Code-Based Cryptography

Tanja Lange
with some slides by Tung Chou and Christiane Peters

Technische Universiteit Eindhoven

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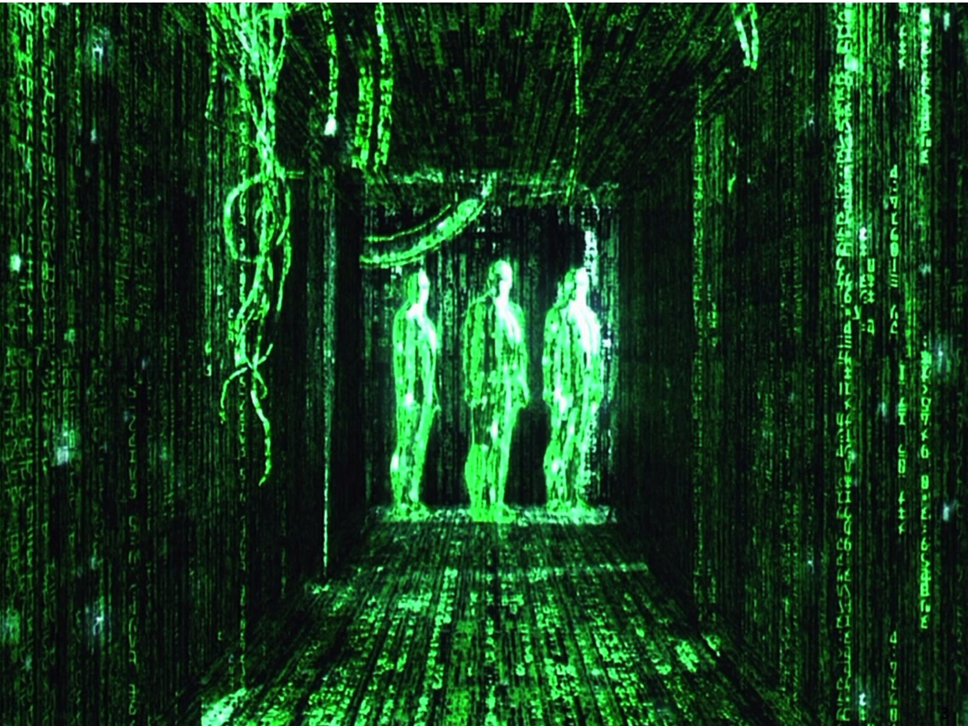
Error correction

- Digital media is exposed to memory corruption.
- Many systems check whether data was corrupted in transit:
  - ISBN numbers have check digit to detect corruption.
  - ECC RAM detects up to two errors and can correct one error. 64 bits are stored as 72 bits: extra 8 bits for checks and recovery.
- In general, k bits of data get stored in n bits, adding some redundancy.
- If no error occurred, these n bits satisfy $n - k$ parity check equations; else can correct errors from the error pattern.
- Good codes can correct many errors without blowing up storage too much;
  offer guarantee to correct t errors (often can correct or at least detect more).
- To represent these check equations we need a matrix.



Hamming code

Parity check matrix ($n = 7, k = 4$):

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

An error-free string of 7 bits $\mathbf{b} = (b_0, b_1, b_2, b_3, b_4, b_5, b_6)$ satisfies these three equations:

$$\begin{array}{cccccccccc} b_0 & +b_1 & & +b_3 & +b_4 & & & = & 0 \\ b_0 & & +b_2 & +b_3 & & +b_5 & & = & 0 \\ & b_1 & +b_2 & +b_3 & & & +b_6 & = & 0 \end{array}$$

If one error occurred at least one of these equations will not hold.
Failure pattern uniquely identifies the error location,
e.g., $1, 0, 1$ means

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e.g., $1, 0, 1$ means b_1 flipped.
In math notation, the failure pattern is $H \cdot \mathbf{b}$.

Coding theory

- Names: code word $\mathbf{c}$, error vector $\mathbf{e}$, received word $\mathbf{b} = \mathbf{c} + \mathbf{e}$.
- Very common to transform the matrix so that the right part has just 1 on the diagonal (no need to store that).

$$H = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

- Many special constructions discovered in 65 years of coding theory:
  - Large matrix H.
  - Fast decoding algorithm to find $\mathbf{e}$ given $\mathbf{s} = H \cdot (\mathbf{c} + \mathbf{e})$, whenever $\mathbf{e}$ does not have too many bits set.
- Given large H, usually very hard to find fast decoding algorithm.
- Use this difference in complexities for encryption.

Code-based encryption

- 1971 Goppa: Fast decoders for many matrices H.
- 1978 McEliece: Use Goppa codes for public-key crypto.
  - Original parameters designed for 2^{64} security.
  - 2008 Bernstein–Lange–Peters: broken in $\approx 2^{60}$ cycles.
  - Easily scale up for higher security.
- 1986 Niederreiter: Simplified and smaller version of McEliece.
- 1962 Prange: simple attack idea guiding sizes in 1978 McEliece.
  The McEliece system (with later key-size optimizations) uses $(c_0 + o(1))\lambda^2(\lg \lambda)^2$-bit keys as $\lambda \to \infty$ to achieve 2^λ security against Prange's attack.
  Here $c_0 \approx 0.7418860694$.

