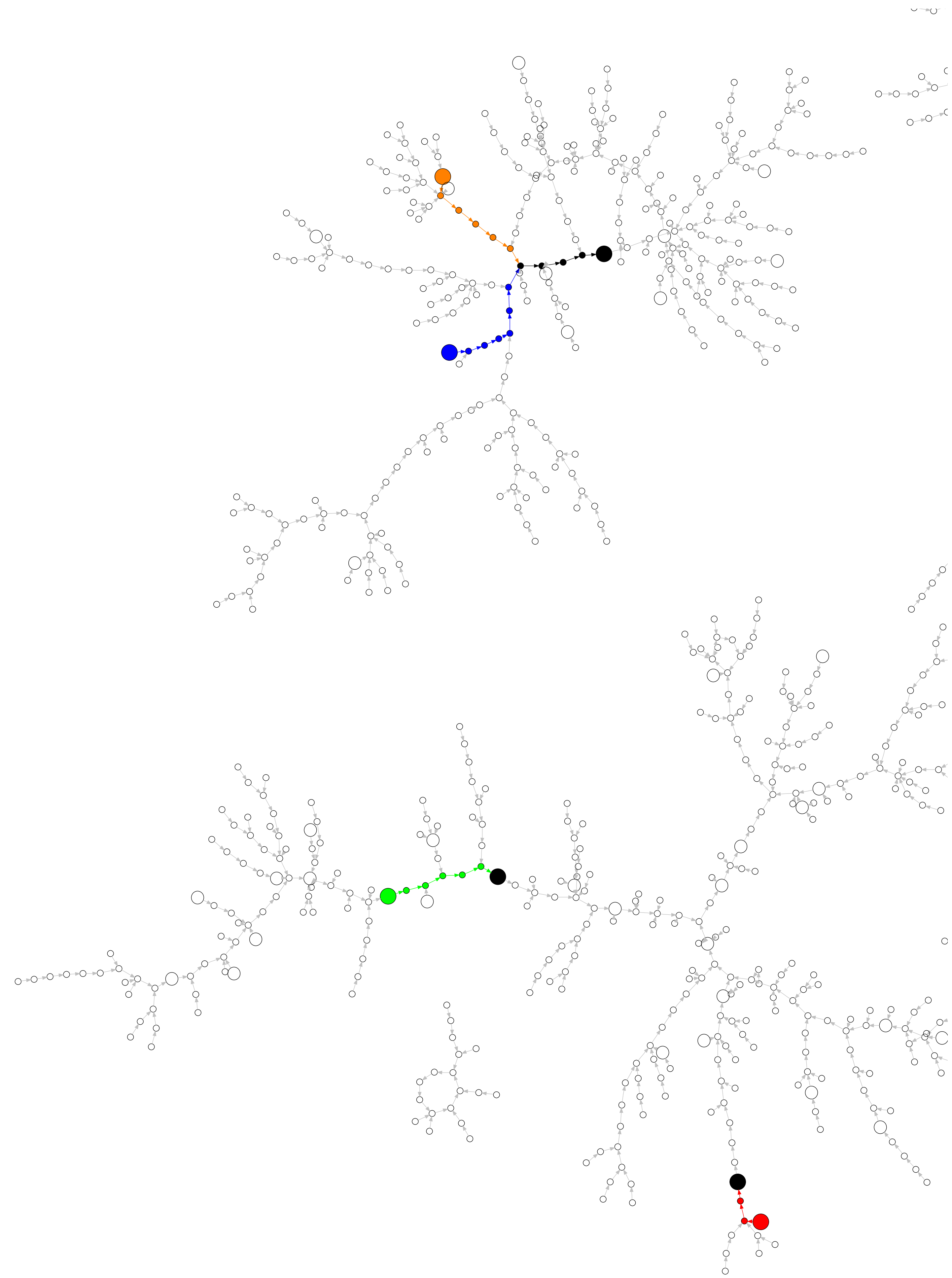

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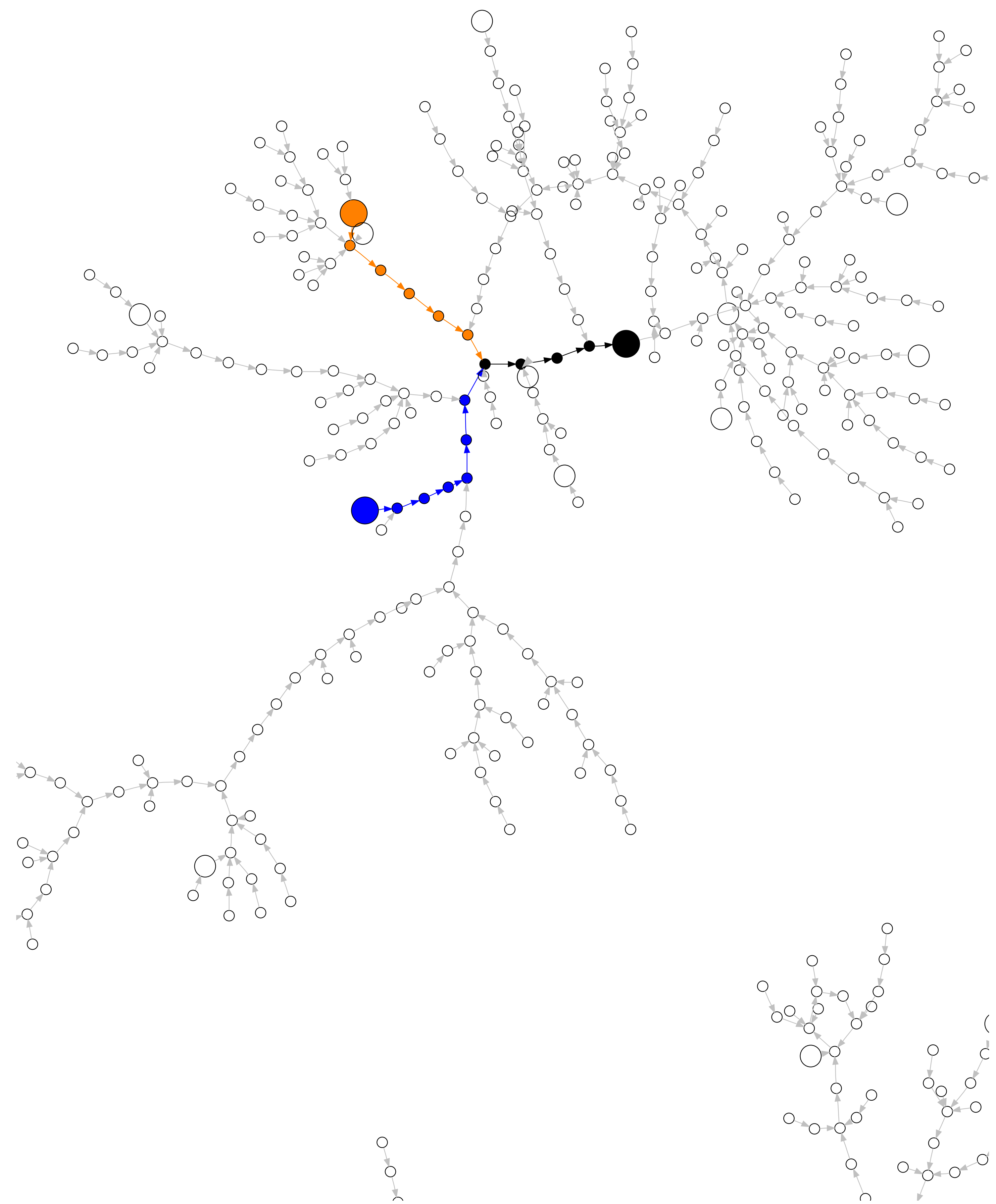

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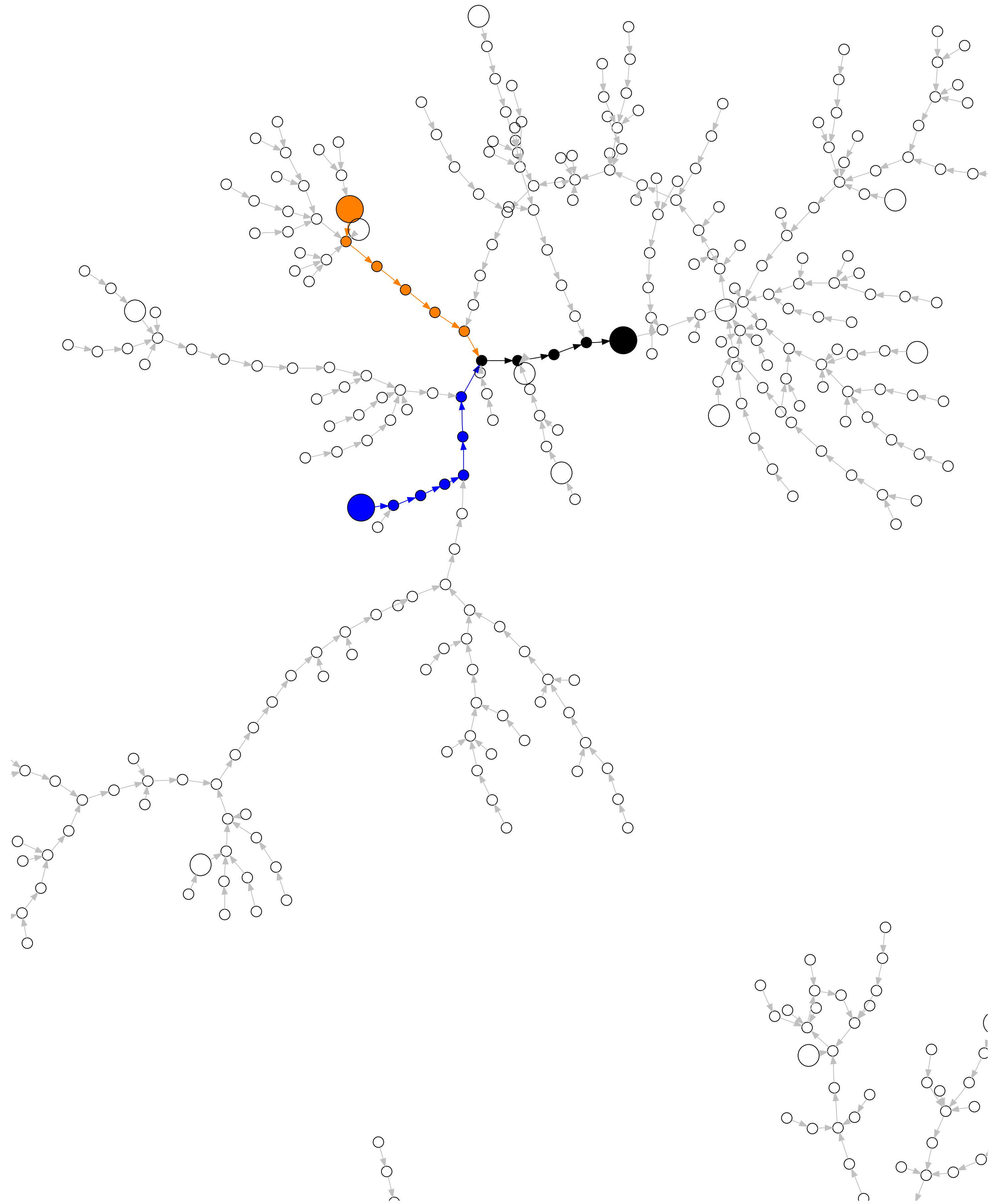

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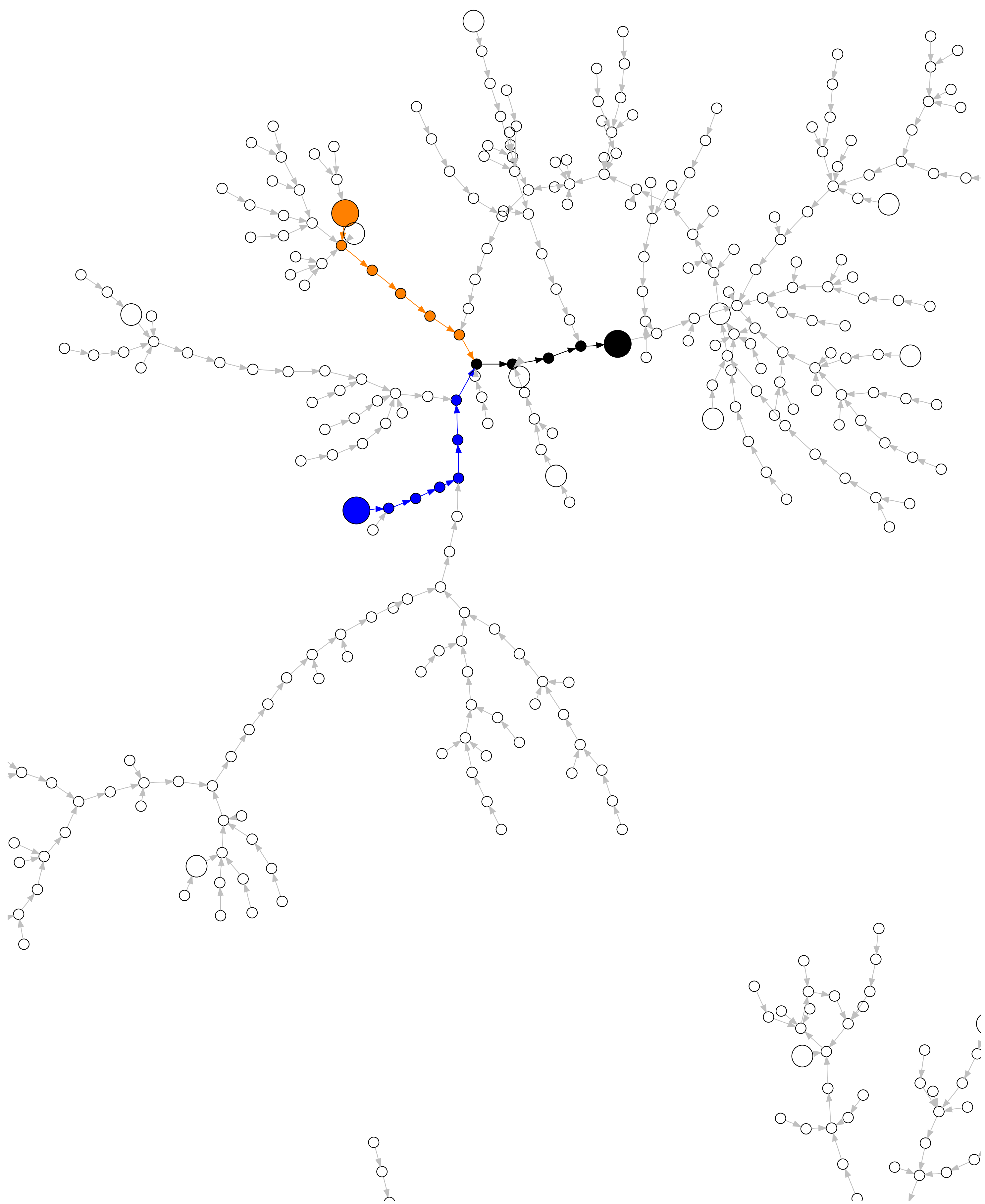

