Signatures and DLP

Tanja Lange
Technische Universiteit Eindhoven

with some slides by
Daniel J. Bernstein

ECDSA

Users can sign messages
using Edwards curves.

Take a point P on an Edwards
curve modulo a prime $q > 2$.

ECDSA signer needs to know
the *order of* P.

There are only finitely many other
points; about q in total.
Adding P to itself will eventually
reach $(0, 1)$; let ℓ be the smallest
integer > 0 with $\ell P = (0, 1)$.
This ℓ is the order of P.

The signature scheme has as system parameters a curve E; a base point P; and a hash function h with output length at least $\lfloor \log_2 \ell \rfloor + 1$.
Alice's secret key is an integer a and her public key is $P_A = aP$.

To sign message m,
Alice computes $h(m)$;
picks random k;
computes $R = kP = (x_1, y_1)$;
puts $r \equiv y_1 \bmod \ell$; computes $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.

The signature on m is (r, s).

Anybody can verify signature
given m and (r, s):
Compute $w_1 \equiv s^{-1}h(m) \bmod \ell$
and $w_2 \equiv s^{-1} \cdot r \bmod \ell$.
Check whether the y-coordinate
of $w_1P + w_2P_A$ equals r modulo ℓ
and if so, accept signature.

Alice's signatures are valid:

$$w_1P + w_2P_A =$$
$$(s^{-1}h(m))P + (s^{-1} \cdot r)P_A =$$
$$(s^{-1}(h(m) + ra))P = kP$$

and so the y-coordinate of this
expression equals r,
the y-coordinate of kP.

Attacker's view on signatures

Anybody can produce an $R = kP$.
Alice's private key is only used in
$s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$.

Can fake signatures if one can break the DLP, i.e., if one can compute a from P_A.
Lectures today and tomorrow deal with methods for breaking DLPs.

Sometimes attacks are easier...

If k is known for some m, (r, s) then

If k is known for some $m, (r, s)$ then $a \equiv (sk - h(m))/r \bmod \ell$.

If two signatures $m_1, (r, s_1)$ and $m_2, (r, s_2)$ have the same value for r:

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If two signatures $m_1, (r, s_1)$ and $m_2, (r, s_2)$ have the same value for r: assume $k_1 = k_2$; observe $s_1 - s_2 = k_1^{-1}(h(m_1) + ra - (h(m_2) + ra))$; compute $k = (s_1 - s_2)/(h(m_1) - h(m_2))$. Continue as above.

If bits of many k's are known (biased PRNG) can attack $s \equiv k^{-1}(h(m) + r \cdot a) \bmod \ell$ as hidden number problem using lattice basis reduction.

Malicious signer

Alice can set up her public key so that two messages of her choice share the same signature,
i.e., she can claim to have signed m_1 or m_2 at will:
$R = (x_1, y_1)$ and $-R = (-x_1, y_1)$ have the same y-coordinate.
Thus, (r, s) fits $R = kP$,
$s \equiv k^{-1}(h(m_1) + ra) \bmod \ell$
and $-R = (-k)P$,
$s \equiv -k^{-1}(h(m_2) + ra) \bmod \ell$ if
$a \equiv -(h(m_1)+h(m_2))/2r \bmod \ell$.

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(Easy tweak: include bit of x_1.)

EdDSA

"High-speed high-security signatures" (Bernstein–Duif–L–Schwabe–Yang, CHES 2011).

Uses $k = \text{hash}(b, m)$;

b is second secret key.

Make h dependent on R and P_A:

$h = \text{hash}(R, P_A, m)$.

No inversions mod ℓ:

$s \equiv k + ha \bmod \ell$.