Security analysis

Some papers studying algorithms for attackers:
1962 Prange; 1981 Clark–Cain, crediting Omura; 1988 Lee–Brickell; 1988 Leon; 1989 Krouk; 1989 Stern; 1989 Dumer; 1990 Coffey–Goodman; 1990 van Tilburg; 1991 Dumer; 1991 Coffey–Goodman–Farrell; 1993 Chabanne–Courteau; 1993 Chabaud; 1994 van Tilburg; 1994 Canteaut–Chabanne; 1998 Canteaut–Chabaud; 1998 Canteaut–Sendrier; 2008 Bernstein–Lange–Peters; 2009 Bernstein–Lange–Peters–van Tilborg; 2009 Bernstein (**post-quantum**); 2009 Finiasz–Sendrier; 2010 Bernstein–Lange–Peters; 2011 May–Meurer–Thomae; 2012 Becker–Joux–May–Meurer; 2013 Hamdaoui–Sendrier; 2015 May–Ozerov; 2016 Canto Torres–Sendrier; 2017 Kachigar–Tillich (**post-quantum**); 2017 Both–May; 2018 Both–May; 2018 Kirshanova (**post-quantum**).

Consequence of security analysis

- The McEliece system (with later key-size optimizations) uses $(c_0 + o(1))\lambda^2(\lg \lambda)^2$-bit keys as $\lambda \to \infty$ to achieve 2^λ security against all these attacks.

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- 256 KB public key for 2^{146} pre-quantum security.
- 512 KB public key for 2^{187} pre-quantum security.
- 1024 KB public key for 2^{263} pre-quantum security.

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- 1024 KB public key for 2^{263} pre-quantum security.
- Post-quantum (Grover): below 2^{263}, above 2^{131}.

Linear codes

A binary linear code C of length n and dimension k is a k-dimensional subspace of $\mathbb{F}_2^n$.
C is usually specified as

- the row space of a generating matrix $G \in \mathbb{F}_2^{k \times n}$

$$C = \{\mathbf{m}G | \mathbf{m} \in \mathbb{F}_2^k\}$$

- the kernel space of a parity-check matrix $H \in \mathbb{F}_2^{(n-k) \times n}$

$$C = \{\mathbf{c} | H\mathbf{c}^\intercal = 0, \;\; \mathbf{c} \in \mathbb{F}_2^n\}$$

Leaving out the $^\intercal$ from now on.

Example

$$G = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$\mathbf{c} = (111)G = (10011)$ is a codeword.

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Same with parity-check matrix:

$$H(\mathbf{c}_1 + \mathbf{c}_2) = H\mathbf{c}_1 + H\mathbf{c}_2 = 0 + 0 = 0.$$

Hamming weight and distance

- The Hamming weight of a word is the number of nonzero coordinates.

$$\mathrm{wt}(1,0,0,1,1) = 3$$

- The Hamming distance between two words in $\mathbb{F}_2^n$ is the number of coordinates in which they differ.

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The Hamming distance between $\mathbf{x}$ and $\mathbf{y}$ equals the Hamming weight of $\mathbf{x} + \mathbf{y}$:

$$d((1,1,0,1,1),(1,0,0,1,1)) = \mathrm{wt}(0,1,0,0,0).$$

Minimum distance

- The minimum distance of a linear code C is the smallest Hamming weight of a nonzero codeword in C.

$$d = \min_{0 \neq \mathbf{c} \in C} \{\mathrm{wt}(\mathbf{c})\} = \min_{\mathbf{b} \neq \mathbf{c} \in C} \{d(\mathbf{b}, \mathbf{c})\}$$

- In code with minimum distance $d = 2t + 1$, any vector $\mathbf{x} = \mathbf{c} + \mathbf{e}$ with $\mathrm{wt}(\mathbf{e}) \leq t$ is uniquely decodable to $\mathbf{c}$; i. e. there is no closer code word.

Decoding problem

Decoding problem: find the closest codeword $\mathbf{c} \in C$ to a given $\mathbf{x} \in \mathbb{F}_2^n$, assuming that there is a unique closest codeword. Let $\mathbf{x} = \mathbf{c} + \mathbf{e}$. Note that finding $\mathbf{e}$ is an equivalent problem.

- If $\mathbf{c}$ is t errors away from $\mathbf{x}$, i.e., the Hamming weight of $\mathbf{e}$ is t, this is called a t-error correcting problem.
- There are lots of code families with fast decoding algorithms, e.g., Reed–Solomon codes, Goppa codes/alternant codes, etc.
- However, the general decoding problem is hard: Information-set decoding (see later) takes exponential time.

The McEliece cryptosystem I

- Let C be a length-n binary Goppa code Γ of dimension k with minimum distance $2t+1$ where $t \approx (n-k)/\log_2(n)$; original parameters (1978) $n = 1024$, $k = 524$, $t = 50$.
- The McEliece secret key consists of a generator matrix G for Γ, an efficient t-error correcting decoding algorithm for Γ; an $n \times n$ permutation matrix P and a nonsingular $k \times k$ matrix S.
- n, k, t are public; but Γ, P, S are randomly generated secrets.
- The McEliece public key is the $k \times n$ matrix $G' = SGP$.