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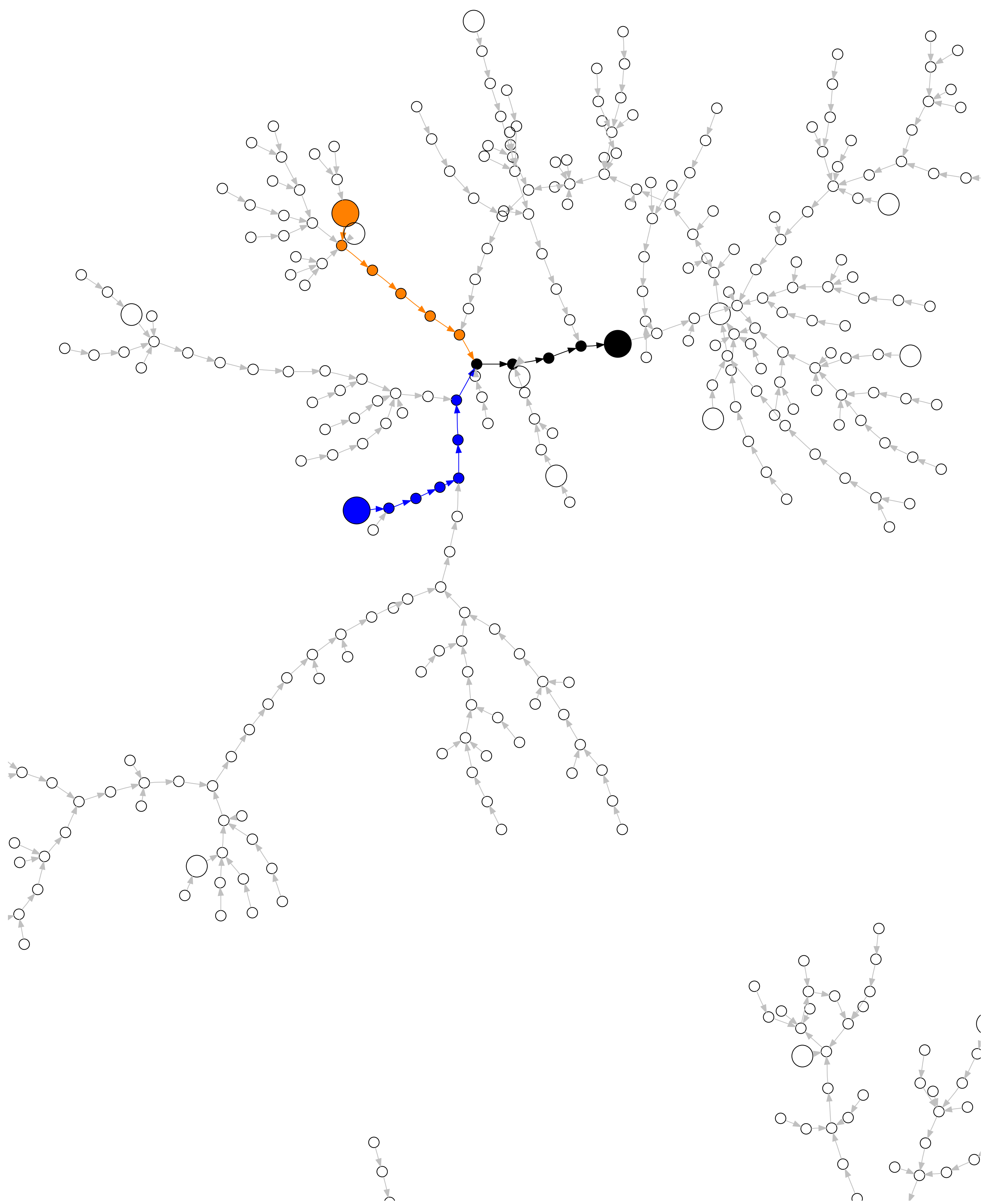

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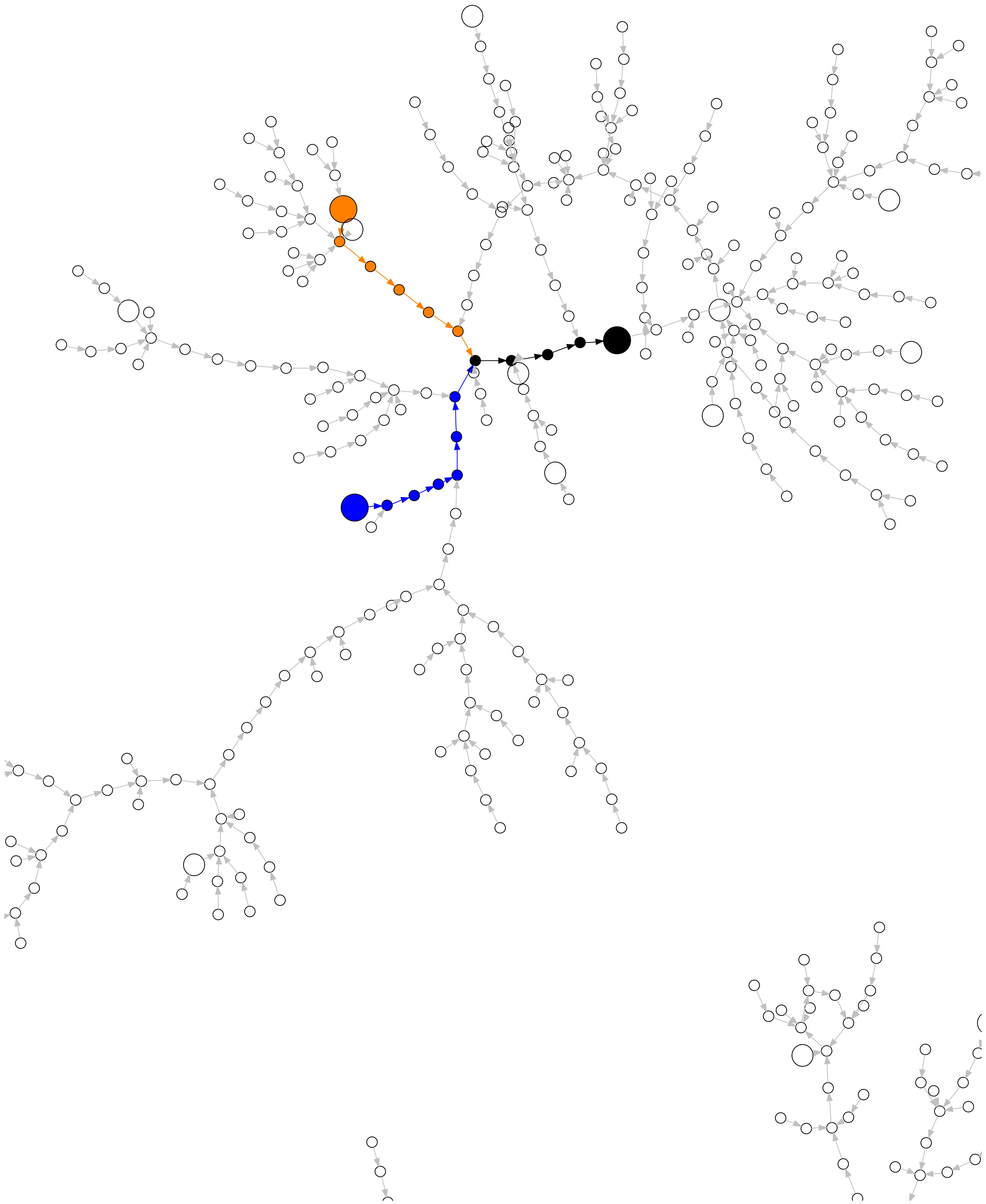


Attacker chooses frequency and definition of distinguished points.

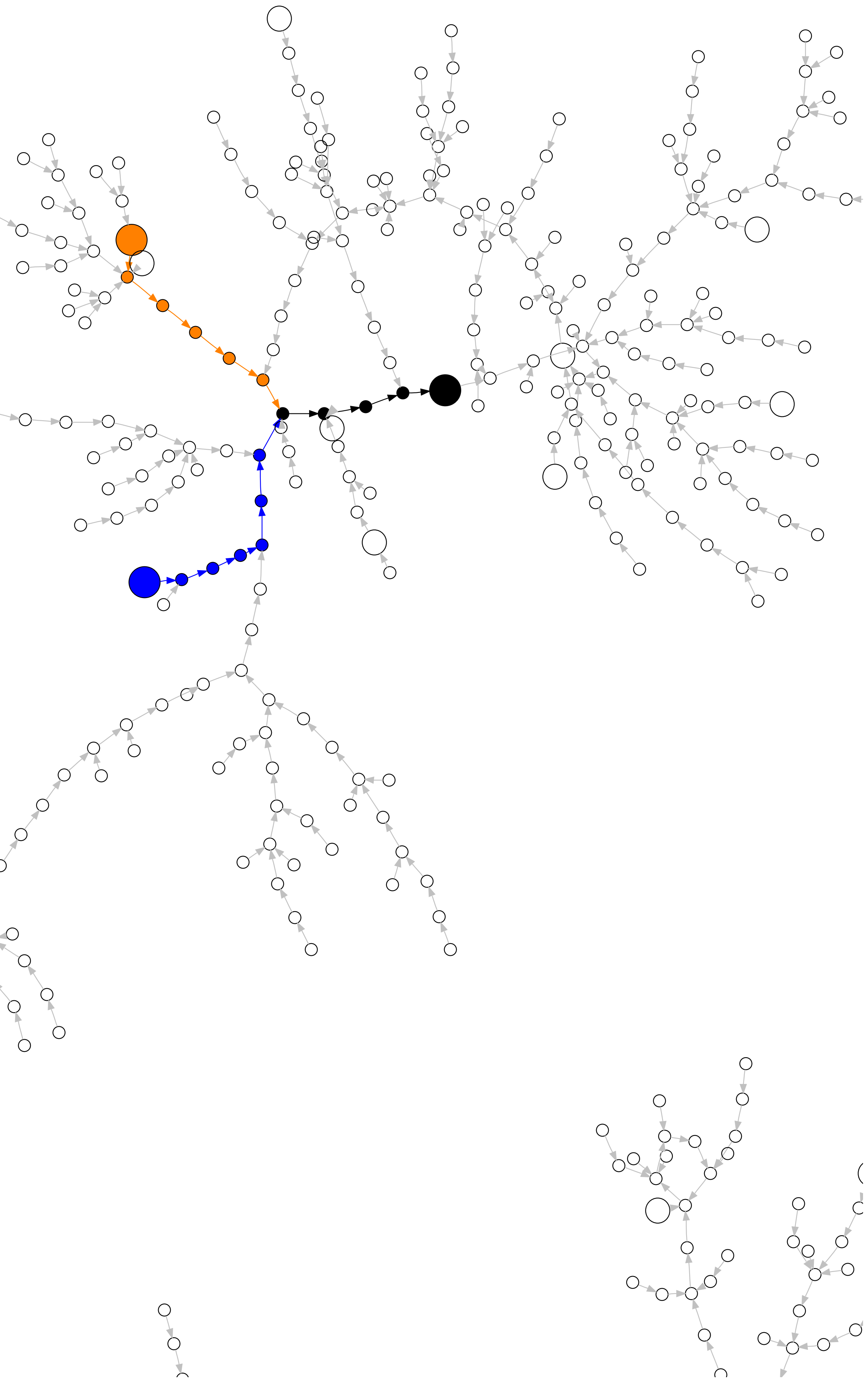
Tradeoffs are possible:

If distinguished points are rare, a small number of very long walks will be performed. This reduces the number of distinguished points sent to the server but increases the delay before a collision is recognized.

If distinguished points are frequent, many shorter walks will be performed.

In any case do not wait for cycle.

Total # of iterations unchanged.



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If distinguished points are rare, a small number of very long walks will be performed. This reduces the number of distinguished points sent to the server but increases the delay before a collision is recognized.

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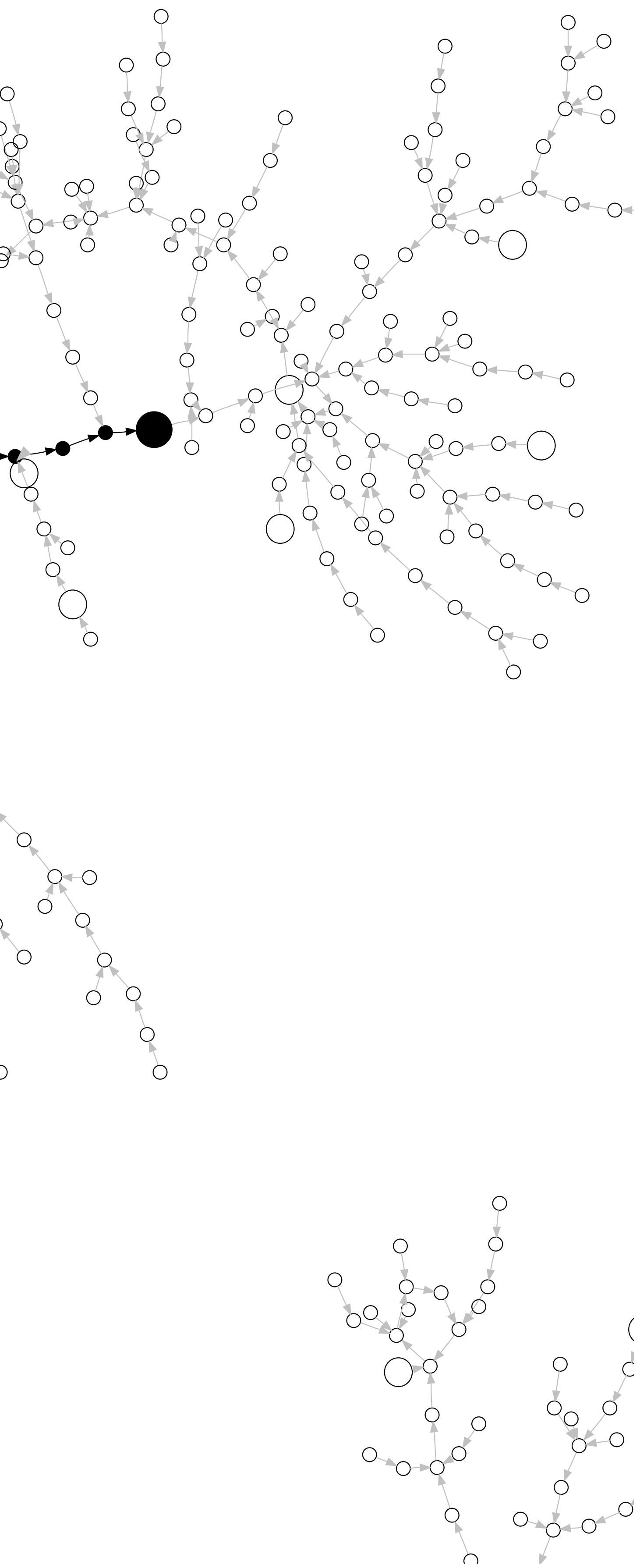

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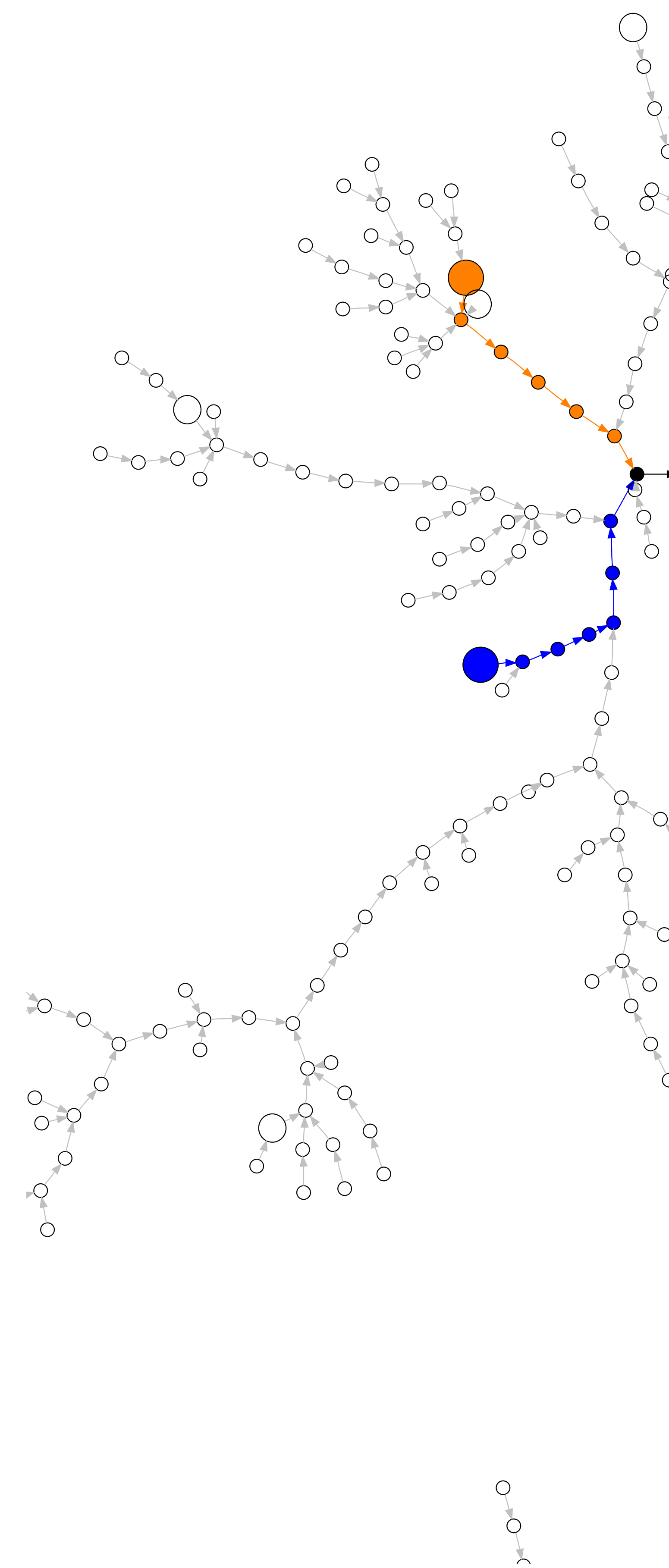

