Batch NFS

Tanja Lange
Technische Universiteit Eindhoven

joint work with
D. J. Bernstein
University of Illinois at Chicago &
Technische Universiteit Eindhoven

Notation

In this talk $\log L$ means $(1 + o(1))(\log N)^{1/3}(\log\log N)^{2/3}$.
L is often written "$L_N(1/3)$" or "$L_N(1/3)^{1+o(1)}$".

In general, $\log L_N(\alpha) = (1 + o(1))(\log N)^{\alpha}(\log\log N)^{1-\alpha}$.

$\alpha = 0$: polynomial time
$\log L_N(\alpha) = (1 + o(1))(\log\log N)$.
$\alpha = 1$: exponential time
$\log L_N(\alpha) = (1 + o(1))(\log N)$.

Exponents of L in this talk are limited to $10^{-6}\mathbf{Z}$.

Breaking RSA-1024

2003 Shamir–Tromer, 2003 Lenstra–Tromer–Shamir–Kortsmit–Dodson–Hughes–Leyland, 2005 Geiselmann–Shamir–Steinwandt–Tromer, 2005 Franke–Kleinjung–Paar–Pelzl–Priplata–Stahlke, etc.: RSA-1024 is breakable in a year by an attack machine costing $<10^9$ dollars.

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Wrong!

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Another example: SSL has used many millions of RSA-1024 keys. Imagine that an attacker has recorded tons of SSL traffic.

Users seem unconcerned:

1. “The attack machine costs more than this RSA key is worth.”

2. “The attack machine isn’t off-the-shelf; it’s only for attackers building ASICs.”

3. For signatures: “We switch keys every month, and the attack machine takes a year.”

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Real quote: “DNSSEC signing keys should be large enough to avoid all known cryptographic attacks during the effectivity period of the key.”

Continuation of quote: “To date, despite huge efforts, no one has broken a regular 1024-bit key; in fact, the best completed attack is estimated to be the equivalent of a 700-bit key. An attacker breaking a 1024-bit signing key would need to expend phenomenal amounts of networked computing power in a way that would not be detected in order to break a single key. Because of this, it is estimated that most zones can safely use 1024-bit keys for at least the next ten years.”

Goal of our “Batch NFS” paper: analyze the *asymptotic* cost, specifically *price-performance ratio*, of breaking *many* RSA keys.

“Many”: e.g. millions.

“Price-performance ratio”: **area-time product** for chips.

“RAM” metric (adding two 64-bit integers has same cost as accessing array of size 2^{64}) is not realistic; “AT” metric is realistic.

“Asymptotic”: We systematically suppress polynomial factors. Our speedups are superpolynomial.

Best result known for *one* key:
time $L^{1.185632}$
using chip area $L^{0.790420}$;
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Our paper also looks more closely at $L^{o(1)}$, analyzing asymptotic speedup from early-abort ECM. Results are not what one would guess from 1982 Pomerance.

Asymptotic consequences:

1. Attack cost per key is reduced, so attacker can target lower-value keys.

2. Primary bottleneck is low-memory factorization—well suited for off-the-shelf graphics cards.

3. Attack time is reduced (and can be reduced more), breaking key rotation.

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"Do the asymptotics really kick in before 1024 bits?" — Maybe not, but no basis for confidence.

Eratosthenes for smoothness

Sieving small integers $i > 0$ using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
5							5	
6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
11								
12	2	2			3			
13								
14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

etc.

The **Q** sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
5							5	
6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
11								
12	2	2			3			
13								
14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

612	2	2			3	3						
613												
614	2											
615					3			5				
616	2	2	2									7
617												
618	2				3							
619												
620	2	2						5				
621					3	3	3					
622	2											
623												7
624	2	2	2	2	3							
625								5	5	5	5	
626	2											
627					3							
628	2	2										
629												
630	2				3	3		5				7
631												

etc.

Have complete factorization of the congruences $i \equiv 611 + i$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$
$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd\{611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2\}$
$= 47$.

$611 = 47 \cdot 13$.