The McEliece cryptosystem II

- Encrypt: Compute $\mathbf{m}G'$ and add a random error vector $\mathbf{e}$ of weight t and length n. Send $\mathbf{y} = \mathbf{m}G' + \mathbf{e}$.
- Decrypt: Compute $\mathbf{y}P^{-1} = \mathbf{m}G'P^{-1} + \mathbf{e}P^{-1} = (\mathbf{m}S)G + \mathbf{e}P^{-1}$. This works because $\mathbf{e}P^{-1}$ has the same weight as $\mathbf{e}$

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  This works because $\mathbf{e}P^{-1}$ has the same weight as $\mathbf{e}$ because P is a permutation matrix.
  Use fast decoding to find $\mathbf{m}S$ and $\mathbf{m}$.
- Attacker is faced with decoding $\mathbf{y}$ to nearest codeword $\mathbf{m}G'$ in the code generated by G'.
  This is general decoding if G' does not expose any structure.

Systematic form

- A systematic generator matrix is a generator matrix of the form $(I_k|Q)$ where I_k is the $k \times k$ identity matrix and Q is a $k \times (n-k)$ matrix (redundant part).
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  Then

$$H(\mathbf{m}G)^{\intercal} = HG^{\intercal}\mathbf{m}^{\intercal} = (Q^{\intercal}|I_{n-k})(I_k|Q)^{\intercal}\mathbf{m}^{\intercal} = 0.$$

Different views on decoding

- The syndrome of $\mathbf{x} \in \mathbb{F}_2^n$ is $\mathbf{s} = H\mathbf{x}$.
  Note $H\mathbf{x} = H(\mathbf{c} + \mathbf{e}) = H\mathbf{c} + H\mathbf{e} = H\mathbf{e}$ depends only on $\mathbf{e}$.
- The syndrome decoding problem is to compute $\mathbf{e} \in \mathbb{F}_2^n$ given $\mathbf{s} \in \mathbb{F}_2^{n-k}$ so that $H\mathbf{e} = \mathbf{s}$ and $\mathbf{e}$ has minimal weight.
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  To decode $\mathbf{x}$ with syndrome decoder, compute $\mathbf{e}$ from $H\mathbf{x}$, then $\mathbf{c} = \mathbf{x} + \mathbf{e}$.
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  To expand syndrome, assume $H = (Q^\intercal | I_{n-k})$.
  Then $\mathbf{x} = (00\ldots0)||\mathbf{s}$ satisfies $\mathbf{s} = H\mathbf{x}$.
- Note that this $\mathbf{x}$ is not a solution to the syndrome decoding problem, unless it has very low weight.

The Niederreiter cryptosystem I

Developed in 1986 by Harald Niederreiter as a variant of the McEliece cryptosystem. This is the schoolbook version.

- Use $n \times n$ permutation matrix P and $n - k \times n - k$ invertible matrix S.
- Public Key: a scrambled parity-check matrix $K = SHP \in \mathbb{F}_2^{(n-k)\times n}$.
- Encryption: The plaintext $\mathbf{e}$ is an n-bit vector of weight t. The ciphertext $\mathbf{s}$ is the $(n-k)$-bit vector

$$\mathbf{s} = K\mathbf{e}.$$

- Decryption: Find a n-bit vector $\mathbf{e}$ with $\mathrm{wt}(\mathbf{e}) = t$ such that $\mathbf{s} = K\mathbf{e}$.
- The passive attacker is facing a t-error correcting problem for the public key, which seems to be random.

The Niederreiter cryptosystem II

- Public Key: a scrambled parity-check matrix $K = SHP$.
- Encryption: The plaintext $\mathbf{e}$ is an n-bit vector of weight t. The ciphertext $\mathbf{s}$ is the $(n-k)$-bit vector

$$\mathbf{s} = K\mathbf{e}.$$

- Decryption using secret key: Compute

$$\begin{aligned} S^{-1}\mathbf{s} &= S^{-1}K\mathbf{e} = S^{-1}(SHP)\mathbf{e} \\ &= H(P\mathbf{e}) \end{aligned}$$

and observe that $\mathrm{wt}(P\mathbf{e}) = 1$, because P permutes.
Use efficient syndrome decoder for H to find $\mathbf{e}' = P\mathbf{e}$ and thus $\mathbf{e} = P^{-1}\mathbf{e}'$.

Note on codes

- McEliece proposed to use binary Goppa codes.
  These are still used today.
- Niederreiter described his scheme using Reed-Solomon codes.
  These were broken in 1992 by Sidelnikov and Chestakov.
- More corpses on the way: concatenated codes, Reed-Muller codes, several Algebraic Geometry (AG) codes, Gabidulin codes, several LDPC codes, cyclic codes.
- Some other constructions look OK (for now).
  NIST competition has several entries on QCMDPC codes.