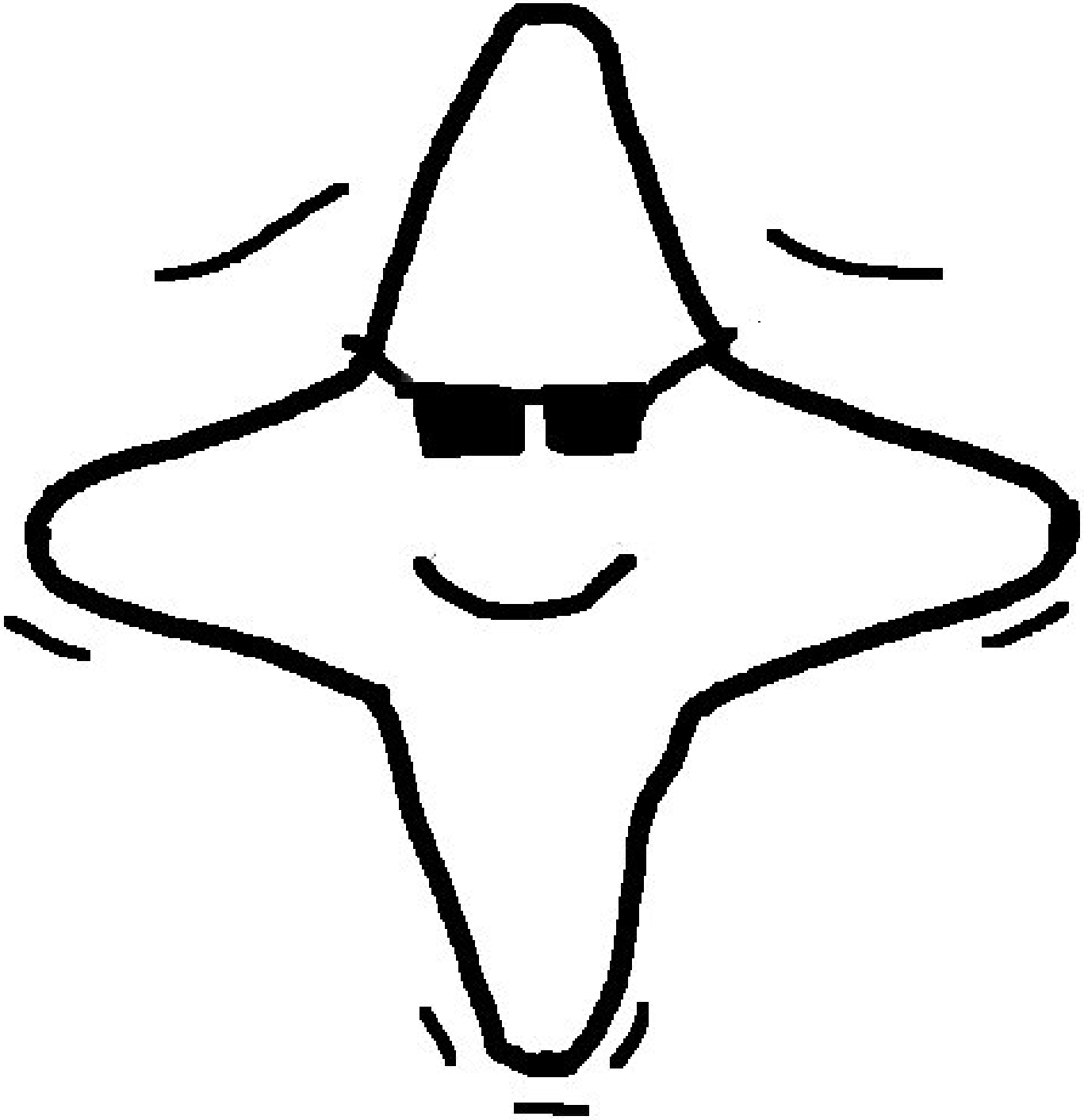
Verification:

does sP equal $R + hP_A$?