The number-field sieve

Generalize $i \equiv i + N \pmod{N}$
$\to a \equiv a + bN \pmod{N}$
$\to a - bm \equiv a - b\alpha \pmod{m - \alpha}$
for root $\alpha \in \mathbf{C}$
of nonzero integer poly.

For any m can find α
so that factoring $m - \alpha$
produces factorization of N.

Optimal choice of $\log m$ is
$(\mu + o(1))(\log N)^{2/3}(\log\log N)^{1/3}$.

RAM cost analysis

1993 Buhler–Lenstra–Pomerance:
Smoothness bound $L^{0.961500}$.
Sieve $L^{1.923000}$ pairs (a, b).
Find $L^{0.961500}$ pairs
with $a - bm$ and $a - b\alpha$ smooth.
Total RAM time $L^{1.923000}$.

1993 Coppersmith:
Total RAM time $L^{1.901884}$
using multiple number fields.

(Multiple number fields
don't seem to combine well
with AT, factory, et al.)

AT cost analysis

Sieving is a disaster
in realistic cost metric.
AT cost $L^{2.403750}$.

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Semi-fix: Reduce smoothness
bounds to rebalance.
AT cost $L^{1.976052}$.
(2001 Bernstein)

The factorization factory

1993 Coppersmith:
There *exists* an algorithm
that factors any integer
with same #bits as N
in RAM time $L^{1.638587}$.

Smoothness bound $L^{0.819290}$.
Smaller than before,
so need more (a, b).

Algorithm *knows* all (a, b)
such that $a - bm$ is smooth.
Note: one m works for all N.
Algorithm uses ECM to check
whether $a - b\alpha_N$ is smooth.

Factorization factory

Finding this algorithm
is slower than running it.
Need to precompute all (a, b)
such that $a - bm$ is smooth.
RAM time $L^{2.006853}$.

The DL situation (See Nadia's talk)

Fixed prime p, DLs in $\langle g \rangle \subseteq \mathbf{F}_p^*$
$L^{1.923000}$ precomputation
to get $\log_g p_i$, small primes p_i.
Barbulescu 2013:
$L^{1.232}$ individual log.

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Barbulescu "DL factory" 2013:
$L^{2.006853}$ precomputation
of smooth $a - bm$,
m depends on $\log_2 p$,
shared over many big primes p.
$L^{1.638587}$ computation per p
(same cost as Coppersmith).
$L^{1.232}$ per individual log.

Back to factorization factory

Finding this algorithm
RAM time $L^{2.006853}$.

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Standard conversion of
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if there are enough targets,
more than $L^{0.368266}$,
then precomputation cost
becomes negligible.

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Standard conversion of precomputation into batching: if there are enough targets, more than $L^{0.368266}$, then precomputation cost becomes negligible.

The big problem: Coppersmith's algorithm has size $L^{1.638587}$. Huge *AT* cost; useless in reality.

Batch NFS

Goal: Optimize AT asymptotics.

1. Generate (a, b) in parallel.
Test $a - bm$ for smoothness.

2. Make many copies of each N,
close to each (a, b) generator.
When smooth $a - bm$ is found,
test each $a - b\alpha_N$ for smoothness.