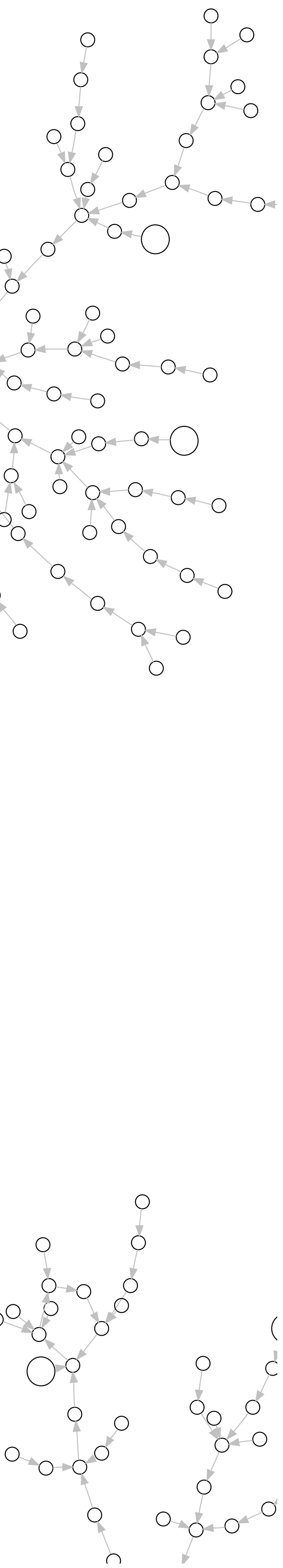

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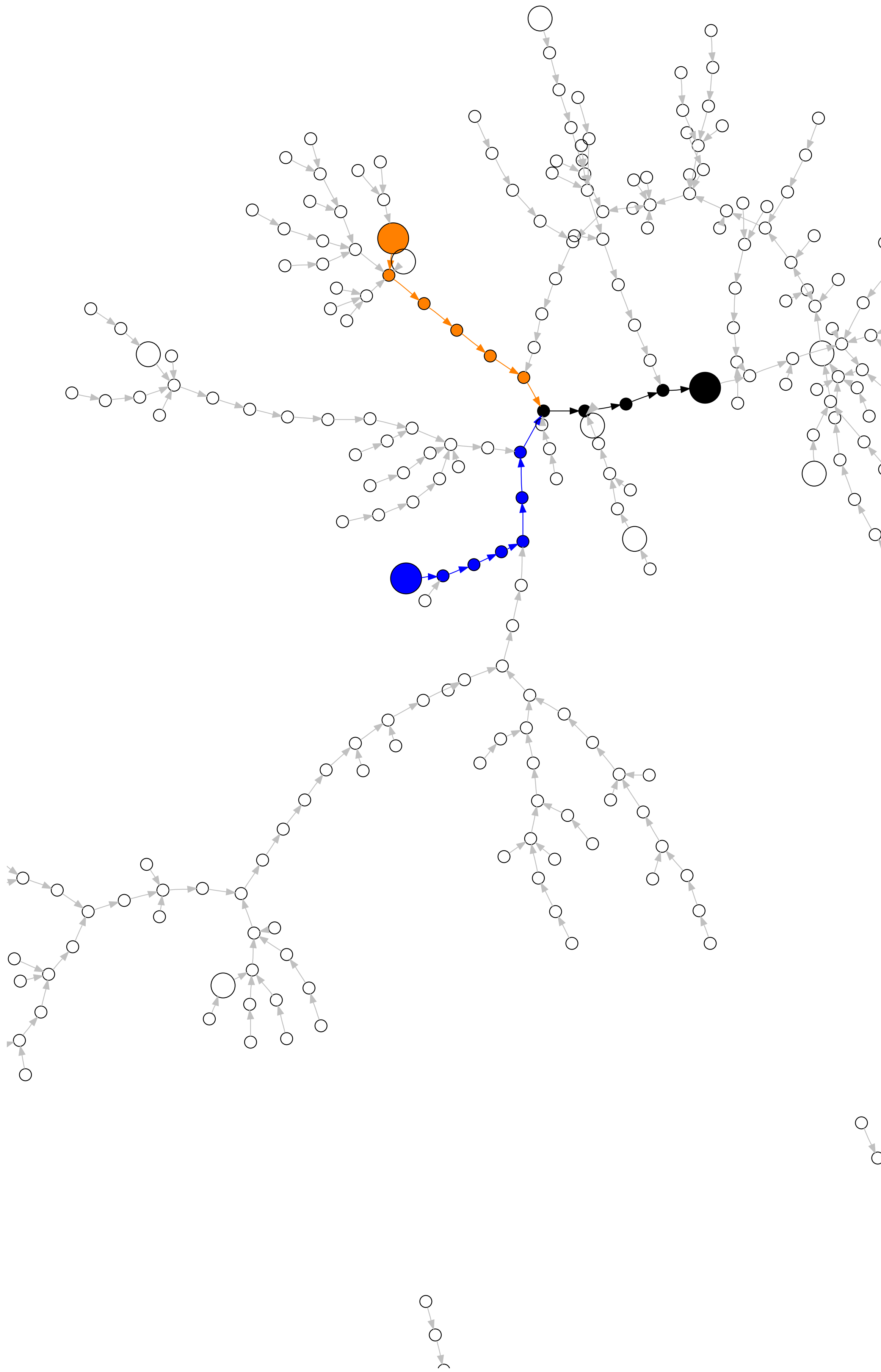

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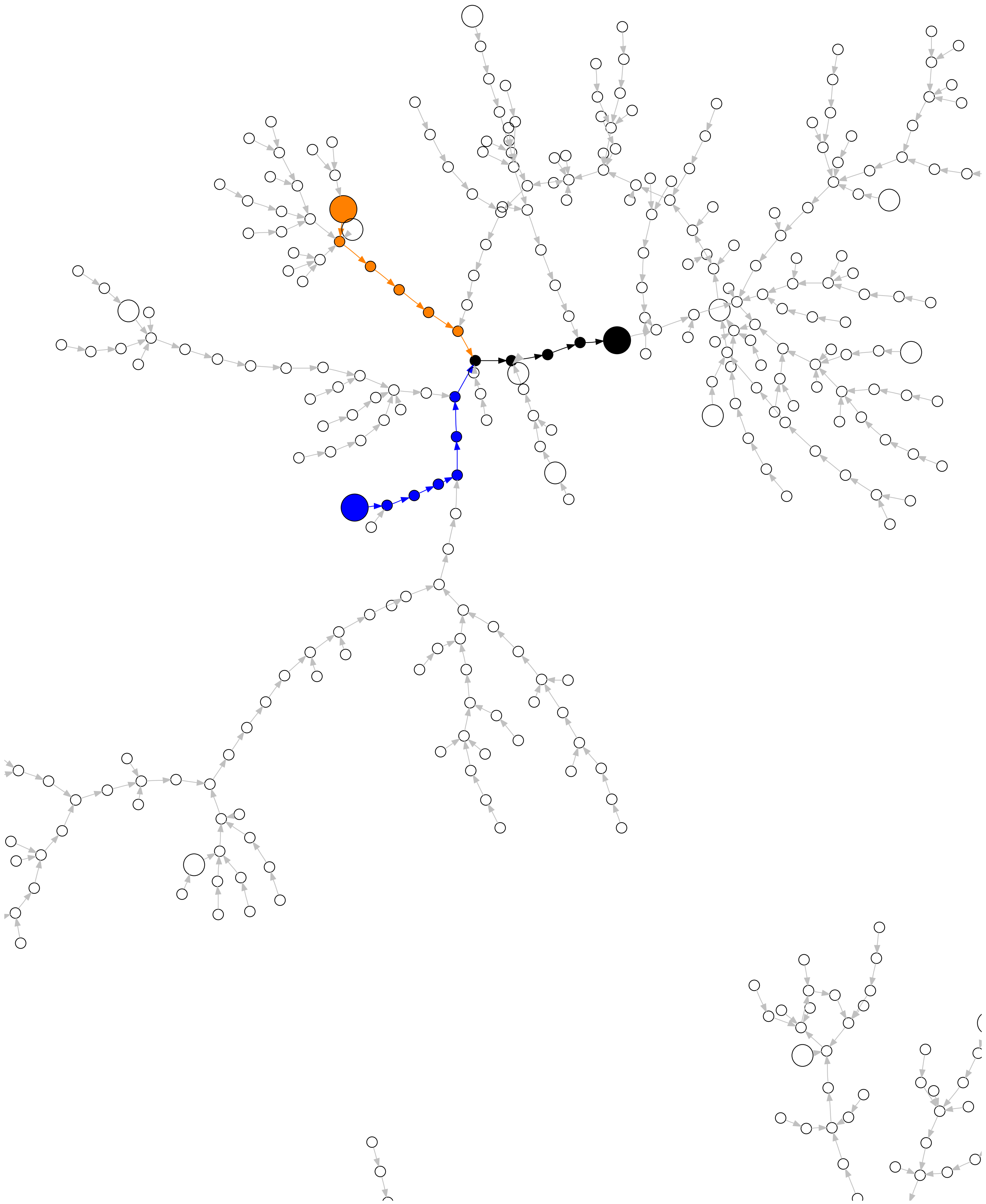

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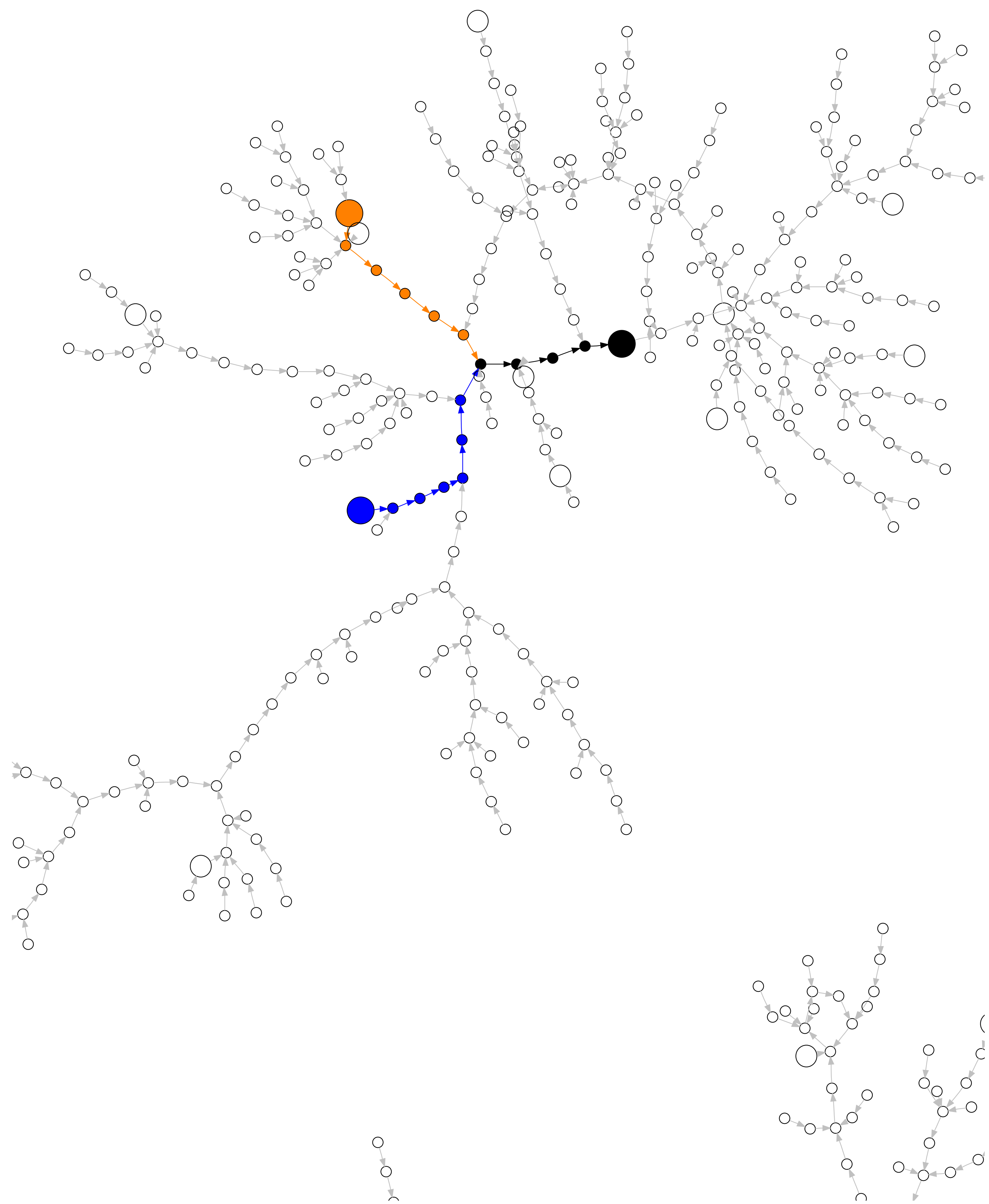

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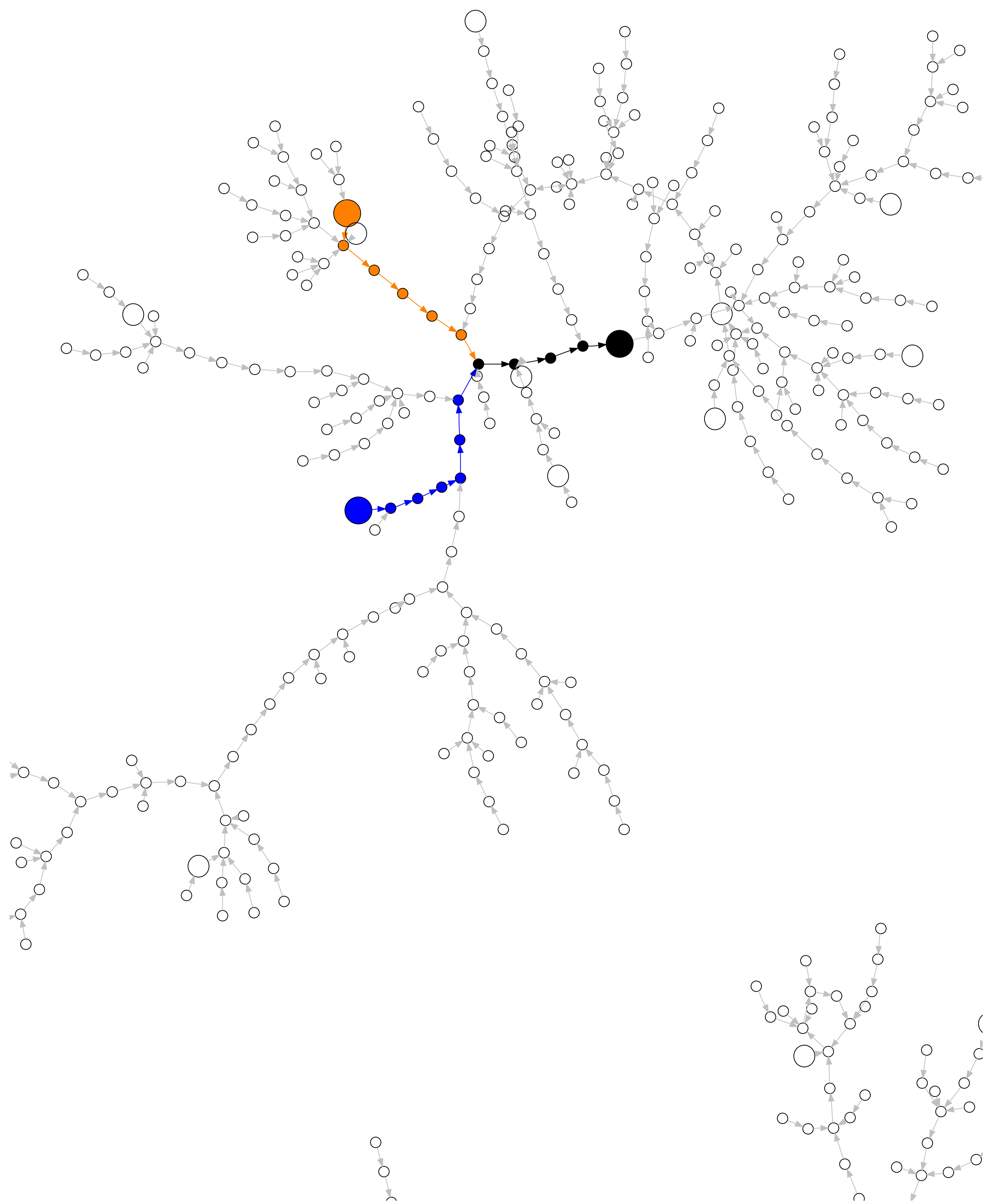