Binary Goppa code

Let $q = 2^m$. A binary Goppa code is often defined by

- a list $L = (a_1, \ldots, a_n)$ of n distinct elements in $\mathbb{F}_q$, called the support.
- a square-free polynomial $g(x) \in \mathbb{F}_q[x]$ of degree t such that $g(a) \neq 0$ for all $a \in L$. $g(x)$ is called the Goppa polynomial.
- E.g. choose $g(x)$ irreducible over $\mathbb{F}_q$.

The corresponding binary Goppa code $\Gamma(L, g)$ is

$$\left\{ \mathbf{c} \in \mathbb{F}_2^n \,\middle|\, S(\mathbf{c}) = \frac{c_1}{x - a_1} + \frac{c_2}{x - a_2} + \cdots + \frac{c_n}{x - a_n} \equiv 0 \bmod g(x) \right\}$$

- This code is linear $S(\mathbf{b} + \mathbf{c}) = S(\mathbf{b}) + S(\mathbf{c})$ and has length n.
- What can we say about the dimension and minimum distance?

Dimension of $\Gamma(L, g)$

- $g(a_i) \neq 0$ implies $\gcd(x - a_i, g(x)) = 1$, thus get polynomials
$$(x - a_i)^{-1} \equiv f_i(x) \equiv \sum_{j=0}^{t-1} f_{i,j} x^j \bmod g(x)$$
via XGCD. All this is over $\mathbb{F}_q = \mathbb{F}_{2^m}$.
- In this form, $S(\mathbf{c}) \equiv 0 \bmod g(x)$ means
$$\sum_{i=1}^{n} c_i \left(\sum_{j=0}^{t-1} f_{i,j} x^j \right) = \sum_{j=0}^{t-1} \left(\sum_{i=1}^{n} c_i f_{i,j} \right) x^j = 0,$$
meaning that for each $0 \leq j \leq t - 1$:
$$\sum_{i=1}^{n} c_i f_{i,j} = 0.$$
- These are t conditions over $\mathbb{F}_q$, so tm conditions over $\mathbb{F}_2$. Giving an $tm \times n$ parity check matrix over $\mathbb{F}_2$.
- Some rows might be linearly dependent, so $k \geq n - tm$.

Nice parity check matrix

Assume $g(x) = \sum_{i=0}^{t} g_i x^i$ monic, i.e., $g_t = 1$.

$$H = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ g_{t-1} & 1 & 0 & \dots & 0 \\ g_{t-2} & g_{t-1} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_1 & g_2 & g_3 & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1^{t-1} & a_2^{t-1} & a_3^{t-1} & \cdots & a_n^{t-1} \end{pmatrix}$$

$$\cdot \begin{pmatrix} \frac{1}{g(a_1)} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{g(a_2)} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{g(a_3)} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{g(a_n)} \end{pmatrix}$$

Minimum distance of $\Gamma(L, g)$. Put $s(x) = S(\mathbf{c})$

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$$\begin{aligned} s(x) &= \sum_{i=1}^{n} c_i/(x - a_i) \\ &= \left(\sum_{i=1}^{n} c_i \prod_{j \neq i} (x - a_j) \right) / \prod_{i=1}^{n} (x - a_i) \equiv 0 \bmod g(x). \end{aligned}$$

- $g(a_i) \neq 0$ implies $\gcd(x - a_i, g(x)) = 1$,
  so $g(x)$ divides $\sum_{i=1}^{n} c_i \prod_{j \neq i} (x - a_j)$.
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  Cancel out those $x - a_i$ with $c_i = 0$.
- The denominator is now $\prod_{i, c_i \neq 0} (x - a_i)$, of degree w.
- The numerator now has degree $w - 1$ and $\deg(g) > w - 1$
  implies that the numerator is $= 0$ (without reduction mod g),
  which is a contradiction to $\mathbf{c} \neq 0$, so $\mathrm{wt}(\mathbf{c}) = w \geq t + 1$.

Better minimum distance for $\Gamma(L, g)$

- Let $\mathbf{c} \neq 0$ have small weight $\mathrm{wt}(\mathbf{c}) = w$.
- Put $f(x) = \prod_{i=1}^{n}(x - a_i)^{c_i}$ with $c_i \in \{0, 1\}$.
- Then the derivative $f'(x) = \sum_{i=1}^{n} c_i \prod_{j \neq i}(x - a_i)^{c_i}$.
- Thus $s(x) = f'(x)/f(x) \equiv 0 \bmod g(x)$.
- As before this implies $g(x)$ divides the numerator $f'(x)$.
- Note that over $\mathbb{F}_{2^m}$:

$$(f_{2i+1}x^{2i+1})' = f_{2i+1}x^{2i}, \quad (f_{2i}x^{2i})' = 0 \cdot f_{2i}x^{2i-1} = 0,$$

  thus $f'(x)$ contains only terms of even degree and $\deg(f') \leq w - 1$. Assume w odd, thus $\deg(f') = w - 1$.
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  thus $f'(x)$ contains only terms of even degree and $\deg(f') \leq w - 1$. Assume w odd, thus $\deg(f') = w - 1$.
- Note that over $\mathbb{F}_{2^m}$: $(x + 1)^2 = x^2 + 1$ and in general

  $$f'(x) = \sum_{i=0}^{(w-1)/2} f_{2i+1}x^{2i} = \left(\sum_{i=0}^{(w-1)/2} \sqrt{f_{2i+1}}x^i \right)^2 = F^2(x).$$

- Since $g(x)$ is square-free, $g(x)$ divides $F(x)$, thus $w \geq 2t + 1$.