3. After all smooths are found,
reorganize: for each N, bring
relevant (a, b) close together.

4. Linear algebra.

Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.	Generate (a, b). Is $a - bm$ smooth? If so, store. Repeat.
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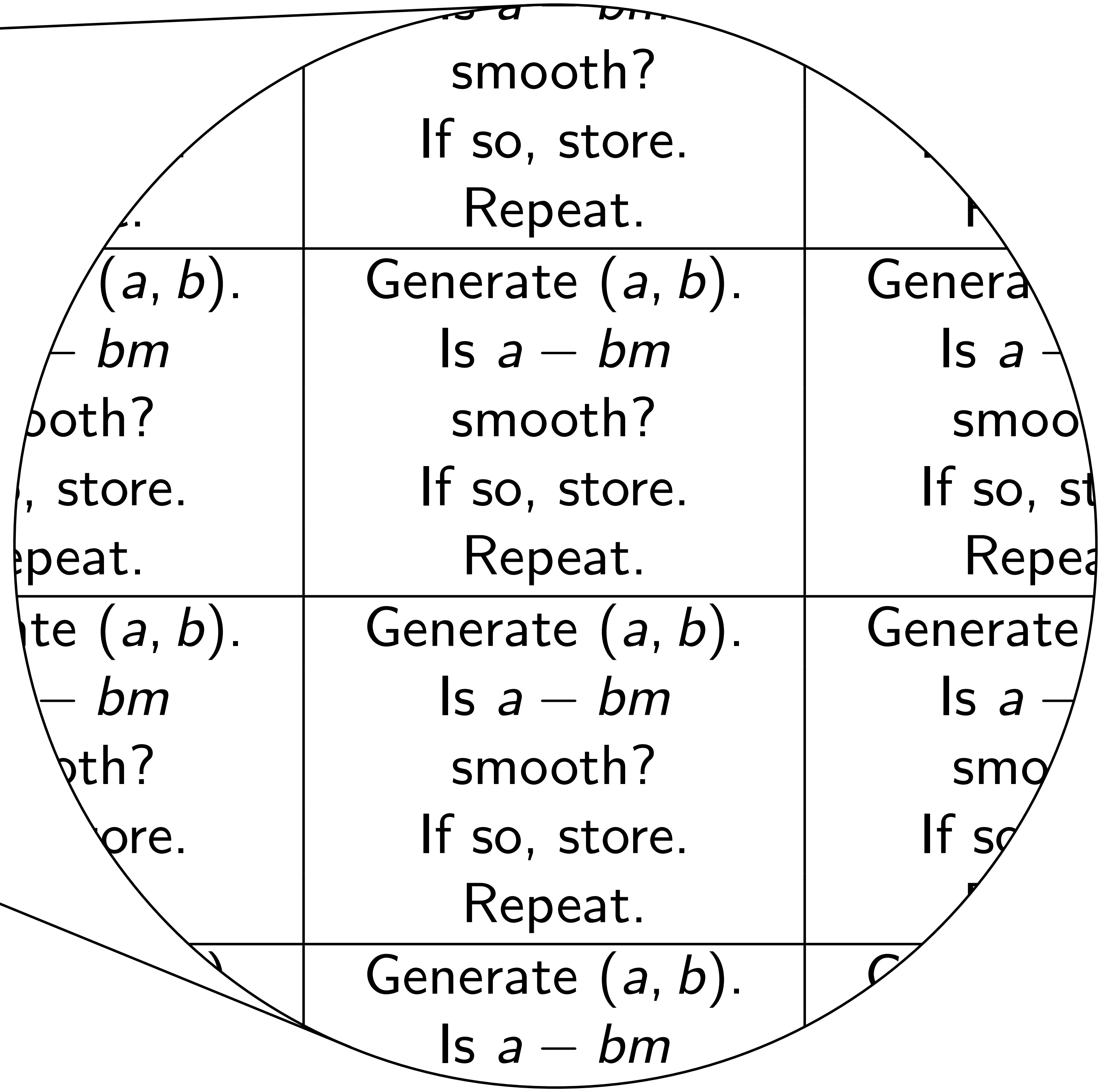
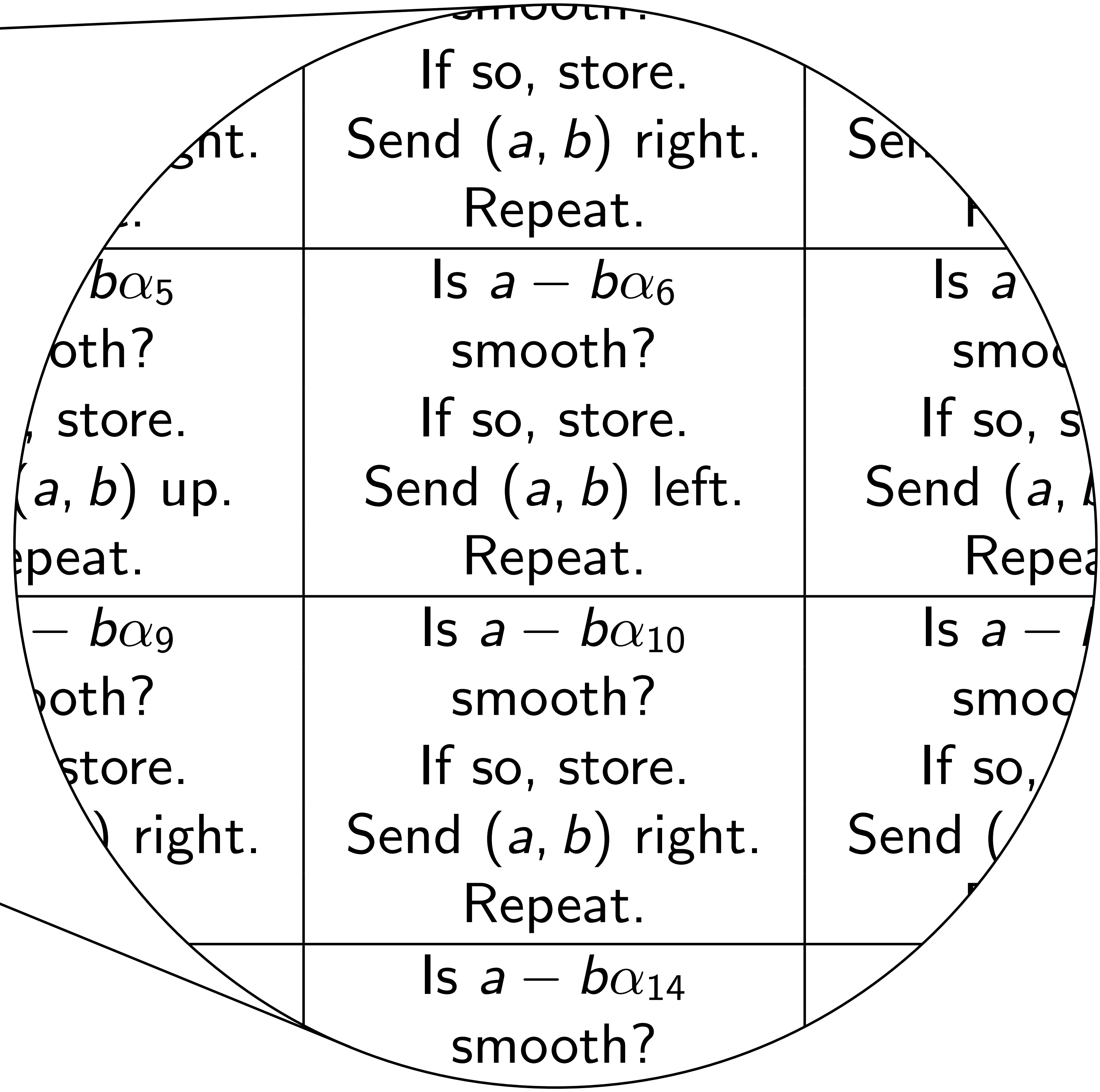