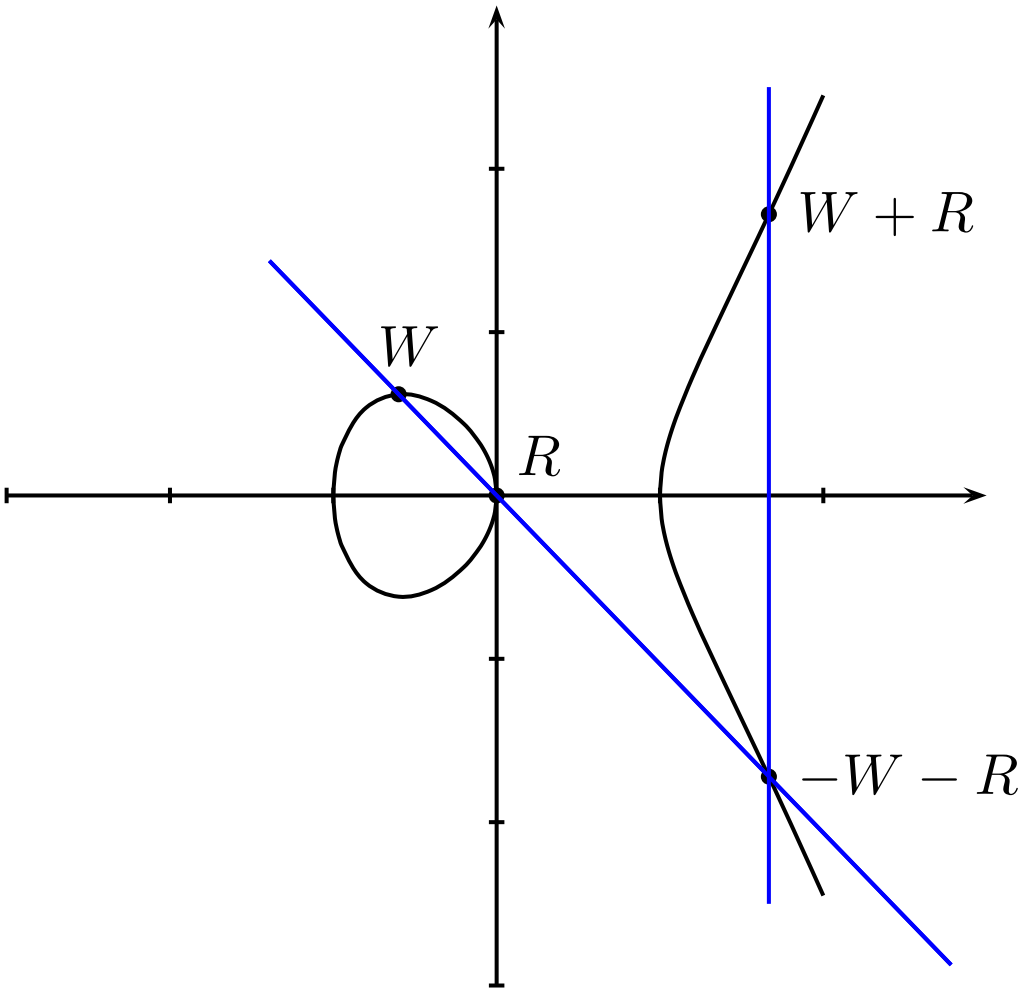
Use Edwards curves!

Very fast signing and verifying.

Edwards curves are cool



Elliptic-curve groups



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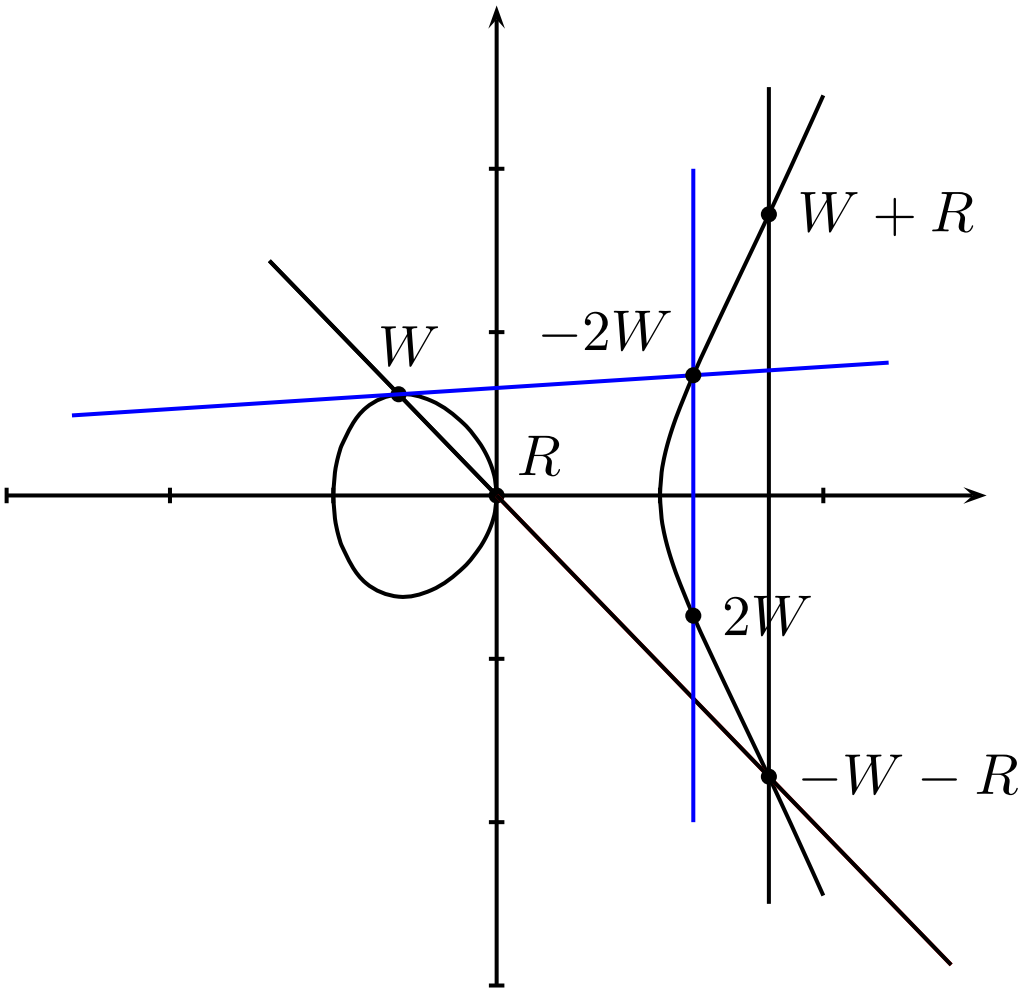