Decoding of $\mathbf{c}+\mathbf{e}$ in $\Gamma(L, g)$

- Decoding works with polynomial arithmetic.
- Fix $\mathbf{e}$. Let $\sigma(x)=\prod_{i, e_i \neq 0}(x-a_i)$. Same as $f(x)$ before for $\mathbf{c}$.
- $\sigma(x)$ is called error locator polynomial. Given $\sigma(x)$ can factor it to retrieve error positions, $\sigma(a_i)=0 \Leftrightarrow$ error in i.
- Split into odd and even terms: $\sigma(x)=A^2(x)+xB^2(x)$.
- Note as before $s(x)=\sigma'(x)/\sigma(x)$ and $\sigma'(x)=B^2(x)$.
- Thus

$$B^2(x) \equiv \sigma(x)s(x) \equiv (A^2(x)+xB^2(x))s(x) \bmod g(x)$$

$$B^2(x)(x+1/s(x)) \equiv A^2(x) \bmod g(x)$$

- Put $v(x) \equiv \sqrt{x+1/s(x)} \bmod g(x)$, then $A(x) \equiv B(x)v(x) \bmod g(x)$.
- Can compute $v(x)$ from $s(x)$.
- Use XGCD on v and g, stop part-way when

$$A(x)=B(x)v(x)+h(x)g(x),$$

with $\deg(A) \leq \lfloor t/2 \rfloor, \deg(B) \leq \lfloor (t-1)/2 \rfloor$.

Reminder: How to hide nice code?

- Do not reveal matrix H related to nice-to-decode code.
- Pick a random invertible $(n-k) \times (n-k)$ matrix S and random $n \times n$ *permutation matrix* P. Put

$$K = SHP.$$

- K is the public key and S and P together with a decoding algorithm for H form the private key.
- For suitable codes K looks like random matrix.
- How to decode syndrome $\mathbf{s} = K\mathbf{e}$?

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- How to decode syndrome $\mathbf{s} = K\mathbf{e}$?
- Computes $S^{-1}\mathbf{s} = S^{-1}(SHP)\mathbf{e} = H(P\mathbf{e})$.
- P permutes, thus $P\mathbf{e}$ has same weight as $\mathbf{e}$.
- Decode to recover $P\mathbf{e}$, then multiply by P^{-1}.

How to hide nice code?

- For Goppa code use secret polynomial $g(x)$.
- Use secret permutation of the a_i, this corresponds to secret permutation of the n positions; this replaces P.
- Use systematic form $K = (K'|I)$ for key;
  - This implicitly applies S.
  - No need to remember S because decoding does not use H.
  - Public key size decreased to $(n - k) \times k$.
- Secret key is polynomial g and support $L = (a_1, \ldots, a_n)$.

McBits (Bernstein, Chou, Schwabe, CHES 2013)

- Encryption is super fast anyways (just a vector-matrix multiplication).
- Main step in decryption is decoding of Goppa code. The McBits software achieves this in constant time.
- Decoding speed at 2^{128} pre-quantum security: $(n; t) = (4096; 41)$ uses 60493 Ivy Bridge cycles.
- Decoding speed at 2^{263} pre-quantum security: $(n; t) = (6960; 119)$ uses 306102 Ivy Bridge cycles.
- Grover speedup is less than halving the security level, so the latter parameters offer at least 2^{128} post-quantum security.
- More at `https://binary.cr.yp.to/mcbits.html`.

Do not use the schoolbook versions!

Sloppy Alice attacks! 1998 Verheul, Doumen, van Tilborg

- Assume that the decoding algorithm decodes up to t errors, i. e. it decodes $\mathbf{y} = \mathbf{c} + \mathbf{e}$ to $\mathbf{c}$ if $\mathrm{wt}(\mathbf{e}) \leq t$.
- Eve intercepts ciphertext $\mathbf{y} = \mathbf{m}G' + \mathbf{e}$.
  Eve poses as Alice towards Bob and sends him tweaks of $\mathbf{y}$.
  She uses Bob's reactions (success of failure to decrypt) to recover $\mathbf{m}$.
- Assume $\mathrm{wt}(\mathbf{e}) = t$. (Else flip more bits till Bob fails).
- Eve sends $\mathbf{y}_i = \mathbf{y} + \mathbf{e}_i$ for $\mathbf{e}_i$ the i-th unit vector.
  If Bob returns error, position i in $\mathbf{e}$ is 0 (so the number of errors has increased to $t + 1$ and Bob fails).
  Else position i in $\mathbf{e}$ is 1.
- After k steps Eve knows the first k positions of $\mathbf{m}G'$ without error. Invert the $k \times k$ submatrix of G' to get $\mathbf{m}$

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- After k steps Eve knows the first k positions of $\mathbf{m}G'$ without error. Invert the $k \times k$ submatrix of G' to get $\mathbf{m}$ assuming it is invertible.
- Proper attack: figure out invertible submatrix of G' at beginning; recover matching k coordinates.