Is $a - b\alpha_1$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_2$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_3$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_4$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_5$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_6$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_7$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_8$ smooth? If so, store. Send (a, b) left. Repeat.
Is $a - b\alpha_9$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{10}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{11}$ smooth? If so, store. Send (a, b) right. Repeat.	Is $a - b\alpha_{12}$ smooth? If so, store. Send (a, b) down. Repeat.
Is $a - b\alpha_{13}$ smooth? If so, store. Send (a, b) up. Repeat.	Is $a - b\alpha_{14}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{15}$ smooth? If so, store. Send (a, b) left. Repeat.	Is $a - b\alpha_{16}$ smooth? If so, store. Send (a, b) left. Repeat.



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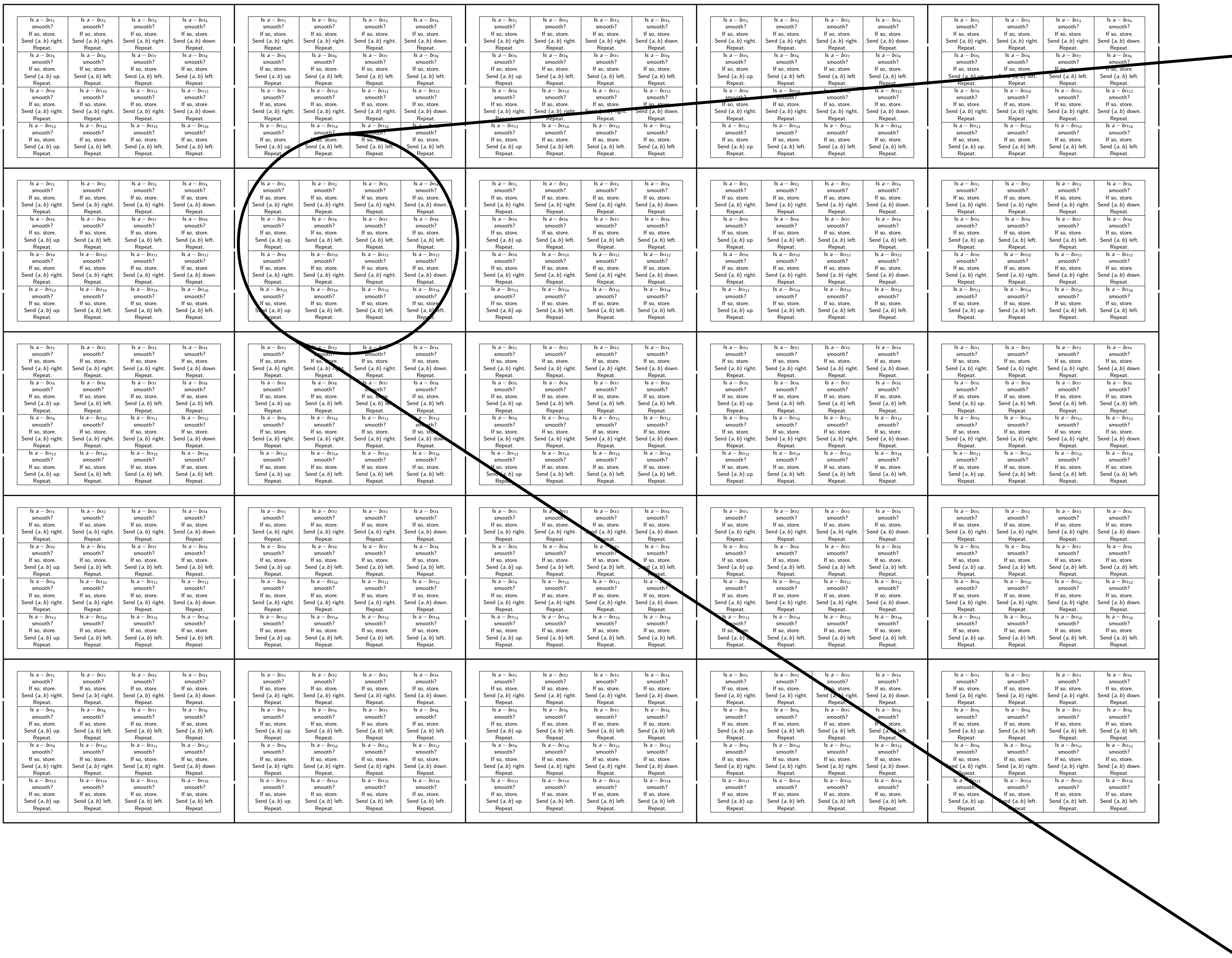