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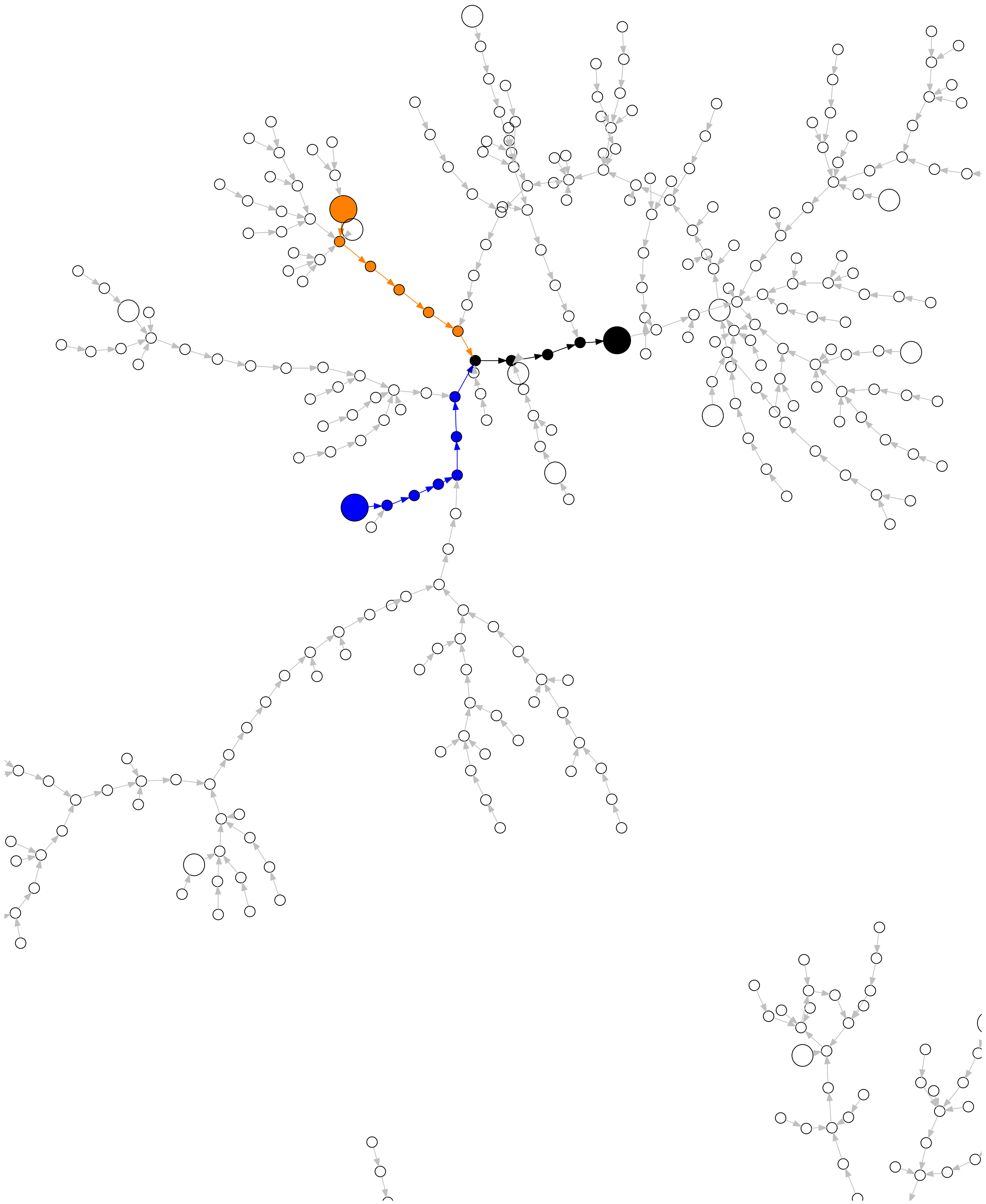

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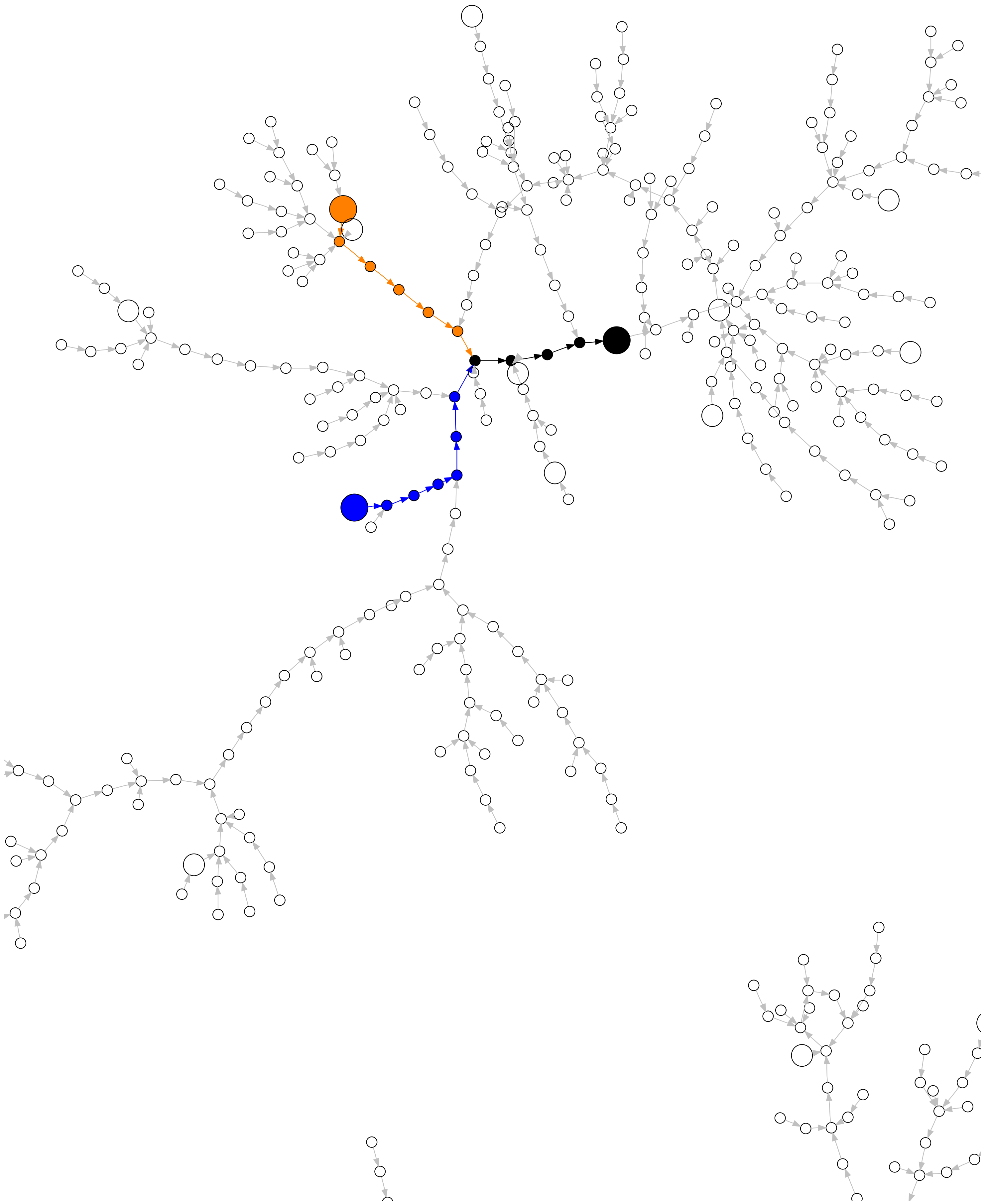

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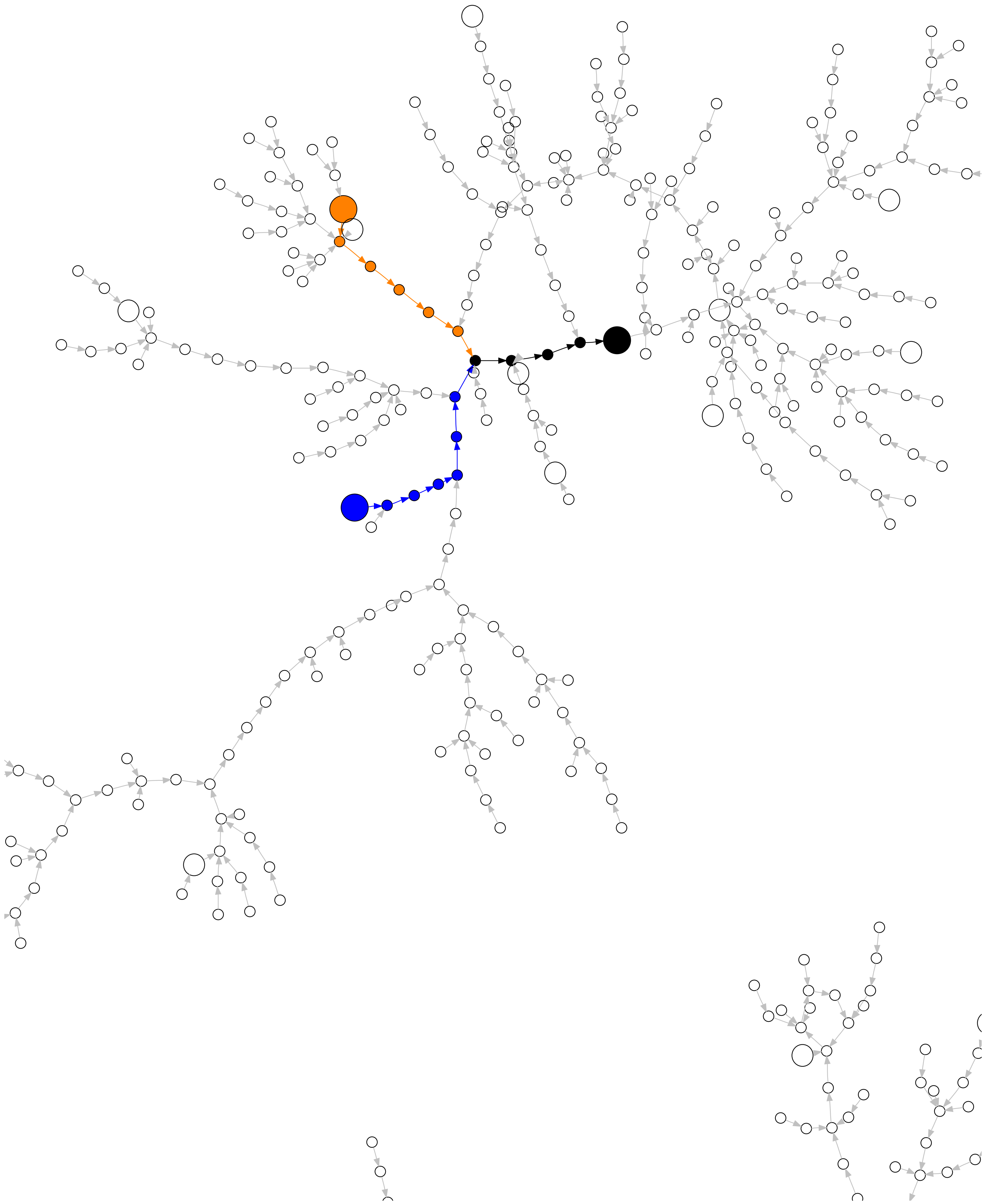

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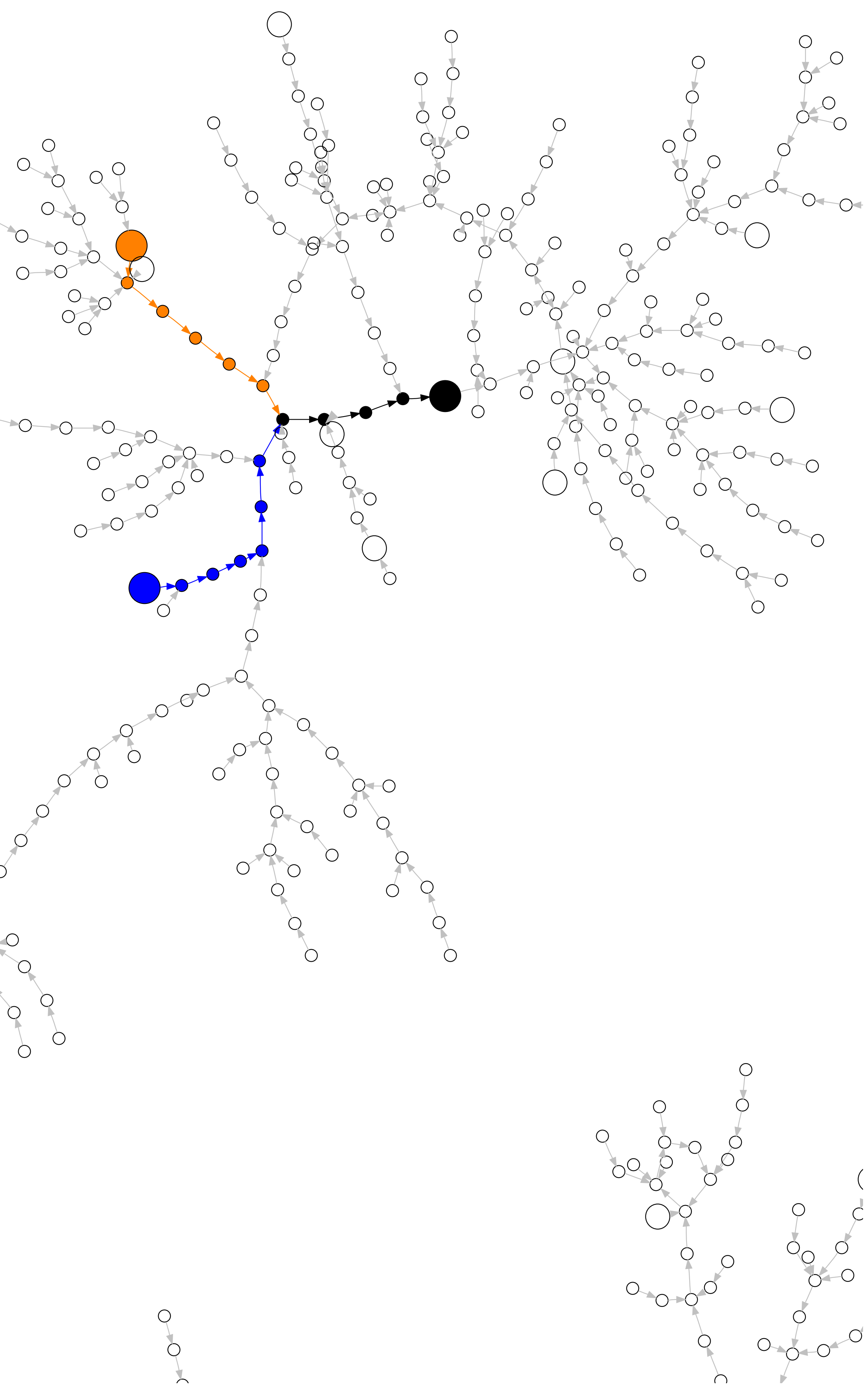