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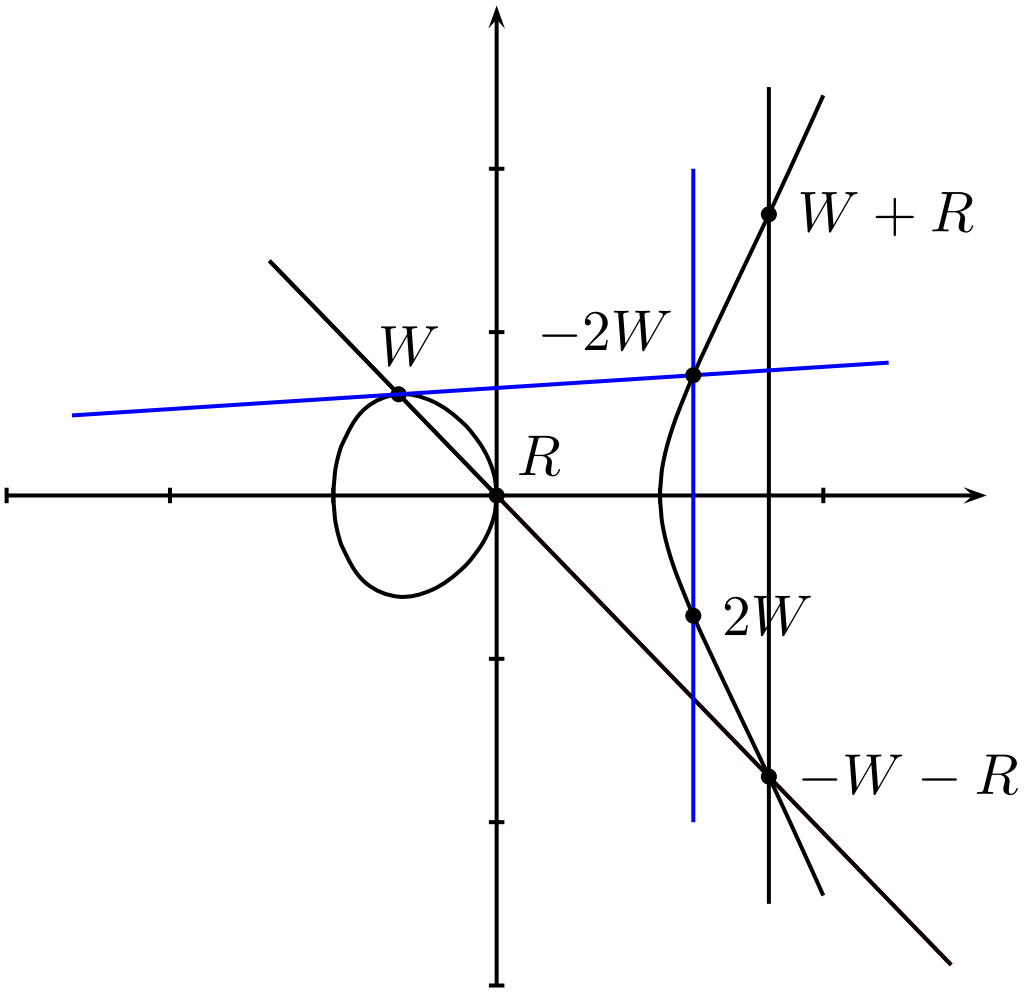
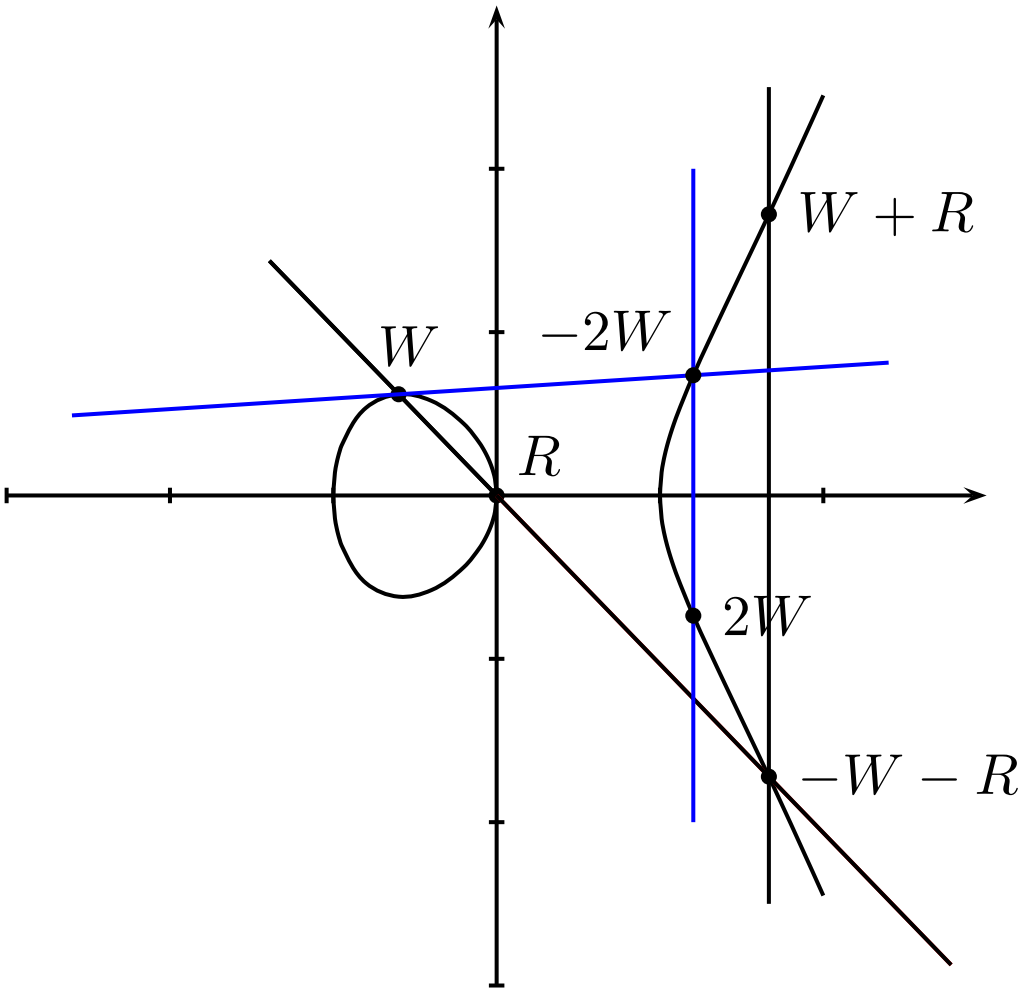