More on sloppy Alice

- This attack has Eve send Bob variations of the same ciphertext; so Bob will think that Alice is sloppy.
- Note, this is more complicated if $\mathbb{F}_q$ instead of $\mathbb{F}_2$ is used.
- Other name: reaction attack.
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More on sloppy Alice

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- Attack also works on Niederreiter version:
  Bitflip cooresponds to sending $\mathbf{s}_i = \mathbf{s} + K_i$,
  where K_i is the i-th column of K.
- More involved but doable (for McEliece and Niederreiter) if decryption requires exactly t errors.

Berson's attack

- Eve knows $\mathbf{y}_1 = \mathbf{m}G' + \mathbf{e}_1$ and $\mathbf{y}_2 = \mathbf{m}G' + \mathbf{e}_2$; these have the same $\mathbf{m}$.

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- Then $\mathbf{y}_1 + \mathbf{y}_2 = \mathbf{e}_1 + \mathbf{e}_2 = \bar{\mathbf{e}}$. This has weight in $[0, 2t]$.
- If $\mathrm{wt}(\bar{\mathbf{e}}) = 2t$:

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  All zero positions in $\bar{\mathbf{e}}$ are error free in both ciphertexts.
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- Else: ignore the $2w = \mathrm{wt}(\bar{\mathbf{e}}) < 2t$ positions in G' and $\mathbf{y}_1$.
  Solve decoding problem for $k \times (n - 2w)$ generator matrix G'' and vector $\mathbf{y}_1'$ with $t - w$ errors; typically much easier.

Formal security notions

- McEliece/Niederreiter are One-Way Encryption (OWE) schemes.
- However, the schemes as presented are not CCA–II secure:
  - Given challenge $\mathbf{y} = \mathbf{m}G' + \mathbf{e}$, Eve can ask for decryptions of anything but $\mathbf{y}$.

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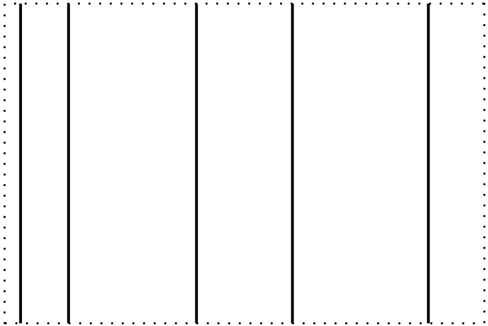
  - This is different from challenge $\mathbf{y}$, so Bob answers.
  - Answer is $\mathbf{m} + \bar{\mathbf{m}}$.
- Fix by using CCA2 transformation (e.g. Fujisaki-Okamoto transform) or (easier) KEM/DEM version:
  pick random $\mathbf{e}$ of weight t, use hash($\mathbf{e}$) as secret key to encrypt and authenticate (for McEliece or Niederreiter).

Generic attack: Brute force

Given K and $\mathbf{s} = K\mathbf{e}$, find $\mathbf{e}$ with $\mathrm{wt}(\mathbf{e}) = t$.

$K =$



Pick any group of t columns of K, add them and compare with $\mathbf{s}$.

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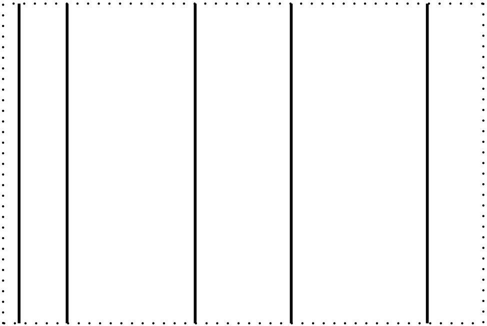
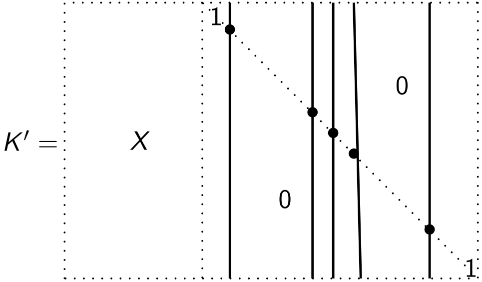
$K =$ 

Pick any group of t columns of K, add them and compare with $\mathbf{s}$.

Cost: $\binom{n}{t}$ sums of t columns.
Can do better so that each try costs only 1 column addition
(after some initial additions).
Cost: $O\binom{n}{t}$ sums of t columns.