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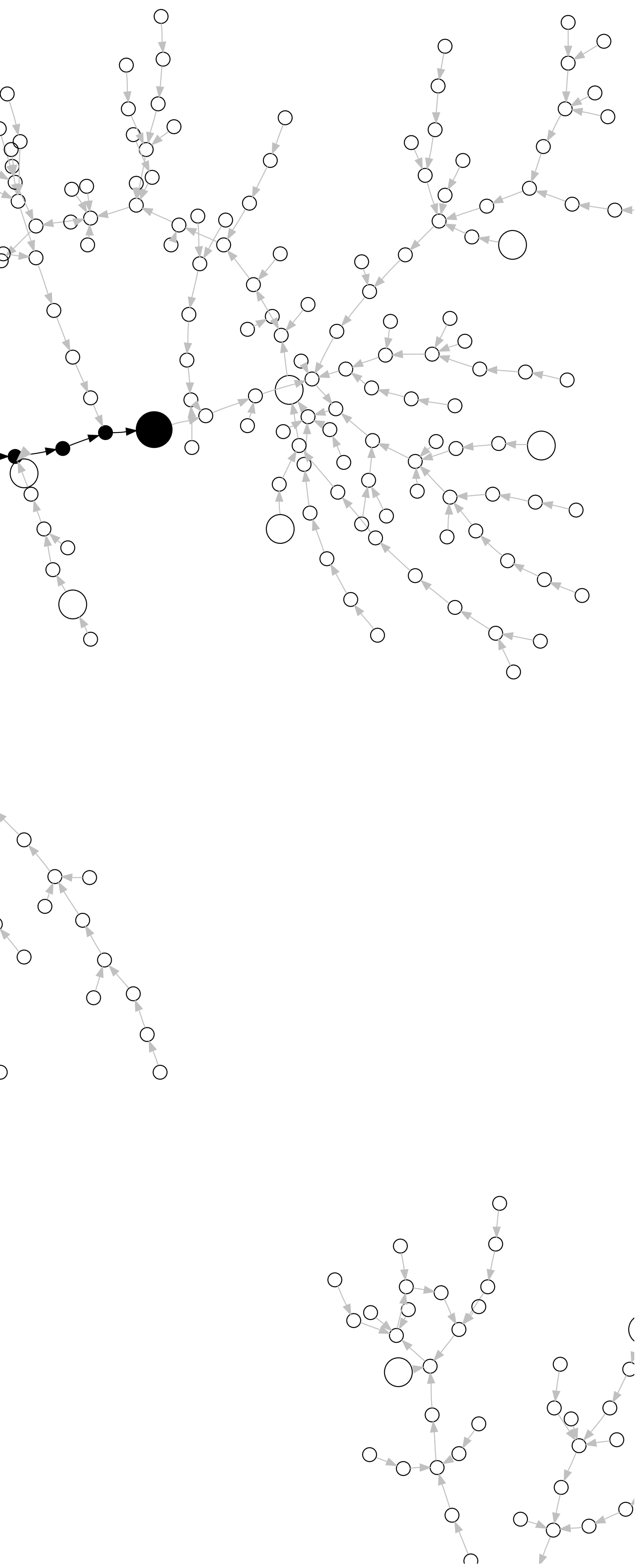

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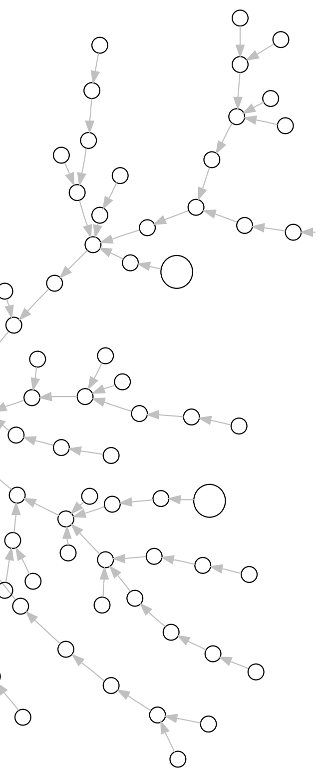

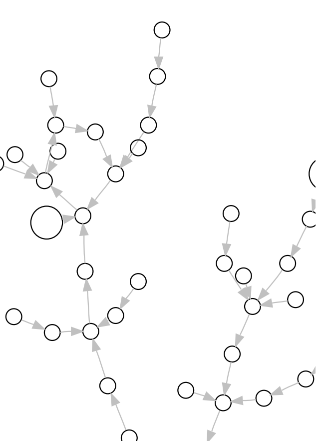

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Easy to update a_i and b_i.

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Randomness of adding walks

Let $h(W) = i$ with probability p_i.

Fix a point T, and let W and W' be two independent uniform random points.

Let $W \neq W'$ both map to T.

This event occurs if simultaneously for $i \neq j$:
$T = W + R_i = W' + R_j$;
$h(W) = i$; $h(W') = j$.

These conditions have probability $1/\ell^2$, p_i, and p_j respectively.

Summing over all (i, j) gives the overall probability
$\left(\sum_{i \neq j} p_i p_j\right) / \ell^2 \quad =$
$\left(\sum_{i,j} p_i p_j - \sum_i p_i^2\right) / \ell^2 \;=$
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$-(| W_i | + c_j P + d_j Q) + c_j P +$
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so $W_{i+3} = W_{i+1}$
so $W_{i+4} = W_{i+2}$ etc.

If h maps to r different values
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or
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Precompute points
$R_0, R_1, \ldots, R_{r-1}$ as known random multiples of P.
Can do full scalar multiplication in inversion-free coordinates!
Start each walk at a point
$W_0 = |\, b_0 Q \,|$,
where b_0 is chosen randomly.
Compute $W_1, W_2, \ldots$ as
$W_{i+1} = |\, W_i + R_{h(W_i)} \,|$.

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Define $|\,(x,y)\,|$ to mean
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or
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where min means
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Doubling the point
makes it escape the cycle.

Define $| (x, y) |$ to mean
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$W_i = |\, 2 \min\{W_{i-1}, W_{i-2}\} \,|$,
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Doubling the point makes it escape the cycle.

Cycles of length 4, 6, or 12 occur far less frequently.
Cycles of length 4, or 6 are detected when checking for cycles of length 12;
so skip individual ones.

Same way of escape:
define $W_i =$
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if trapped and $W_i = W_{i-1}$ otherwise.

Do not store all these points

When checking for cycle, store only potential entry po W_{i-13} (one coordinate, for comparison) and the smallest point encountered s (to escape).

For large DLP look for larger cycles; in general, look for fruitless cycles of even lengt up to $\approx (\log \ell)/(\log r)$.

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How to choose w?

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For large DLP look for larger cycles; in general, look for fruitless cycles of even lengths up to $\approx (\log \ell)/(\log r)$.

How to choose w?

Fruitless cycles of length 2 a
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These cycles persist
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Do not choose w
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When checking for cycle, store only potential entry point W_{i-13} (one coordinate, for comparison) and the smallest point encountered since (to escape).

For large DLP look for larger cycles; in general, look for fruitless cycles of even lengths up to $\approx (\log \ell)/(\log r)$.

How to choose w?

Fruitless cycles of length 2 appear with probability $\approx 1/(2r)$. These cycles persist until detected. After w iterations, probability of cycle $\approx w/(2r)$, wastes $\approx w/2$ iterations (on average) if it does appear.

Do not choose w as small as possible! If a cycle has *not* appeared then the check wastes an iteration.

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How to choose w?

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How to choose w?

Fruitless cycles of length 2 appear with probability $\approx 1/(2r)$.

These cycles persist until detected.

After w iterations, probability of cycle $\approx w/(2r)$, wastes $\approx w/2$ iterations (on average) if it does appear.

Do not choose w as small as possible!

If a cycle has *not* appeared then the check wastes an iteration.

The overall loss is

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How to choose w?

Fruitless cycles of length 2 appear with probability $\approx 1/(2r)$.
These cycles persist until detected.
After w iterations, probability of cycle $\approx w/(2r)$, wastes $\approx w/2$ iterations (on average) if it does appear.

Do not choose w as small as possible!
If a cycle has *not* appeared then the check wastes an iteration.

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Loss rapidly disappears as c increases.
Can use lcm of cycle lengths
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Concrete example: 112-bit D

Use $r = 2048$. Check for 2- every 48 iterations. Check for larger cycles much frequently. Unify the checks for 4-cycles 6-cycles into a check for 12- every 49152 iterations.

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New record

Announced 29 Nov 2016, most work by Ruben Niederl (@cryptocephaly on twitte

Elliptic curve over $\mathbf{F}_{2^{127}}$,
DLP in subgroup of order 2^1
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$\sqrt{\pi 2^{117.35}/4}/2^{30} \sim 379\,821$
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Elliptic curve over $\mathbf{F}_{2^{127}}$,
DLP in subgroup of order $2^{117.35}$.
Used parallel Pollard rho,
DP criterion: 30 top bits equal 0.

Expected
$\sqrt{\pi 2^{117.35}/4}/2^{30} \sim 379\,821\,956$
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Computations ran on 64 to 576
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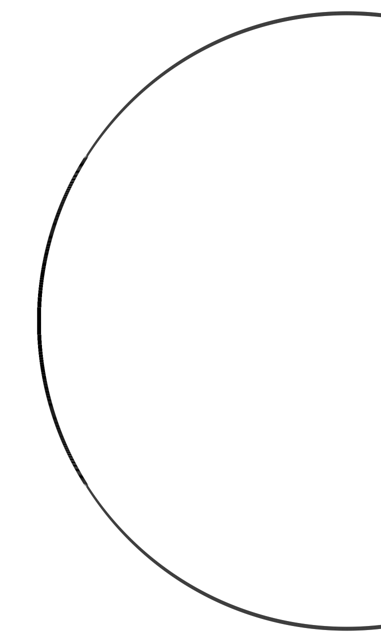
DLs in intervals



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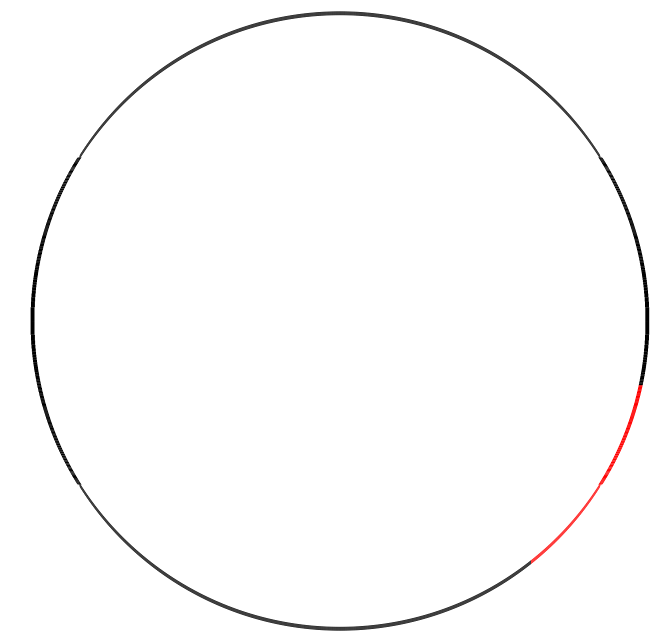

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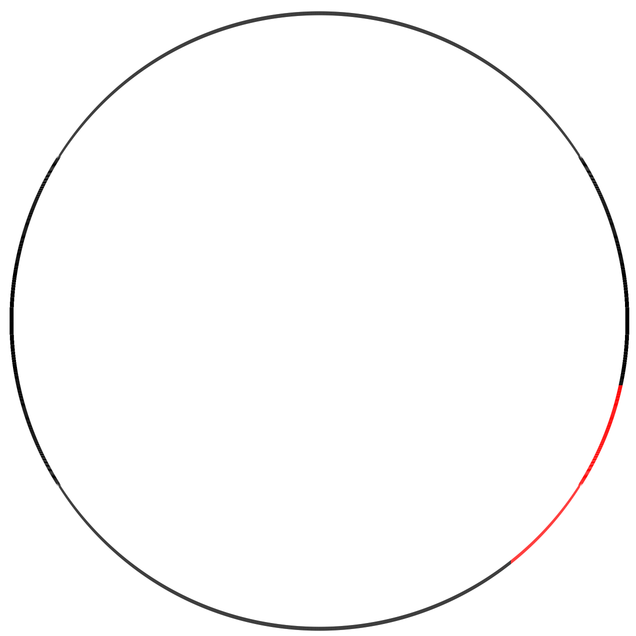

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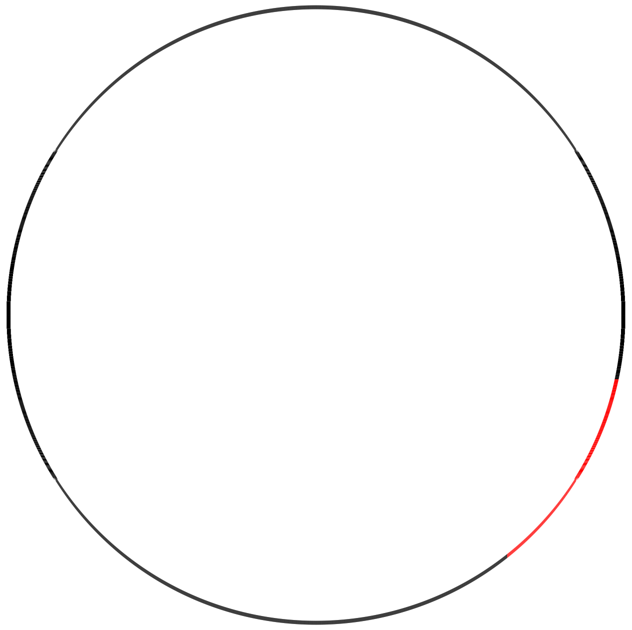

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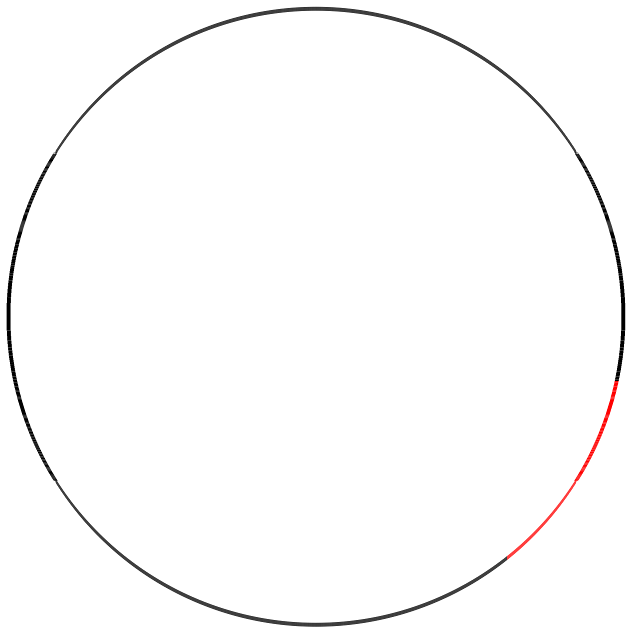

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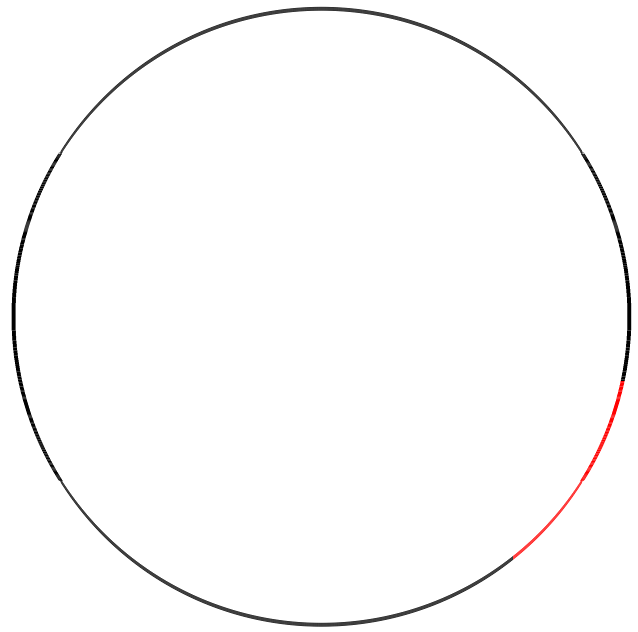