Following algorithms will need a unique representative per point. For that Weierstrass curves are the speed leader.

Elliptic-curve groups



Following algorithms will need a unique representative per point.

For that Weierstrass curves are the speed leader. . . . and I thought turtles were defensive.

The discrete-logarithm problem

Define $p = 1000003$ and
consider the Weierstrass curve
$y^2 = x^3 - x$ over $\mathbf{F}_p$.
This curve has
$1000004 = 2^2 \cdot 53^2 \cdot 89$
points and $P = (101384, 614510)$
is a point of order $2 \cdot 53^2 \cdot 89$.

The discrete-logarithm problem

Define $p = 1000003$ and consider the Weierstrass curve $y^2 = x^3 - x$ over $\mathbf{F}_p$.

This curve has $1000004 = 2^2 \cdot 53^2 \cdot 89$ points and $P = (101384, 614510)$ is a point of order $2 \cdot 53^2 \cdot 89$.

In general, point counting over $\mathbf{F}_p$ runs in time polynomial in $\log p$.

Number of points in $[p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$.

The group is isomorphic to $\mathbf{Z}/n \times \mathbf{Z}/m$, where $n|m$ and $n|(p-1)$.

Can we find an integer
$n \in \{1, 2, 3, \ldots, 500001\}$
such that $nP = (670366, 740819)$?

This point was generated as a multiple of P; could also be outside cyclic group.

Could find n by brute force.
Is there a faster way?

Understanding brute force

Can compute successively
$1P = (101384, 614510)$,
$2P = (102361, 628914)$,
$3P = (77571, 87643)$,
$4P = (650289, 31313)$,
$500001P = -P$.
$500002P = \infty$.

At some point we'll find n
with $nP = (670366, 740819)$.

Maximum cost of computation:
$\leq$ 500001 additions of P;
$\leq$ 500001 nanoseconds on a CPU
that does 1 ADD/nanosecond.

This is negligible work
for $p \approx 2^{20}$.

But users can
standardize a larger p,
making the attack slower.

