Quantum circuits for the CSIDH: optimizing quantum evaluation of isogenies

Daniel J. Bernstein, Tanja Lange,
Chloe Martindale, Lorenz Panny

https://quantum.isogeny.org

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Slow. Not obviously not post-quantum.


[ˈsiːˌsaɪd]

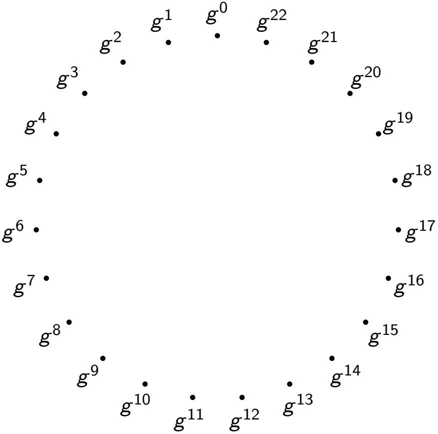
CSIDH: An Efficient Post-Quantum Commutative Group Action

Wouter Castryck, Tanja Lange, Chloe Martindale, Lorenz Panny, Joost Renes 2018

- Closest thing we have in PQC to normal DH key exchange: Keys can be reused, keys can be blinded; no difference between initiator & responder.
- Public keys are represented by some $A \in \mathbf{F}_p$; p fixed prime.
- Alice computes and distributes her public key A.
  Bob computes and distributes his public key B.
- Alice and Bob do computations on each other's public keys to obtain shared secret.
- Fancy math: computations start on some elliptic curve
  $$E_A : y^2 = x^3 + Ax^2 + x,$$
  use *isogenies* to move to a different curve.
- Computations need arithmetic (add, mult, div) modulo p and elliptic-curve computations.

Square-and-multiply

Reminder: DH in group with $\#G = 23$. Alice computes g^{13}.



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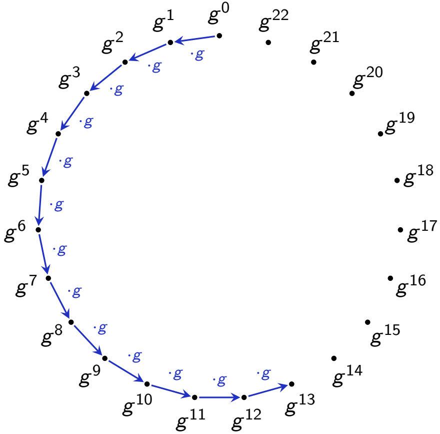

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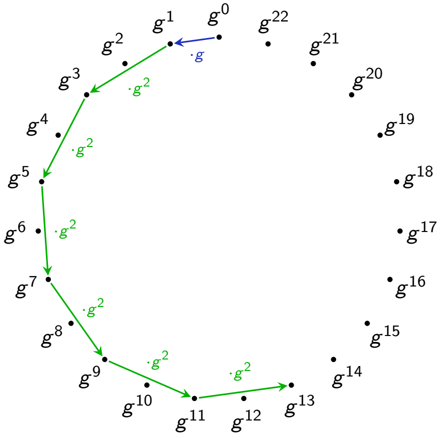

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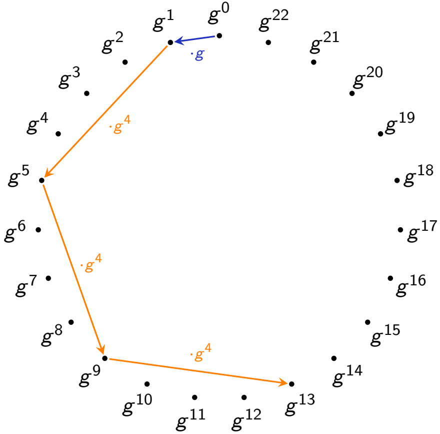