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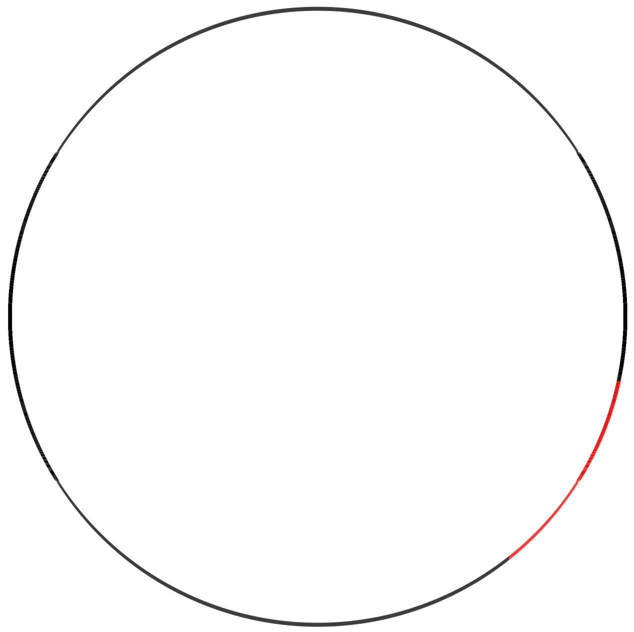

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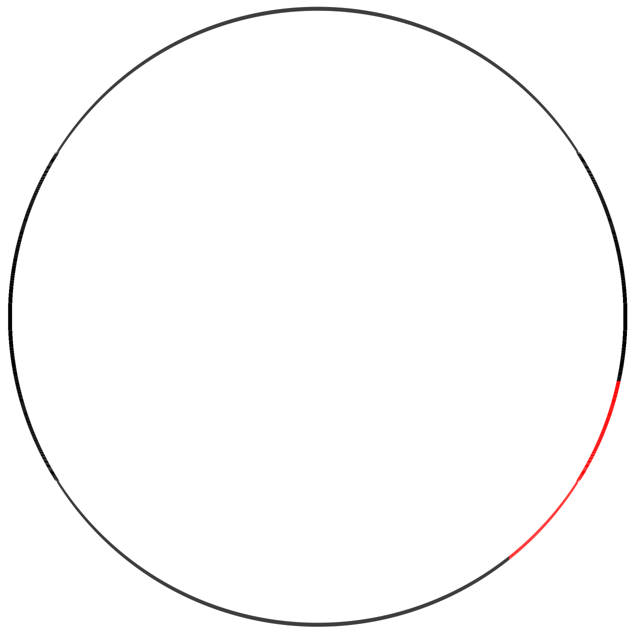

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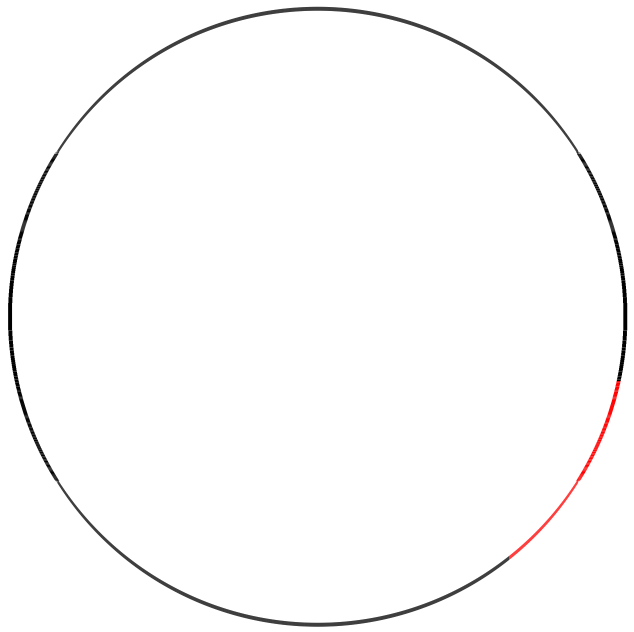
Real kangaroos sleep

DLs in intervals



Want to use knowledge that DL is in a small interval $[a, b]$, much smaller than ℓ.

We can use this in baby-step giant-step algorithm.

How to use this in a memory-less algorithm?

Standard interval method: Pollard's kangaroo method.

Pollard's kangaroos do small jumps around the interval.

Real kangaroos sleep

(at least outside Australia).

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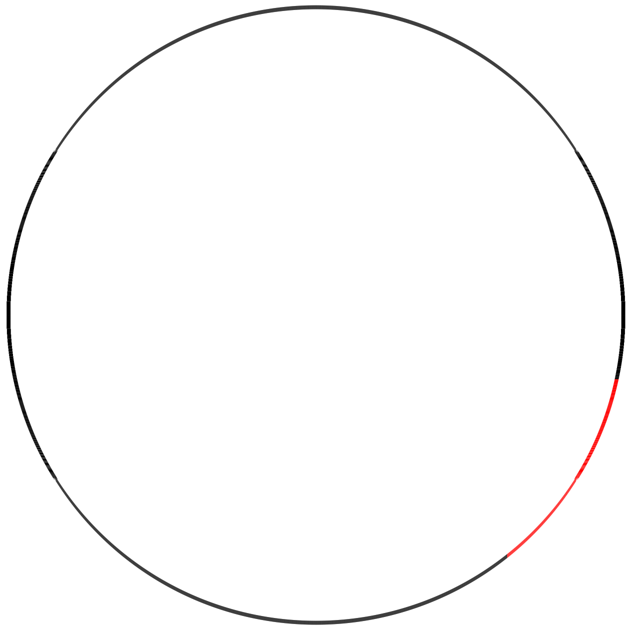

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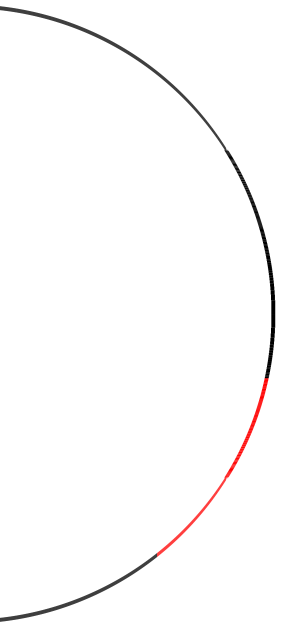

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Kangaroo method

in Australia

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Main actor:

Standard interval method:
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Real kangaroos sleep

(at least outside Australia).

The tame kangaroo

starts at a known multiple of P, e.g. bP.

Standard interval method:
Pollard's kangaroo method.

Pollard's kangaroos do small jumps around the interval.

Real kangaroos sleep

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The tame kangaroo jumps.

Jumps are determined by current position.

Standard interval method:
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Real kangaroos sleep

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The tame kangaroo jumps.

Jumps are determined by current position.
Average jump distance is $\sqrt{b-a}$.

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Real kangaroos sleep

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The tame kangaroo jumps.

Jumps are determined by current position.
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d interval method:
s kangaroo method.

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The tame kangaroo jumps.

Jumps are determined
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The tame kangaroo jumps.

Jumps are determined by current position.

Average jump distance is $\sqrt{b-a}$.

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The tame kangaroo installs and waits.

The tame kangaroo jumps.

Jumps are determined by current position. Average jump distance is $\sqrt{b-a}$.

The tame kangaroo stops

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The tame kangaroo installs a trap and waits.

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The tame kangaroo stops

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The tame kangaroo installs a trap and waits.

The wild kangaroo

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Follows the same i

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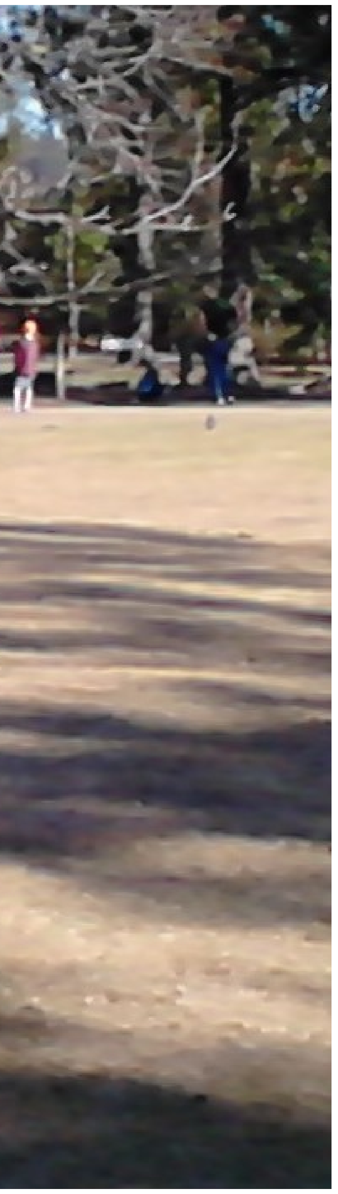

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The wild kangaroo

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Know $Q = nP$ with $n \in [a,$

Hope that the paths of the t
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Eventually the wild kangaroo
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The wild kangaroo

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Follows the same instructions for jumps.

But we don't know where the starting point Q is.
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Hope that the paths of the tame and wild kangaroo intersect.

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Same story in mat

Kangaroo = seque
Starting point X_0
Distance $d_0 = 0$.
Step set: $S = \{s_1$
with s_i on average
$s = \beta\sqrt{b-a}$.
Hash function
$H : \langle P \rangle \to \{1, 2, \ldots$
Update function
$d_{i+1} = d_i + s_{H(X_i}$
$X_{i+1} = X_i + s_{H(X}$

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Same story in math

Kangaroo = sequence $X_i \in$
Starting point $X_0 = s_0 P$.
Distance $d_0 = 0$.
Step set: $S = \{s_1 P, \ldots, s_L$
with s_i on average
$s = \beta\sqrt{b - a}$.
Hash function
$H : \langle P \rangle \to \{1, 2, \ldots, L\}$.
Update function
$d_{i+1} = d_i + s_{H(X_i)}, \quad i = 0$
$X_{i+1} = X_i + s_{H(X_i)} P, \quad i =$

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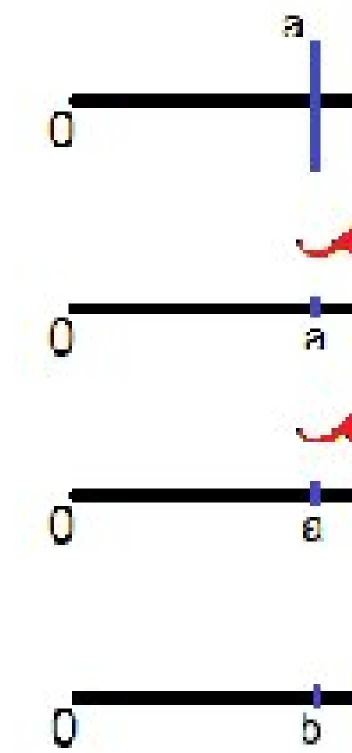

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$X'_0 = Q = nP$.
Trap: distance d_N
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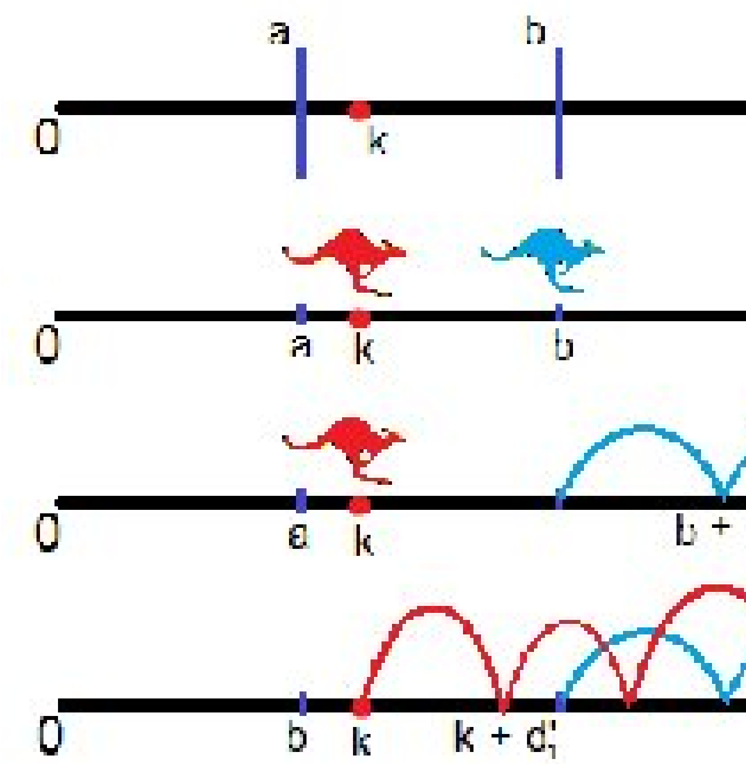

Picture credit:
Christine van Vred

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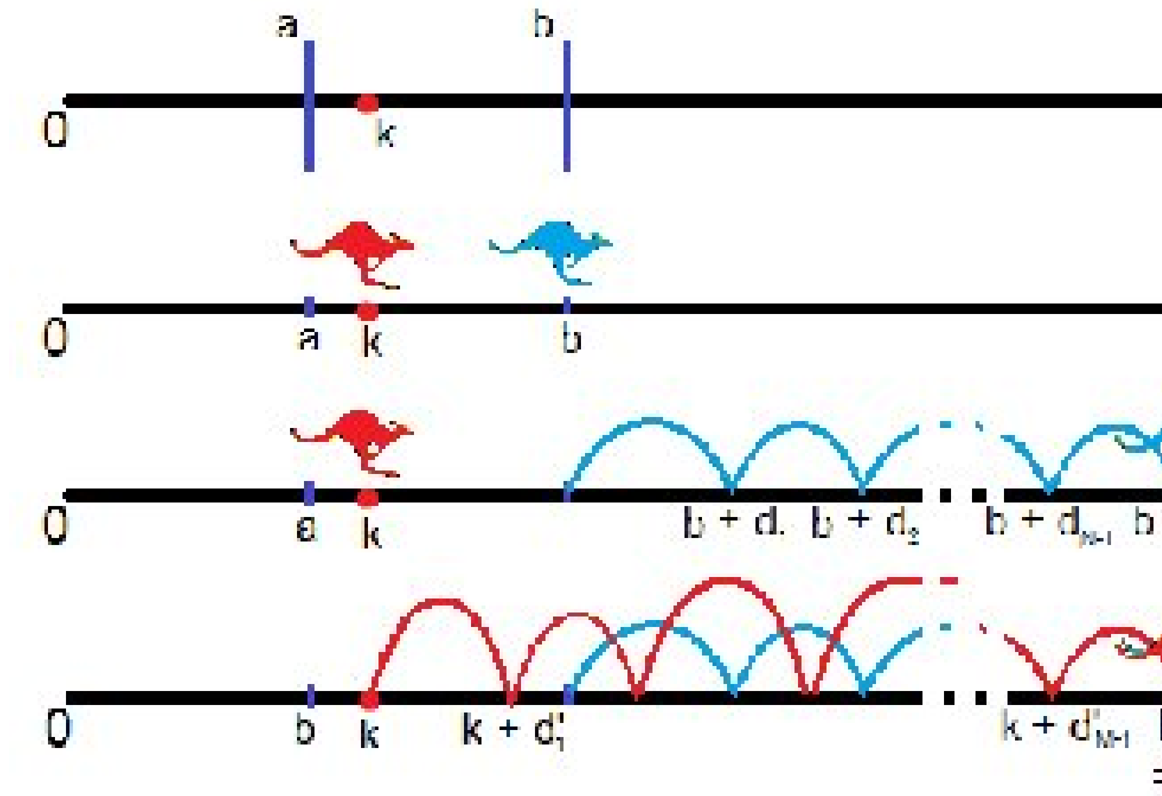
Tame kangaroo starts at
$X_0 = bP$,
wild kangaroo starts at
$X_0' = Q = nP$.
Trap: distance d_N,
endpoint $X_N = (b + d_N)P$.