Attack cost scales linearly:
$\approx 2^{50}$ ADDs for $p \approx 2^{50}$,
$\approx 2^{100}$ ADDs for $p \approx 2^{100}$, etc.

(Not exactly linearly:
cost of ADDs grows with p.
But this is a minor effect.)

Computation has a good chance
of finishing earlier.
Chance scales linearly:
1/2 chance of 1/2 cost;
1/10 chance of 1/10 cost; etc.

"So users should choose large n."

That's pointless. We can apply
"random self-reduction":
choose random r, say 69961;
compute $rP = (593450, 987590)$;
compute $(r + n)P$ as
$(593450, 987590)+(670366, 740819)$;
compute discrete log;
subtract r mod 500002; obtain n.

Computation can be parallelized.

One low-cost chip can run
many parallel searches.
Example, 2^6 €: one chip,
2^{10} cores on the chip,
each 2^{30} ADDs/second?
Maybe; see SHARCS workshops
for detailed cost analyses.

Attacker can run
many parallel chips.
Example, 2^{30} €: 2^{24} chips,
so 2^{34} cores,
so 2^{64} ADDs/second,
so 2^{89} ADDs/year.

Multiple targets and giant steps

Computation can be applied
to many targets at once.

Given 100 DL targets n_1P,
n_2P, ..., $n_{100}P$:
Can find *all* of $n_1, n_2, \ldots, n_{100}$
with $\leq$ 500002 ADDs.

Simplest approach: First build
a sorted table containing
n_1P, ..., $n_{100}P$.
Then check table for
$1P$, $2P$, etc.

Interesting consequence #1:
Solving all 100 DL problems
isn't much harder than
solving one DL problem.

Interesting consequence #2:
Solving *at least one*
out of 100 DL problems
is much easier than
solving one DL problem.

When did this computation
find its *first* n_i?
Typically $\approx 500002/100$ mults.

Can use random self-reduction
to turn a single target
into multiple targets.

Given nP:
Choose random $r_1, r_2, \ldots, r_{100}$.
Compute $r_1P + nP$,
$r_2P + nP$, etc.

Solve these 100 DL problems.
Typically $\approx \ell/100$ mults
to find *at least one*
$r_i + n \bmod \ell$,
immediately revealing n.

Also spent some ADDs
to compute each $r_i P$:
$\approx \lg p$ ADDs for each i.

Faster: Choose $r_i = ir_1$
with $r_1 \approx \ell/100$.
Compute $r_1 P$;
$r_1 P + nP$;
$2r_1 P + nP$;
$3r_1 P + nP$; etc.
Just 1 ADD for each new i.

$\approx 100 + \lg \ell + \ell/100$ ADDs
to find n given nP.

Faster: Increase 100 to $\approx \sqrt{\ell}$.
Only $\approx 2\sqrt{\ell}$ ADDs
to solve one DL problem!
"Shanks baby-step-giant-step
discrete-logarithm algorithm."

Example: $p = 1000003, \ell = 500002$, $P = (101384, 614510)$,
$Q = nP = (670366, 740819)$.
Compute $708P$=(393230, 421116).
Then compute 707 targets:
$708P + Q = (342867, 153817)$,
$2 \cdot 708P + nP = (430321, 994742)$,
$3 \cdot 708P + nP = (423151, 635197)$,
$\ldots, 706 \cdot 708P + nP = (534170, 450849)$.

Build a sorted table of targets:

$600 \cdot 708P + Q = (799978, 929249)$,

$219 \cdot 708P + Q = (425475, 793466)$,

$679 \cdot 708P + Q = (996985, 191440)$,

$242 \cdot 708P + Q = (262804, 347755)$,

$27 \cdot 708P + Q = (785344, 831127)$,

...

$317 \cdot 708P + Q = (599785, 189116)$.

Look up P, $2P$, $3P$, etc. in table.

$620P = (950652, 688508)$; find

$596 \cdot 708P + Q = (950652, 688508)$

in the table of targets;

so $620 = 596 \cdot 708 + n \bmod 500002$;

deduce $n = 78654$.

Factors of the group order

P has order $2 \cdot 53^2 \cdot 89$.

Given $Q = nP$, find $n = \log_P Q$:

$R = (53^2 \cdot 89)P$ has order 2, and
$S = (53^2 \cdot 89)Q$ is multiple of R.
Compute $n_1 = \log_R S \equiv n \bmod 2$.

$R = (2 \cdot 53 \cdot 89)P$ has order 53,
and
$S = (2 \cdot 53 \cdot 89)Q$ is multiple of R.
Compute $n_2 = \log_R S \equiv n \bmod 53$.
This is a DLP in a group
of size 53.