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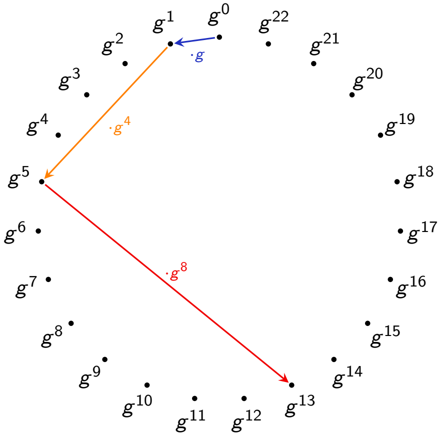

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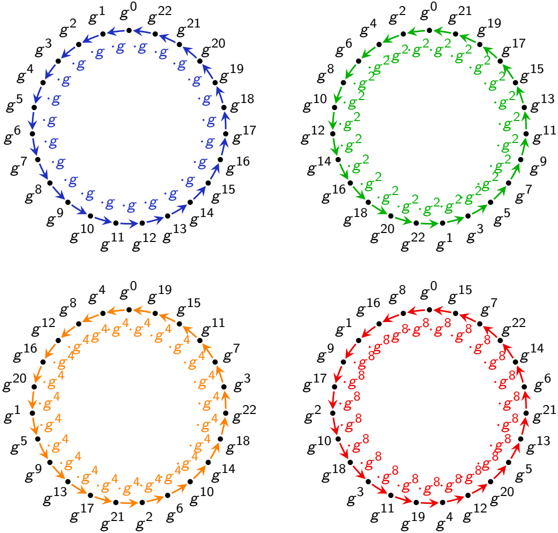

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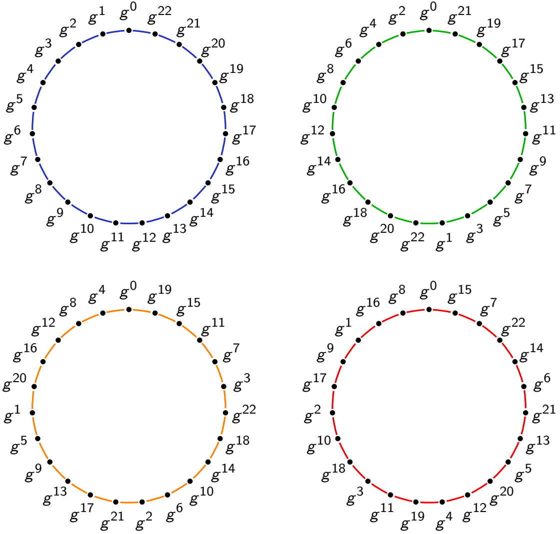

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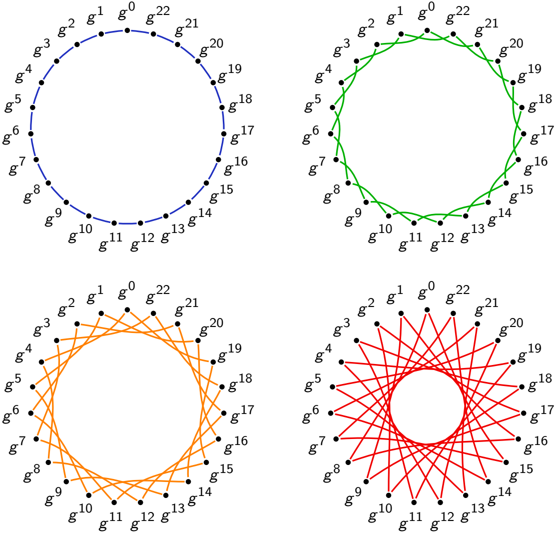