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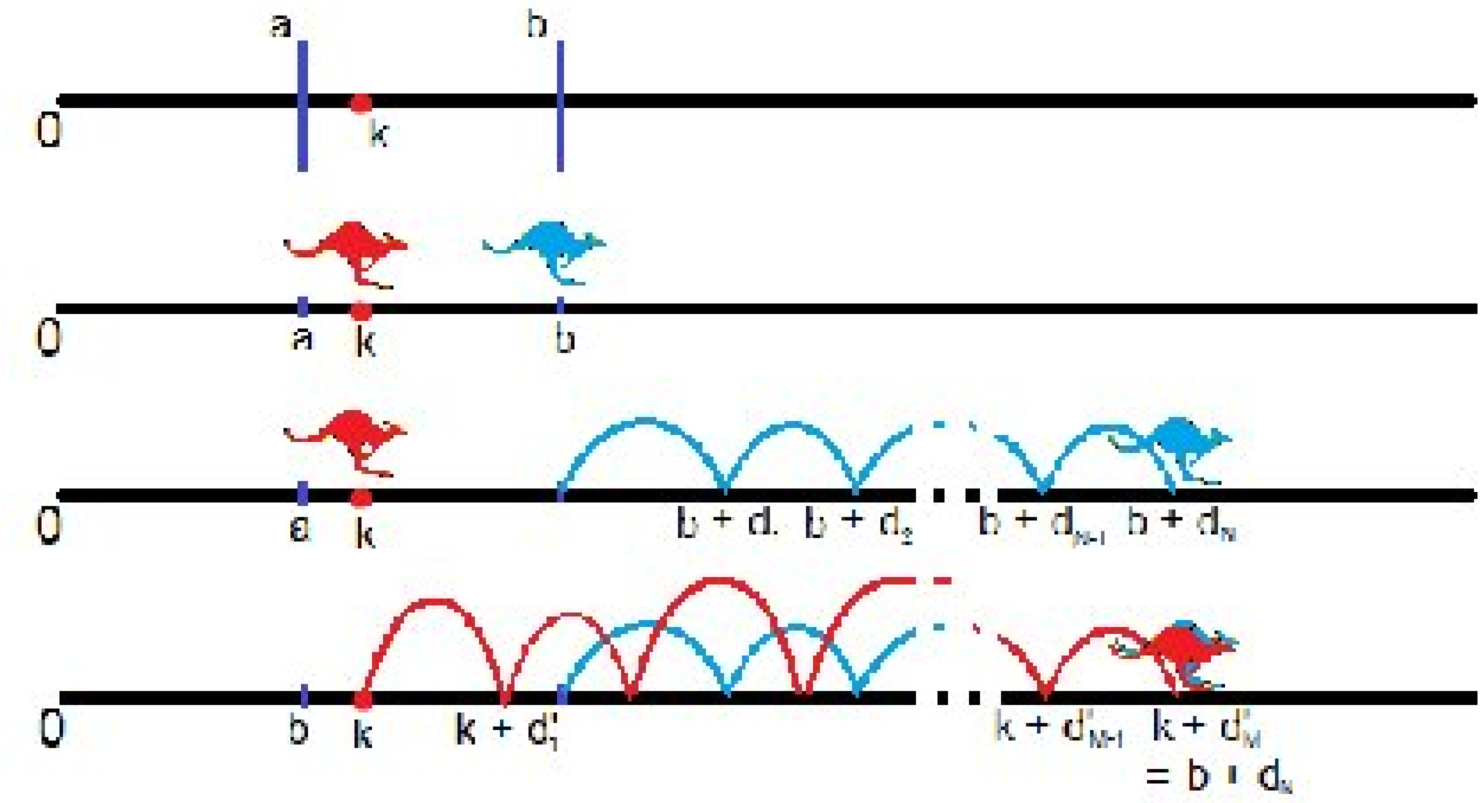

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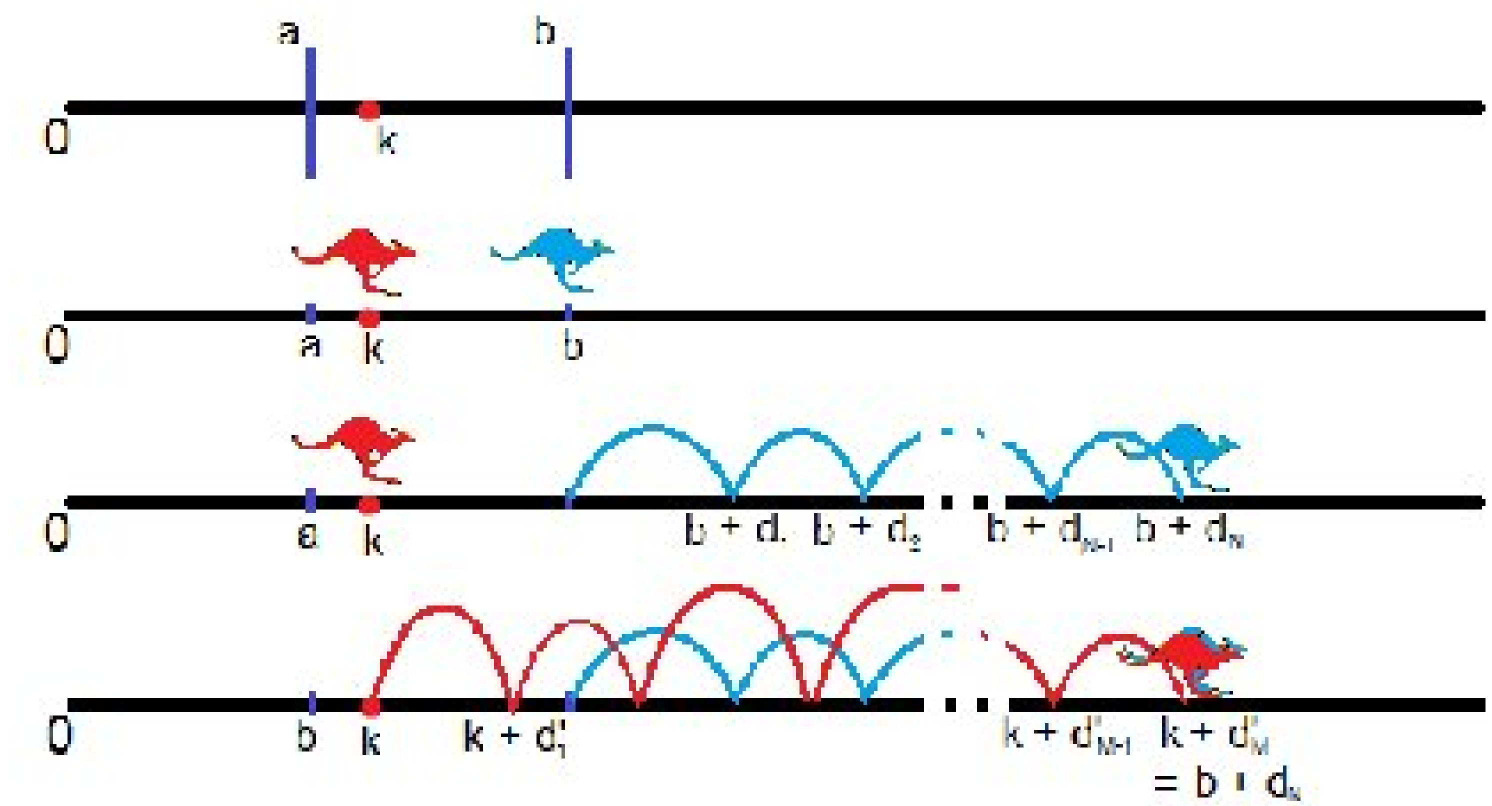

Picture credit:

Christine van Vredendaal.

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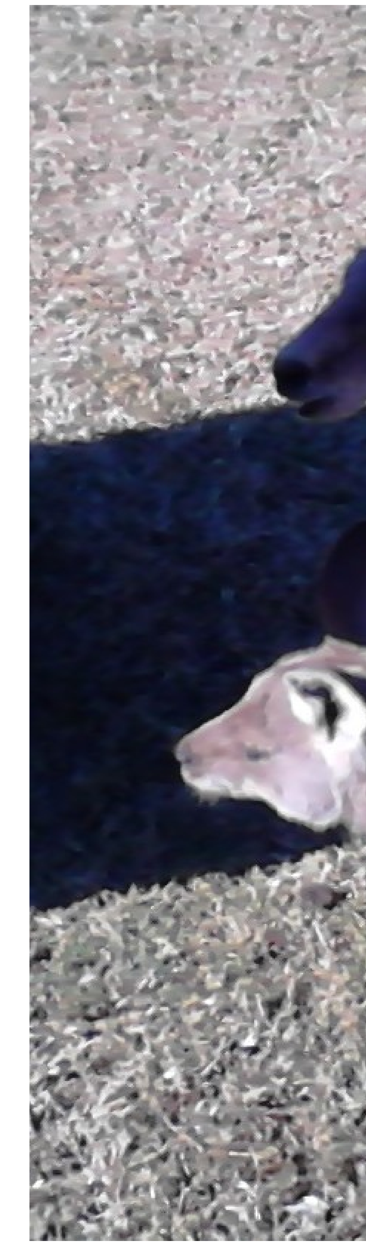

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Tame kangaroo starts at

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wild kangaroo starts at

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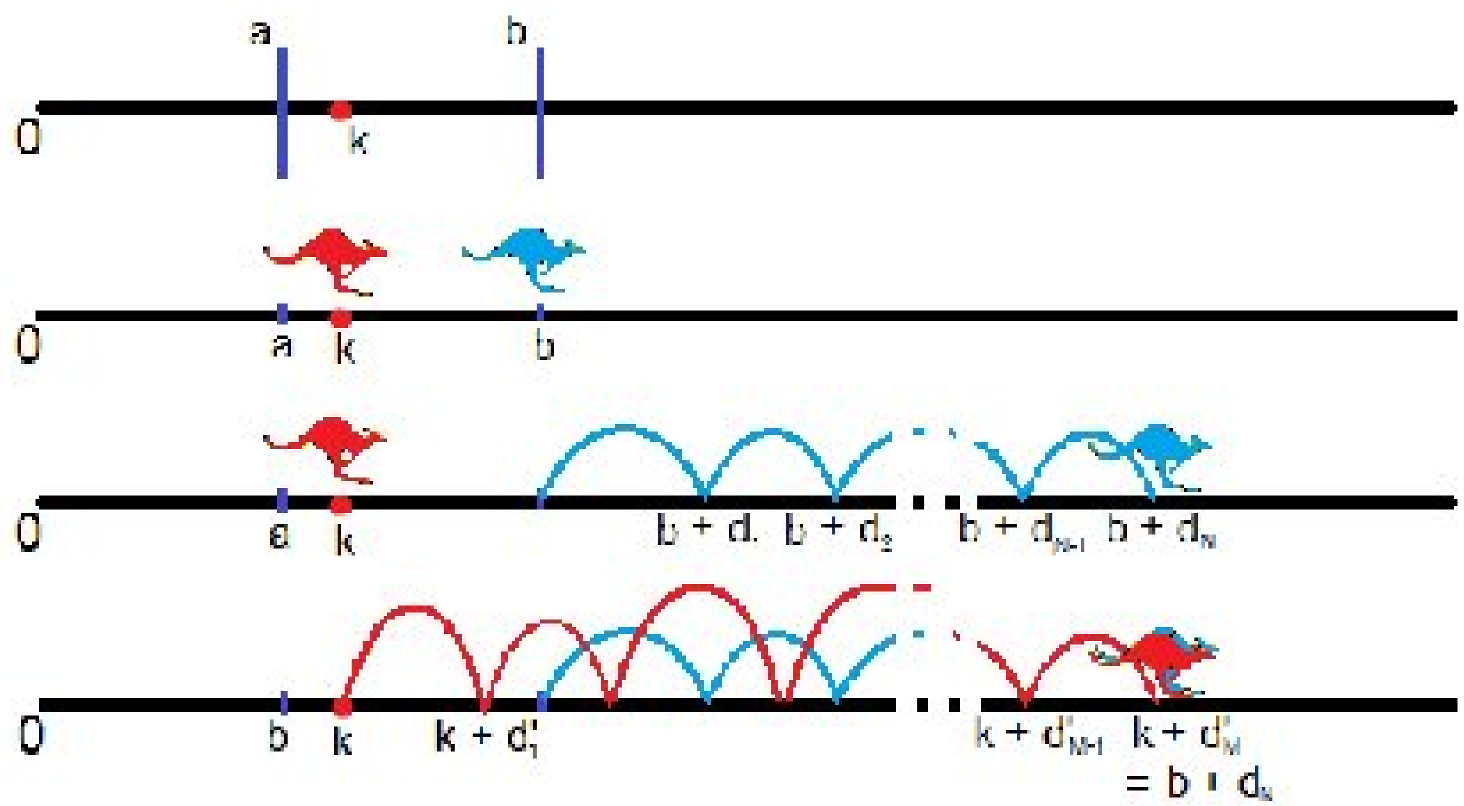

Picture credit:

Christine van Vredendaal.

Parallel kangaroo

Use an entire herd

of tame kangaroos

all starting

around $((b - a)/2$

$\langle P \rangle$.

$P\}$,

$, 1, 2, \ldots,$

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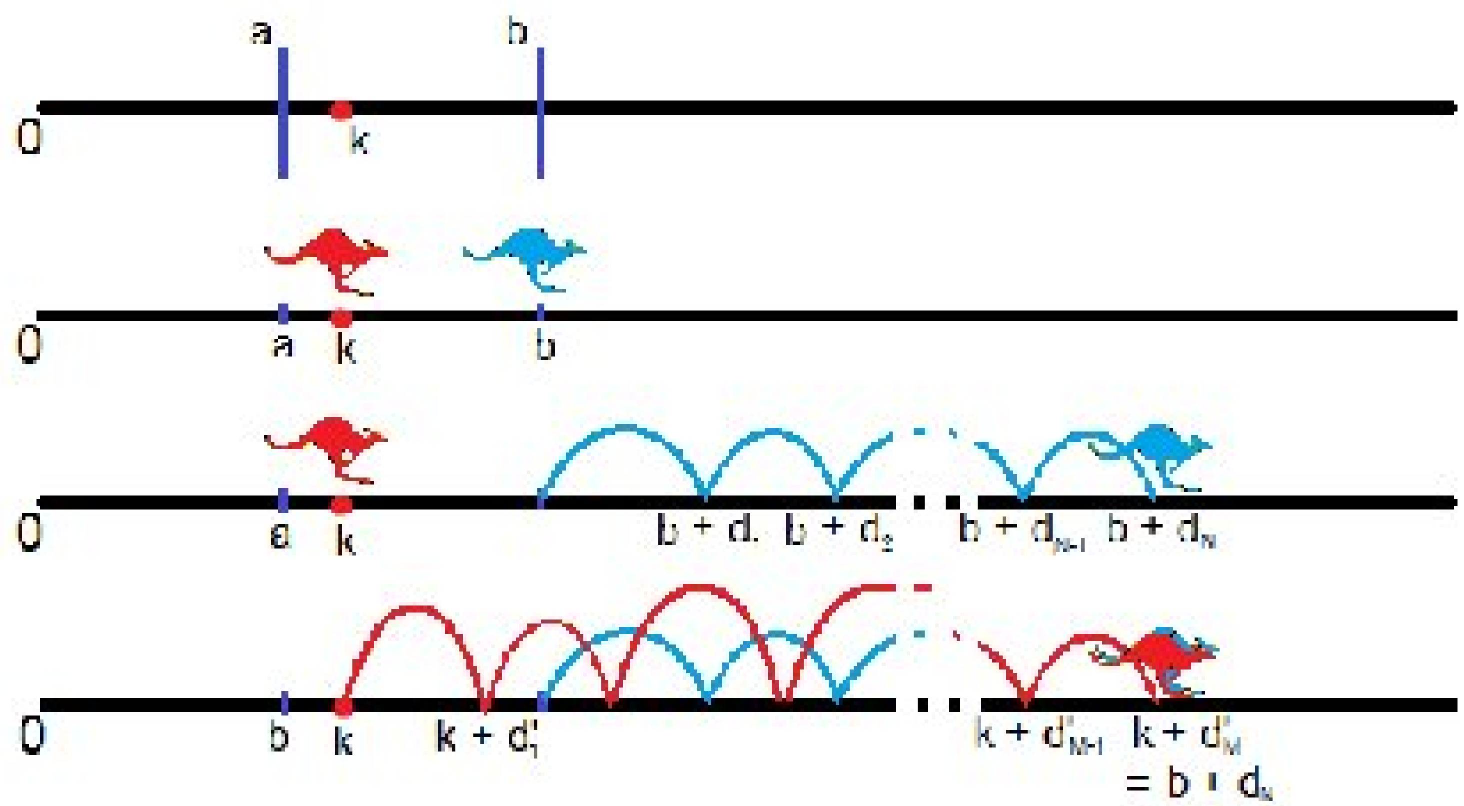
Tame kangaroo starts at
$X_0 = bP$,
wild kangaroo starts at
$X'_0 = Q = nP$.
Trap: distance d_N,
endpoint $X_N = (b + d_N)P$.