$T = (2 \cdot 89)(Q - n_2 P)$ is also a multiple of R.
Compute $n_3 = \log_R T \equiv n \bmod 53$.
Now $n_2 + 53n_3 \equiv n \bmod 53^2$.

$R = (2 \cdot 53^2)P$ has order 89, and $S = (2 \cdot 53^2)Q$ is multiple of R.
Compute $n_4 = \log_R S \equiv n \bmod 89$.

Use Chinese Remainder Theorem
$n \equiv n_1 \bmod 2$,
$n \equiv n_2 + 53n_3 \bmod 53^2$,
$n \equiv n_4 \bmod 89$,
to determine n modulo $2 \cdot 53^2 \cdot 89$.

This "Pohlig-Hellman method" converts an order-ab DL into an order-a DL, an order-b DL, and a few scalar multiplications.

Here $(53^2 \cdot 89)P = (1, 0)$ and $(53^2 \cdot 89)Q = \infty$, thus $n_1 = 0$.

$(2 \cdot 53 \cdot 89)P = (539296, 488875)$,
$(2 \cdot 53 \cdot 89)Q = (782288, 572333)$.
A search quickly finds $n_2 = 2$.
$(2 \cdot 89)(Q - 2P) = \infty$, thus $n_3 = 0$ and $n_2 + 53n_3 = 2$.

$(2 \cdot 53^2)P = (877560, 947848)$ and $(2 \cdot 53^2)Q = (822491, 118220)$. Compute $n_4 = 67$, e.g. using BSGS.

Use Chinese Remainder Theorem

$n \equiv 0 \bmod 2$,

$n \equiv 2 \bmod 53^2$,

$n \equiv 67 \bmod 89$,

to determine $n = 78654$.

Pohlig-Hellman method reduces security of discrete logarithm problem in group generated by P to security of largest prime order subgroup.