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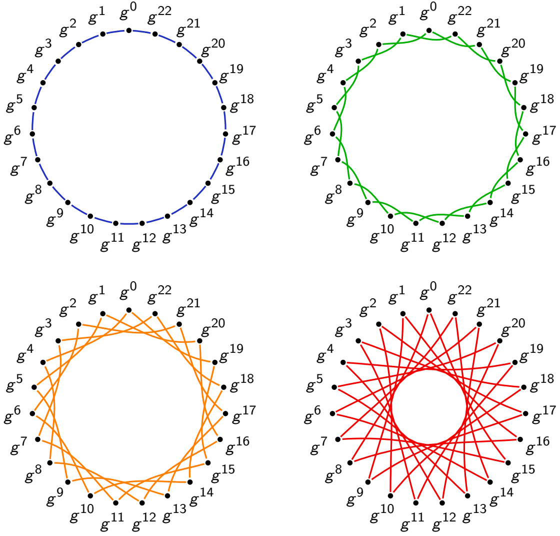

Cycles are *compatible*: [right, then left] = [left, then right], etc.

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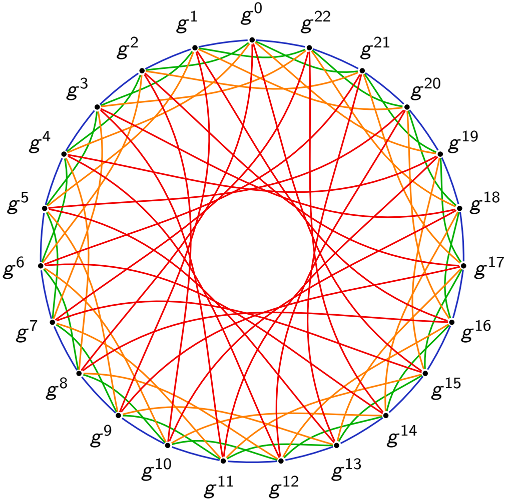
Union of cycles: rapid mixing



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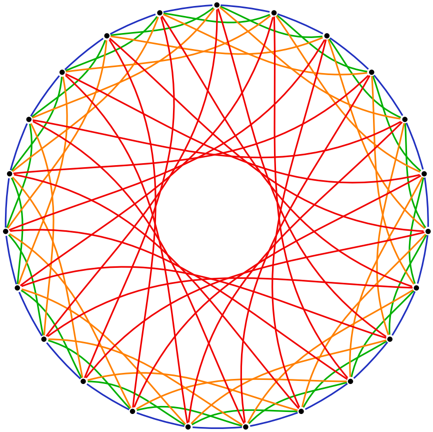

CSIDH: Nodes are now *elliptic curves* and edges are *isogenies*.

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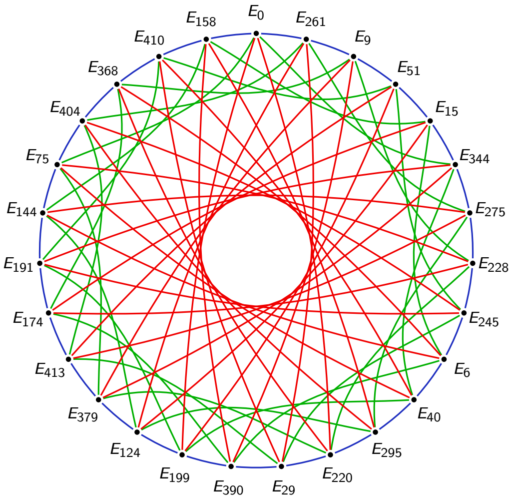
Graphs of elliptic curves



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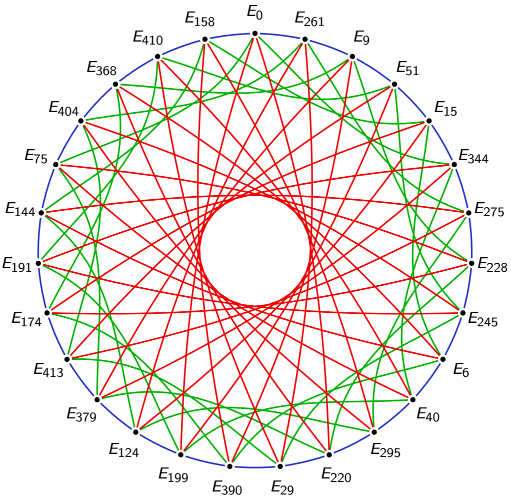

Nodes: Supersingular elliptic curves $E_A\colon y^2 = x^3 + Ax^2 + x$ over $\mathbf{F}_{419}$.