Picture credit:
Christine van Vredendaal.

Parallel kangaroo method

Use an entire herd

of tame kangaroos,
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Tame kangaroo starts at
$X_0 = bP$,
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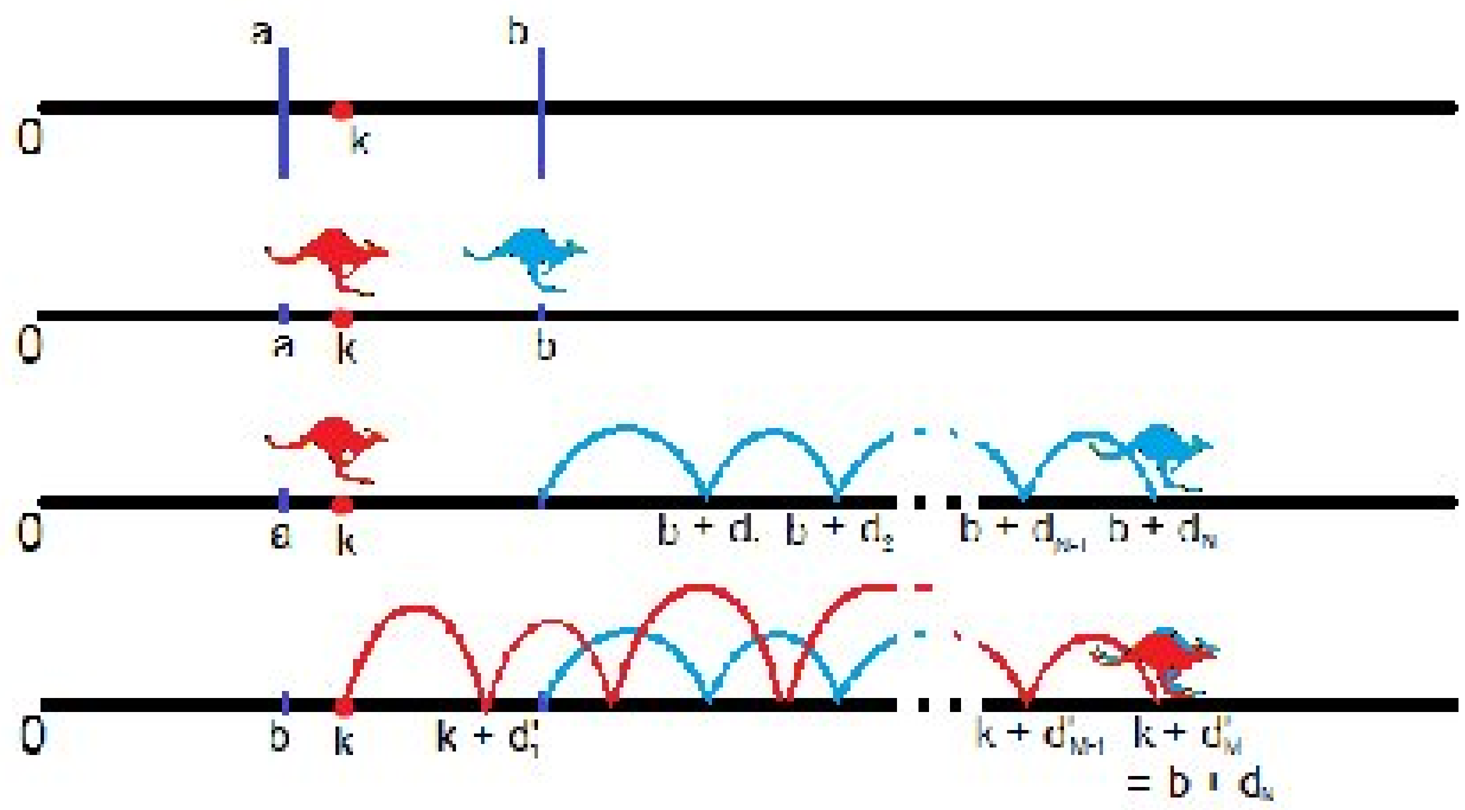
$X'_0 = Q = nP$.
Trap: distance d_N,
endpoint $X_N = (b + d_N)P$.



Picture credit:
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Parallel kangaroo method

Use an entire herd

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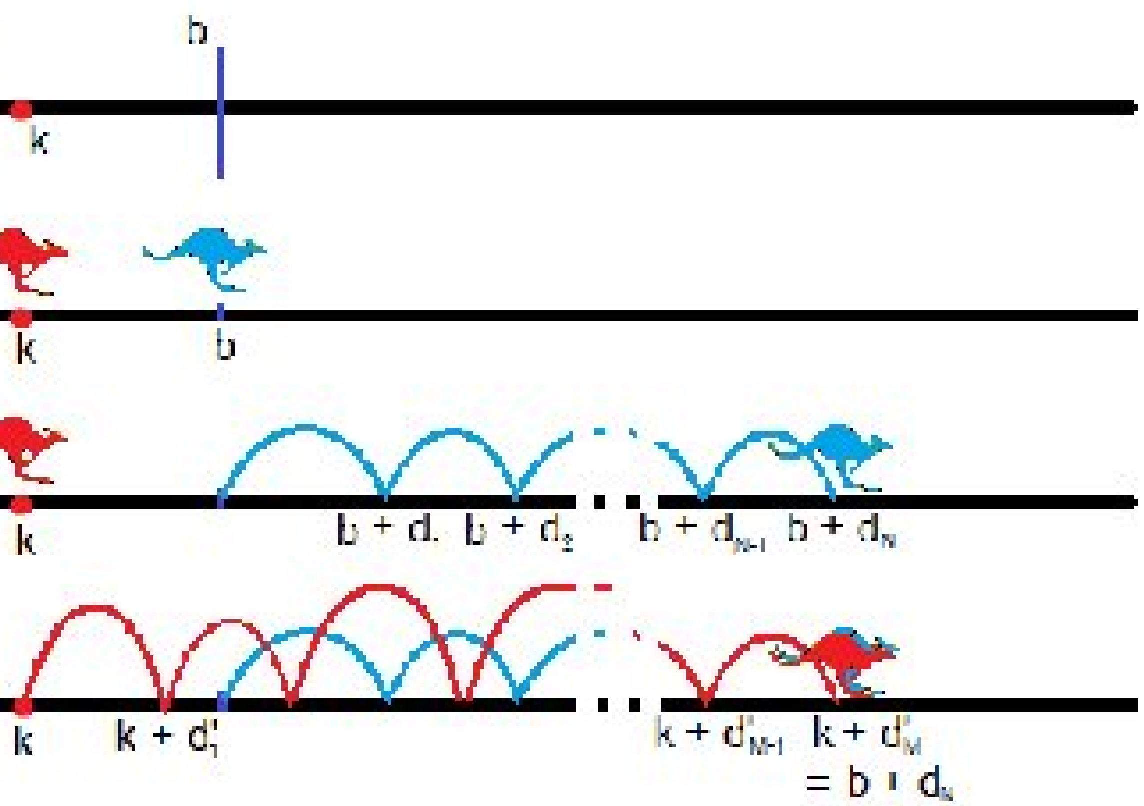

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Parallel kangaroo method

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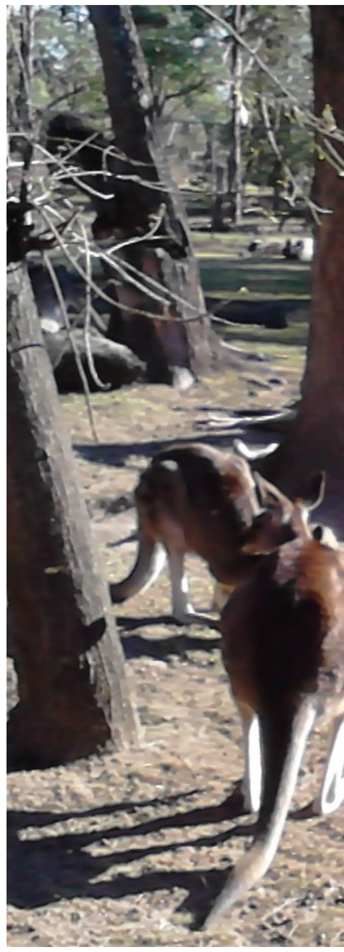

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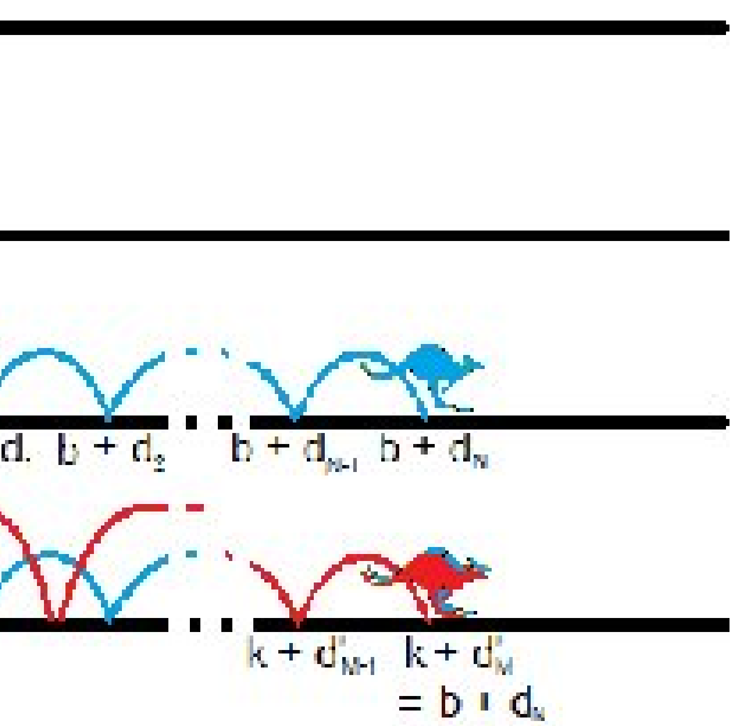

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Parallel kangaroo method

Use an entire herd

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around $((b - a)/2)P$ ...

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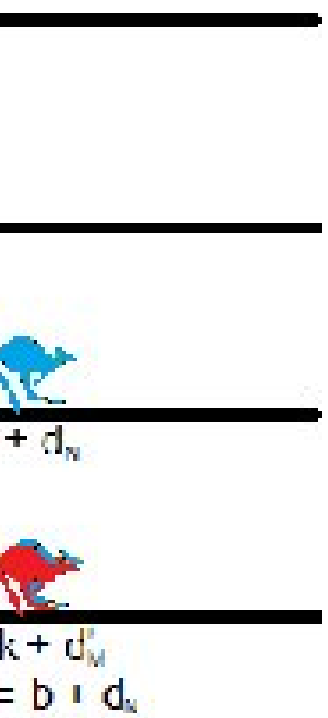

Parallel kangaroo method

Use an entire herd

of tame kangaroos,
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around $((b-a)/2)P$ ...

... and define certain spots a
distinguished points

Also start a herd of
wild kangaroos around Q.
Hope that one wild and
one tame kangaroo
meet at one distinguished po

Parallel kangaroo method

Use an entire herd

of tame kangaroos,
all starting
around $((b-a)/2)P$ ...

...and define certain spots as distinguished points

Also start a herd of
wild kangaroos around Q.
Hope that one wild and
one tame kangaroo
meet at one distinguished point.

kangaroo method

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. . . and define certain spots as distinguished points

Also start a herd of
wild kangaroos around Q.
Hope that one wild and
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meet at one distinguished point.

Pairings

Let $(G_1,$
be group
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…and define certain spots as distinguished points

Also start a herd of
wild kangaroos around Q.
Hope that one wild and
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meet at one distinguished point.

Pairings

Let $(G_1, +), (G_2, +$
be groups of prime
$e : G_1 \times G_2 \rightarrow G_T$
be a map satisfyin
$e(P + Q, R') = e($
$e(P, R' + S') = e($

Request further th
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$e(P, R') = 1$ for al
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Such an e is called
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. . . and define certain spots as distinguished points

Also start a herd of wild kangaroos around Q. Hope that one wild and one tame kangaroo meet at one distinguished point.

Pairings

Let $(G_1, +), (G_2, +)$ and $(G$
be groups of prime order ℓ a
$e : G_1 \times G_2 \to G_T$
be a map satisfying
$e(P + Q, R') = e(P, R')e(Q$
$e(P, R' + S') = e(P, R')e(P$

Request further that e is non-degenerate in the first argument, i.e., if for some P
$e(P, R') = 1$ for all $R' \in G_2$,
then P is the identity in G_1

Such an e is called a *bilinea* or *pairing*.

. . . and define certain spots as distinguished points

Also start a herd of
wild kangaroos around Q.
Hope that one wild and
one tame kangaroo
meet at one distinguished point.

Pairings

Let $(G_1, +), (G_2, +)$ and $(G_T, \cdot)$ be groups of prime order ℓ and let
$e : G_1 \times G_2 \to G_T$
be a map satisfying
$e(P + Q, R') = e(P, R')e(Q, R')$,
$e(P, R' + S') = e(P, R')e(P, S')$.

Request further that e is non-degenerate in the first argument, i.e., if for some P $e(P, R') = 1$ for all $R' \in G_2$, then P is the identity in G_1

Such an e is called a *bilinear map* or *pairing*.

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These maps are important since the only pairings we know how to compute are variants of *Weil pairing* and *Tate pairing* which have $e(P, P) = 1$.

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Distortion maps

For supersingular curves there exist maps
$\phi : E(\mathbf{F}_q) \to E(\mathbf{F}_{q^k})$
i.e. maps $G_1 \to G_2$, giving
$\tilde{e}(P, P) \neq 1$ for $\tilde{e}(P, P) = e(P, \phi(P))$.
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$(x,y) \mapsto (-x, \sqrt{-}$

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Example from Tue

$p = 1000003 \equiv$

$y^2 = x^3 - x$ over

Has $1000004 = p$

$P = (101384, 614$

of order 500002.

$nP = (670366, 74$

Construct $\mathbf{F}_{p^2}$ as

$\phi(P) = (898619, 6$

Invoke magma and

$e(P, \phi(P)) = 3872$

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Solve with index c

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Example from Tuesday:

$p = 1000003 \equiv 3 \bmod 4$ a

$y^2 = x^3 - x$ over $\mathbf{F}_p$.

Has $1000004 = p + 1$ points

$P = (101384, 614510)$ is a

of order 500002.

$nP = (670366, 740819)$.

Construct $\mathbf{F}_{p^2}$ as $\mathbf{F}_p(i)$.

$\phi(P) = (898619, 614510i)$.

Invoke magma and compute

$e(P, \phi(P)) = 387265 + 276$

$e(Q, \phi(P)) = 609466 + 807$

Solve with index calculus to

$n = 78654$.

(Btw. this is the clock).

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Example from Tuesday:

$p = 1000003 \equiv 3$ mod 4 and

$y^2 = x^3 - x$ over $\mathbf{F}_p$.

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Invoke magma and compute

$e(P, \phi(P)) = 387265 + 276048i$;

$e(Q, \phi(P)) = 609466 + 807033i$.

Solve with index calculus to get

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Menezes, Okamoto, and Va
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For $p = 2$ have $k \leq 4$.
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Over $\mathbf{F}_p$, $p \geq 5$ have $k \leq 2$.
These bounds are attained.

Not only supersingular curve
MNT curves are non-supersi
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For $\mathbf{F}_{p^n}$ watch out that pairing can map to $\mathbf{F}_{p^{km}}$ with $m < n$.
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Weil descent:
Maps DLP in E over $\mathbf{F}_{p^{mn}}$ to DLP on variety J over $\mathbf{F}_{p^n}$.
J has larger dimension; elements represented as polynomials of low degree. $\Rightarrow$ index calculus.

This is efficient if dimension of J is not too big.

Particularly nice to compute with J if it is the Jacobian of a hyperelliptic curve C.
For genus g get complexity $\tilde{O}(p^{2-\frac{2}{g+1}})$ with the factor base described before, since polynomials have degree $<= g$.