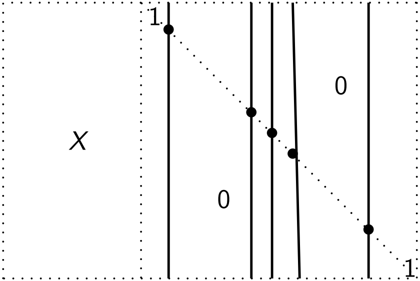
Generic attack: Information-set decoding, 1962 Prange



1. Permute K and bring to systematic form $K' = (X|I_{n-k})$.
   (If this fails, repeat with other permutation).
2. Then $K' = UKP$ for some permutation matrix P and U the matrix that produces systematic form.
3. This updates $\mathbf{s}$ to $U\mathbf{s}$.
4. If $\mathrm{wt}(U\mathbf{s}) = t$ then $\mathbf{e}' = (00\ldots0)||U\mathbf{s}$.
   Output unpermuted version of $\mathbf{e}'$.
5. Else return to 1 to rerandomize.

Cost:

Generic attack: Information-set decoding, 1962 Prange

$K' =$ 

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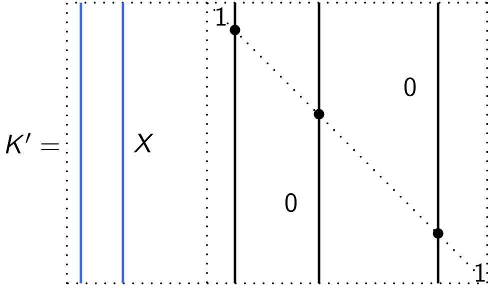
5. Else return to 1 to rerandomize.

Cost: $O(\binom{n}{t}/\binom{n-k}{t})$ matrix operations.

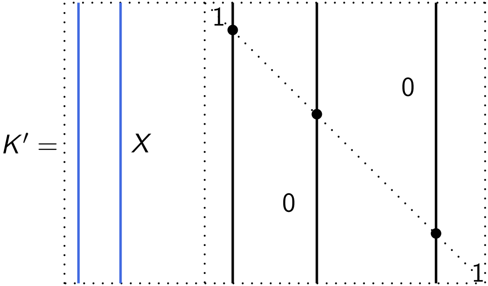
Lee–Brickell attack



1. Permute K and bring to systematic form $K' = (X|I_{n-k})$. (If this fails, repeat with other permutation). $\mathbf{s}$ is updated.
2. For small p, pick p of the k columns on the left, compute their sum $X\mathbf{p}$. ($\mathbf{p}$ is the vector of weight p).
3. If $\mathrm{wt}(\mathbf{s} + X\mathbf{p}) = t - p$ then put $\mathbf{e}' = \mathbf{p}||(\mathbf{s} + X\mathbf{p})$. Output unpermuted version of $\mathbf{e}'$.
4. Else return to 2 or return to 1 to rerandomize.

Cost:

Lee–Brickell attack



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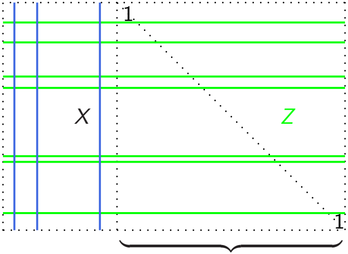
   Output unpermuted version of $\mathbf{e}'$.
4. Else return to 2 or return to 1 to rerandomize.

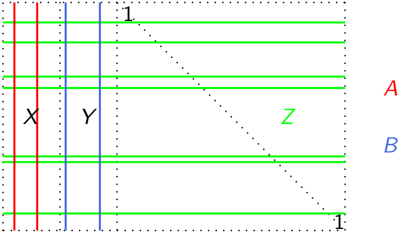
Cost: $O(\binom{n}{t}/(\binom{k}{p}\binom{n-k}{t-p}))$ [matrix operations$+\binom{k}{p}$ column additions].

Leon's attack



- Setup similar to Lee-Brickell's attack.
- Random combinations of p vectors will be dense, so have $\mathrm{wt}(\mathbf{s} + X\mathbf{p}) \sim k/2$.
- Idea: Introduce early abort by checking only ℓ positions (selected by set Z, green lines in the picture). This forms $\ell \times k$ matrix X_Z, length-ℓ vector $\mathbf{s}_Z$.
- Inner loop becomes:
  1. Pick $\mathbf{p}$ with $\mathrm{wt}(\mathbf{p}) = p$.
  2. Compute $X_Z\mathbf{p}$.
  3. If $\mathbf{s}_Z + X_Z\mathbf{p} \neq 0$ goto 1.
  4. Else compute $X\mathbf{p}$.
     - 4.1 If $\mathrm{wt}(\mathbf{s} + X\mathbf{p}) = t - p$ then put $\mathbf{e}' = \mathbf{p}||(\mathbf{s} + X\mathbf{p})$. Output unpermuted version of $\mathbf{e}'$.
     - 4.2 Else return to 1 or rerandomize K.
- Note that $\mathbf{s}_Z + X_Z\mathbf{p} = 0$ means that there are no ones in the positions specified by Z. Small loss in success, big speedup.