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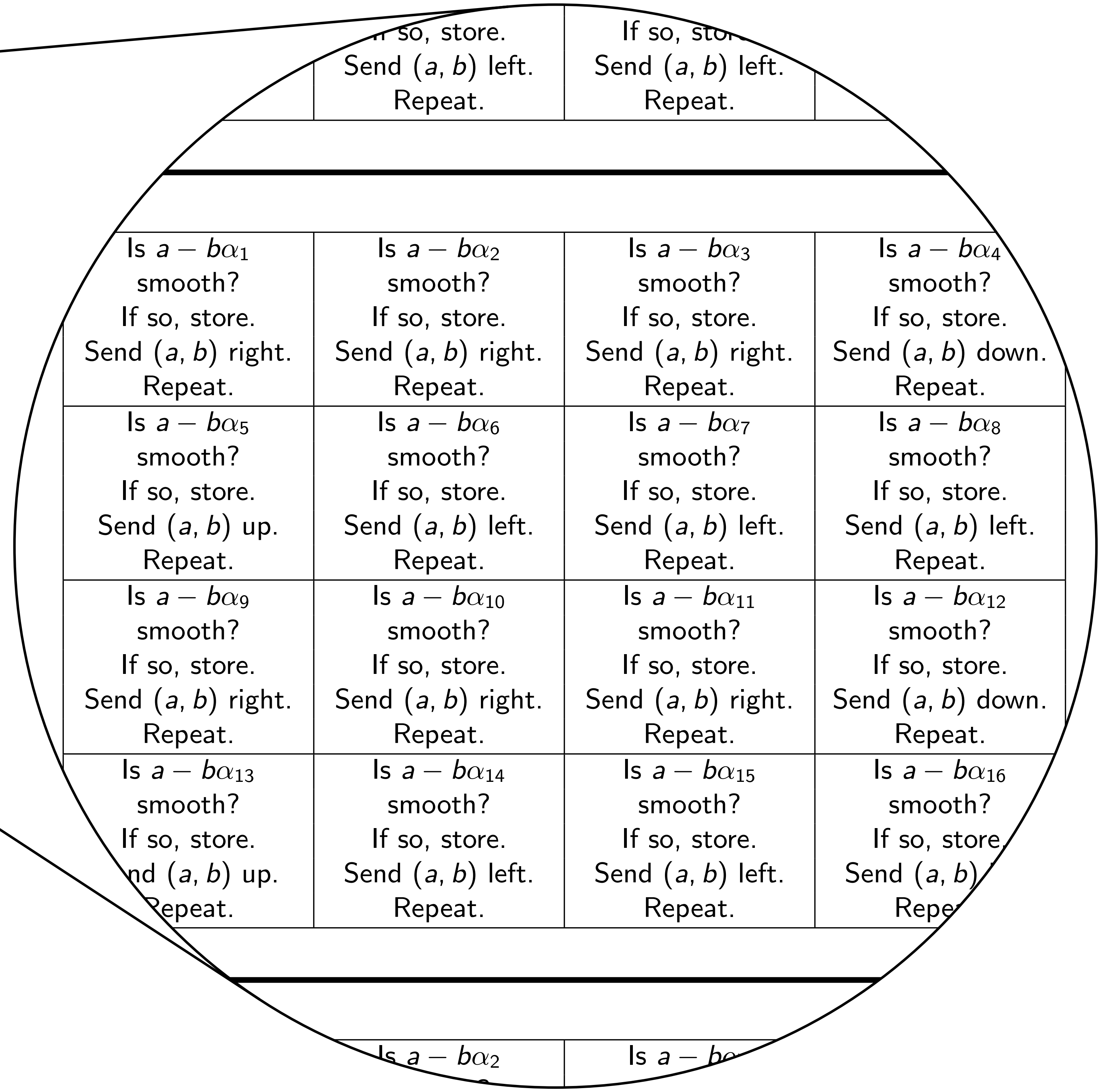