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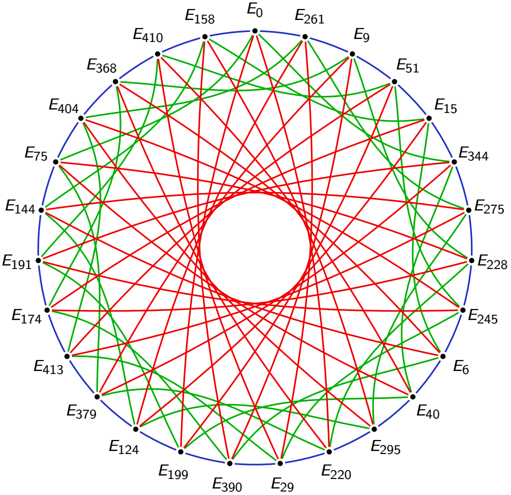

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Edges: 3-, 5-, and 7-isogenies.

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Encryption systems with small public keys

Key bits where all known attacks take 2^λ operations
(naive serial attack metric, ignoring memory cost):

	pre-quantum	post-quantum
SIDH, SIKE	$(24 + o(1))\lambda$	$(36 + o(1))\lambda$
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CRS, CSIDH	$(4 + o(1))\lambda$	superlinear
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Hard problem in CSIDH:
Given curves E_0 and $E = \varphi(E_0)$ find isogeny φ.
Also: φ needs to be quickly computable, $\varphi = [P_1]^{a_1} \cdots [P_d]^{a_d}$.

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Subexp 2010 Childs–Jao–Soukharev attack (on CRS):
This problem can be seen as a hidden-shift problem.
2003 Kuperberg or 2004 Regev or 2011 Kuperberg solves this in subexponentially many queries.

Attack works for any commutative group action, thus also CSIDH.

Major questions

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- What about memory, using parallel AT metric?

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Automatic tallies of nonlinear ops (AND, OR), linear ops (XOR, NOT).

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Building confidence in correctness of output:
1. Compare output to Sage script for CSIDH.
2. Generating-function analysis of *exact* error rates.

Compare to experiments with noticeable error rates.

Case study: one CSIDH-512 query

Consider query with exponents uniform over $\{-5, \ldots, 5\}^{74}$ for the same 74 isogenies as in the constructive use.
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Variations in 512, $\{-5,\ldots,5\}$, 2^{-32}: see paper.

Case study: full CSIDH-512 attack

CSIDH-512 user has inputs $[P_1]^{a_1} \cdots [P_d]^{a_d}$ with $(a_1, \ldots, a_d) \in \{-5, \ldots, 5\}^{74}$ but Kuperberg assumes $[P_1]^a$ with uniform $a \in \mathbf{Z}/N$.

- Approach 1: Compute lattice $L = \mathrm{Ker}(a_1, \ldots, a_d \mapsto [P_1]^{a_1} \cdots [P_d]^{a_d})$.
  Given $a \in \mathbf{Z}^d$, find close $v \in L$:
  distance $\exp((\log N)^{1/2+o(1)})$ using time $\exp((\log N)^{1/2+o(1)})$.
- Approach 2: Increase d up to $\exp((\log N)^{1/2+o(1)})$.
  Search randomly for small relations.
  Time $\exp((\log N)^{1/2+o(1)})$ to compute group action.
- Approach 3 (ours): Uniform $(a_1, \ldots, a_d)$ in $\{-c, \ldots, c\}^d$.
  Choose c somewhat larger than users do.
  Not much slowdown in action.
  Surely $g = [P_1]^{a_1} \cdots [P_d]^{a_d}$ is nearly uniformly distributed.
  Need more analysis of impact of these redundant representations upon Kuperberg's algorithm.