Stern's attack



- Setup similar to Leon's and Lee-Brickell's attacks.
- Use the early abort trick, so specify set Z.
- Improve chances of finding $\mathbf{p}$ with $\mathbf{s} + X_Z\mathbf{p} = 0$:
  - Split left part of K' into two disjoint subsets X and Y.
  - Let $A = \{\mathbf{a} \in \mathbb{F}_2^{k/2} | \mathrm{wt}(\mathbf{a}) = p\}$, $B = \{\mathbf{b} \in \mathbb{F}_2^{k/2} | \mathrm{wt}(\mathbf{b}) = p\}$.
  - Search for words having exactly p ones in X and p ones in Y and exactly $w - 2p$ ones in the remaining columns.
  - Do the latter part as a collision search:
    Compute $\mathbf{s}_Z + X_Z\mathbf{a}$ for all (many) $\mathbf{a} \in A$, sort.
    Then compute $Y_Z\mathbf{b}$ for $\mathbf{b} \in B$ and look for collisions; expand.
  - Iterate until word with $\mathrm{wt}(\mathbf{s} + X\mathbf{a} + Y\mathbf{b}) = 2p$ is found for some X, Y, Z.
- Select p, ℓ, and the subset of A to minimize overall work.

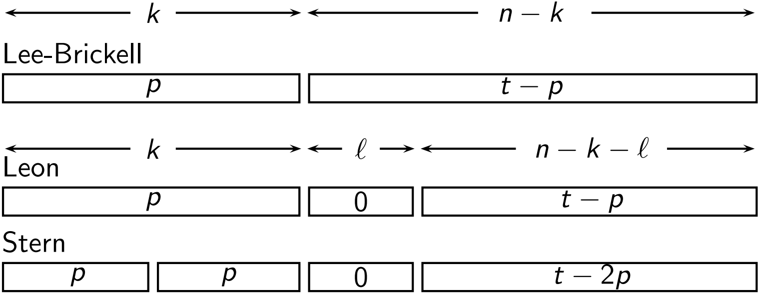
Running time in practice

2008 Bernstein, Lange, Peters.

- Wrote attack software against original McEliece parameters, decoding 50 errors in a $[1024, 524]$ code.
- Lots of optimizations, e.g. cheap updates between $\mathbf{s}_Z + X_Z\mathbf{a}$ and next value for $\mathbf{a}$; optimized frequency of K randomization.
- Attack on a single computer with a 2.4GHz Intel Core 2 Quad Q6600 CPU would need, on average, 1400 days (2^{58} CPU cycles) to complete the attack.
- About 200 computers involved, with about 300 cores.
- Most of the cores put in far fewer than 90 days of work; some of which were considerably slower than a Core 2.
- Computation used about 8000 core-days.
- Error vector found by Walton cluster at SFI/HEA Irish Centre of High-End Computing (ICHEC).

Information-set decoding

Methods differ in where the "errors" are allowed to be.



Running time is exponential for Goppa parameters n, k, d.

Information-set decoding

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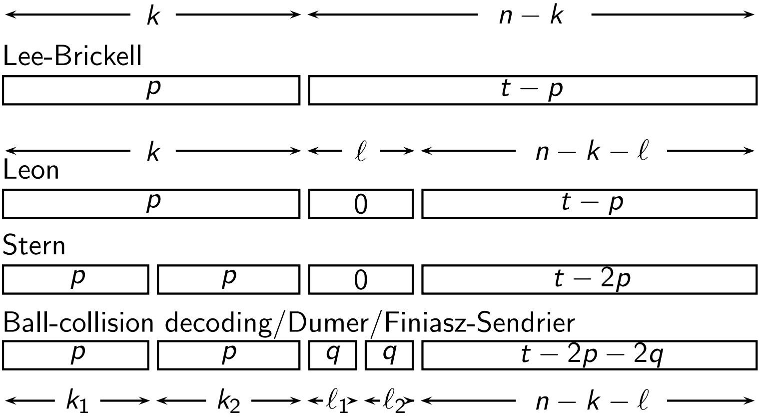

2011 May-Meurer-Thomae and 2012 Becker-Joux-May-Meurer refine multi-level collision search. No change in exponent for Goppa parameters n, k, d.

Improvements

- Increase n: The most obvious way to defend McEliece's cryptosystem is to increase the code length n.
- Allow values of n between powers of 2: Get considerably better optimization of (e.g.) the McEliece public-key size.
- Use list decoding to increase t: Unique decoding is ensured by CCA2-secure variants.
- Decrease key size by using fields other than $\mathbb{F}_2$ (wild McEliece).
- Decrease key size & be faster by using other codes. Needs security analysis: some codes have too much structure.

More exciting codes

- We distinguish between generic attacks (such as information-set decoding) and structural attacks (that use the structure of the code).
- Gröbner basis computation is a generally powerful tool for structural attacks.
- Cyclic codes need to store only top row of matrix, rest follows by shifts. Quasi-cyclic: multiple cyclic blocks.
- QC Goppa: too exciting, too much structure.
- Interesting candidate: Quasi-cyclic Moderate-Density Parity-Check (QC-MDPC) codes, due to Misoczki, Tillich, Sendrier, and Barreto (2012).
  Very efficient but practical problem if the key is reused (Asiacrypt 2016).
- Hermitian codes, general algebraic geometry codes.
- Please help us update https://pqcrypto.org/code.html.