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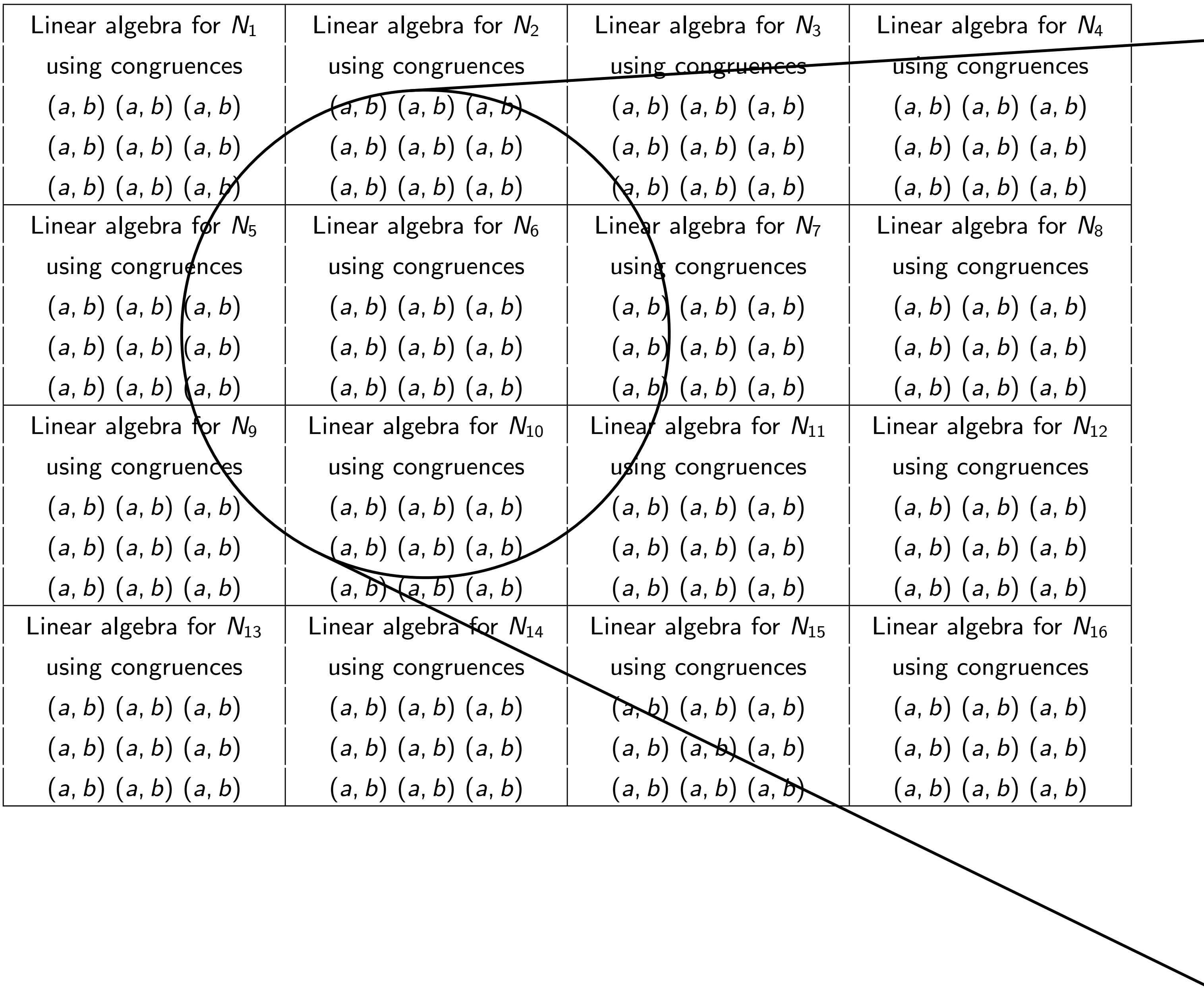
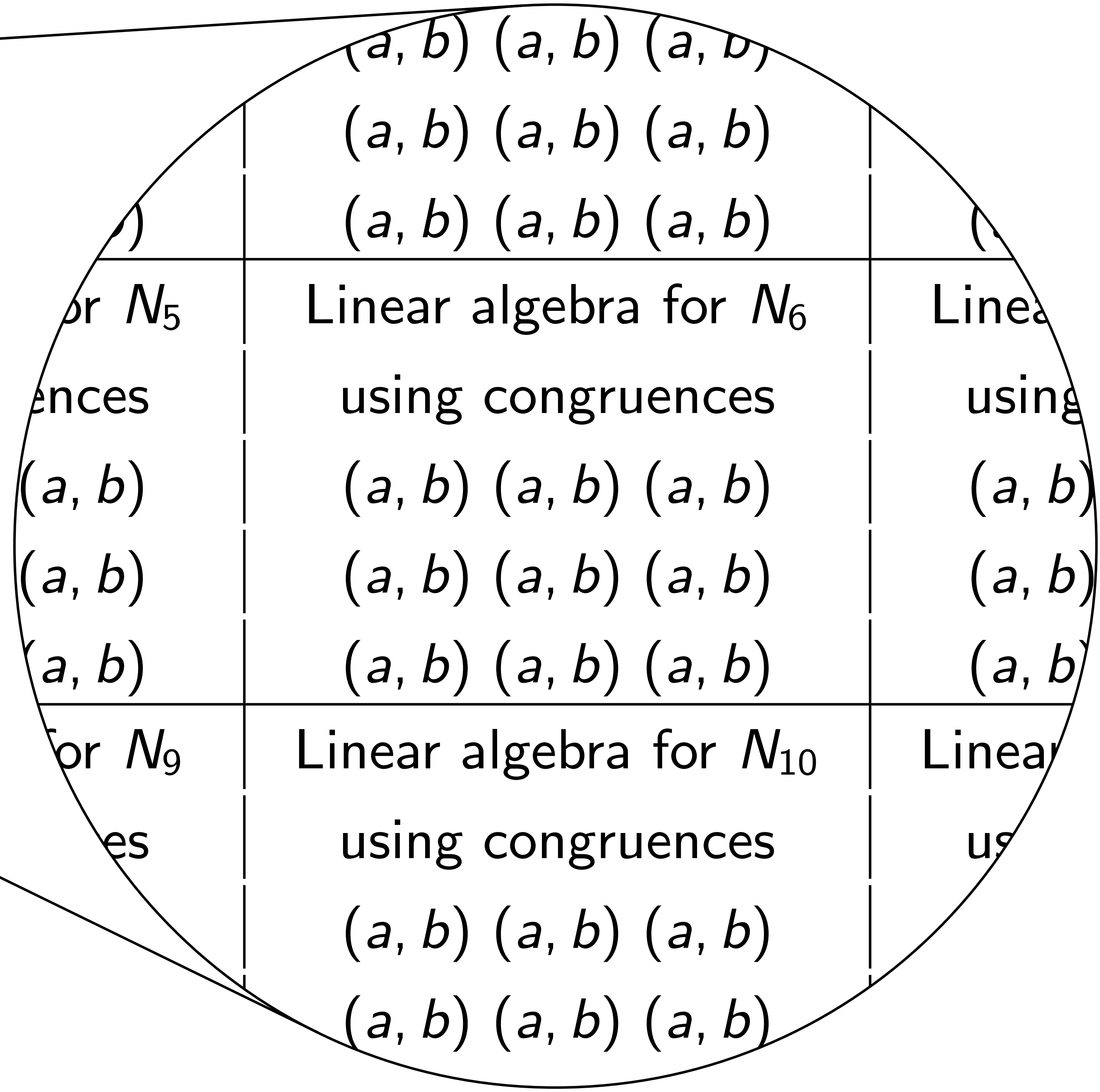


Linear algebra for N_1	Linear algebra for N_2	Linear algebra for N_3	Linear algebra for N_4
using congruences	using congruences	using congruences	using congruences
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
Linear algebra for N_5	Linear algebra for N_6	Linear algebra for N_7	Linear algebra for N_8
using congruences	using congruences	using congruences	using congruences
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
Linear algebra for N_9	Linear algebra for N_{10}	Linear algebra for N_{11}	Linear algebra for N_{12}
using congruences	using congruences	using congruences	using congruences
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
Linear algebra for N_{13}	Linear algebra for N_{14}	Linear algebra for N_{15}	Linear algebra for N_{16}
using congruences	using congruences	using congruences	using congruences
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)
(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)	(a, b) (a, b) (a, b)



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or N_5 ences (a, b) (a, b) $a, b)$	Linear algebra for N_6 using congruences $(a, b) \; (a, b) \; (a, b)$ $(a, b) \; (a, b) \; (a, b)$ $(a, b) \; (a, b) \; (a, b)$	Linea using (a, b) (a, b) $(a, b$
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