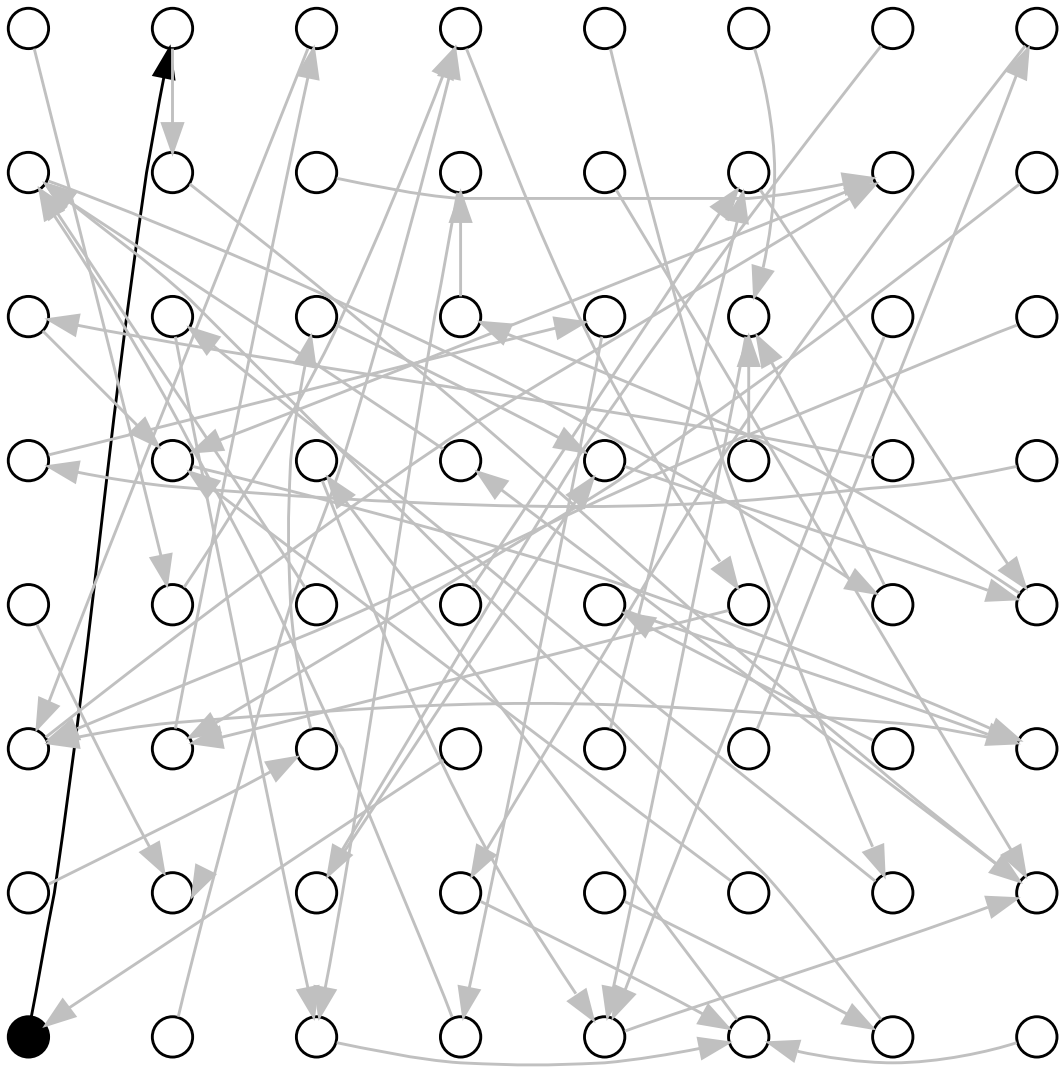
The rho method

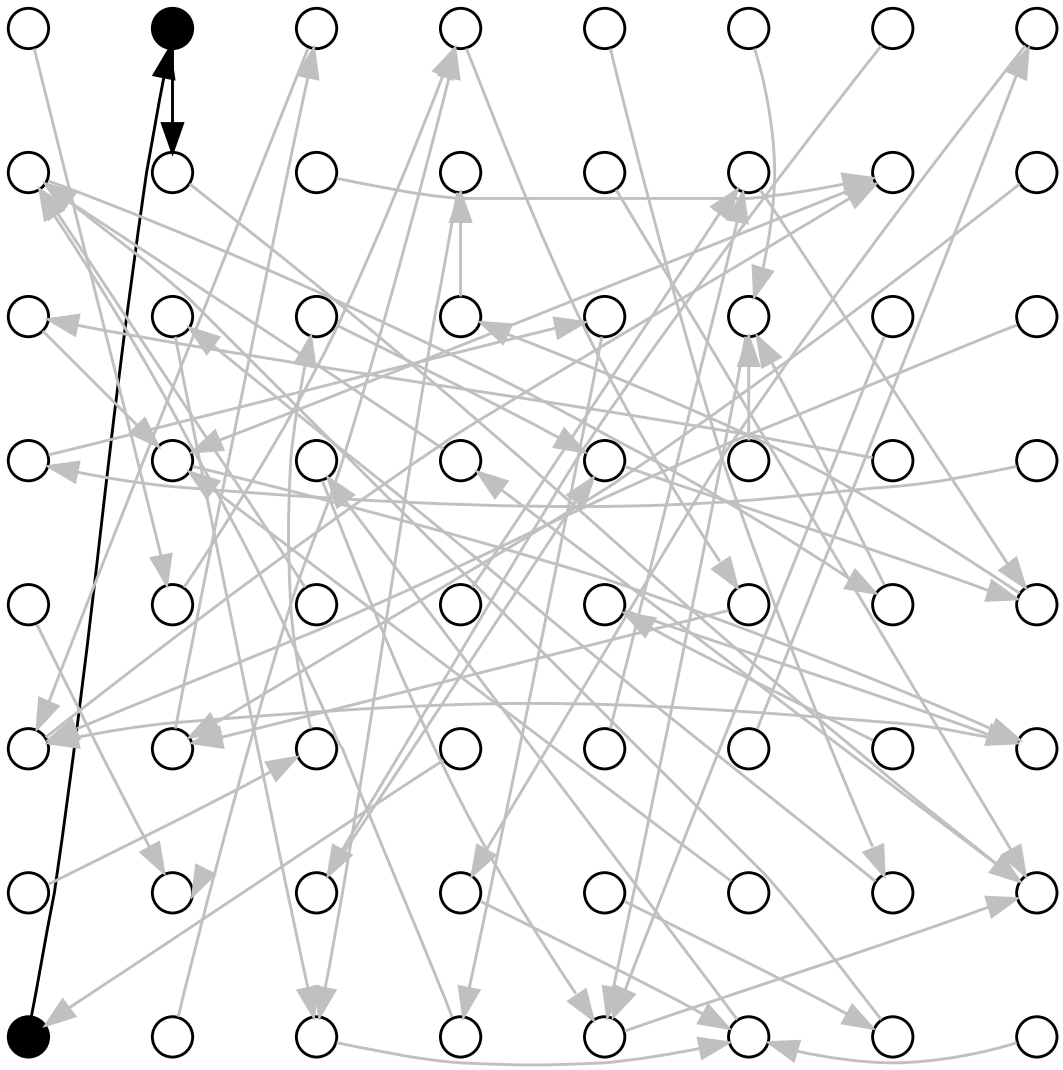
Simplified, non-parallel rho:

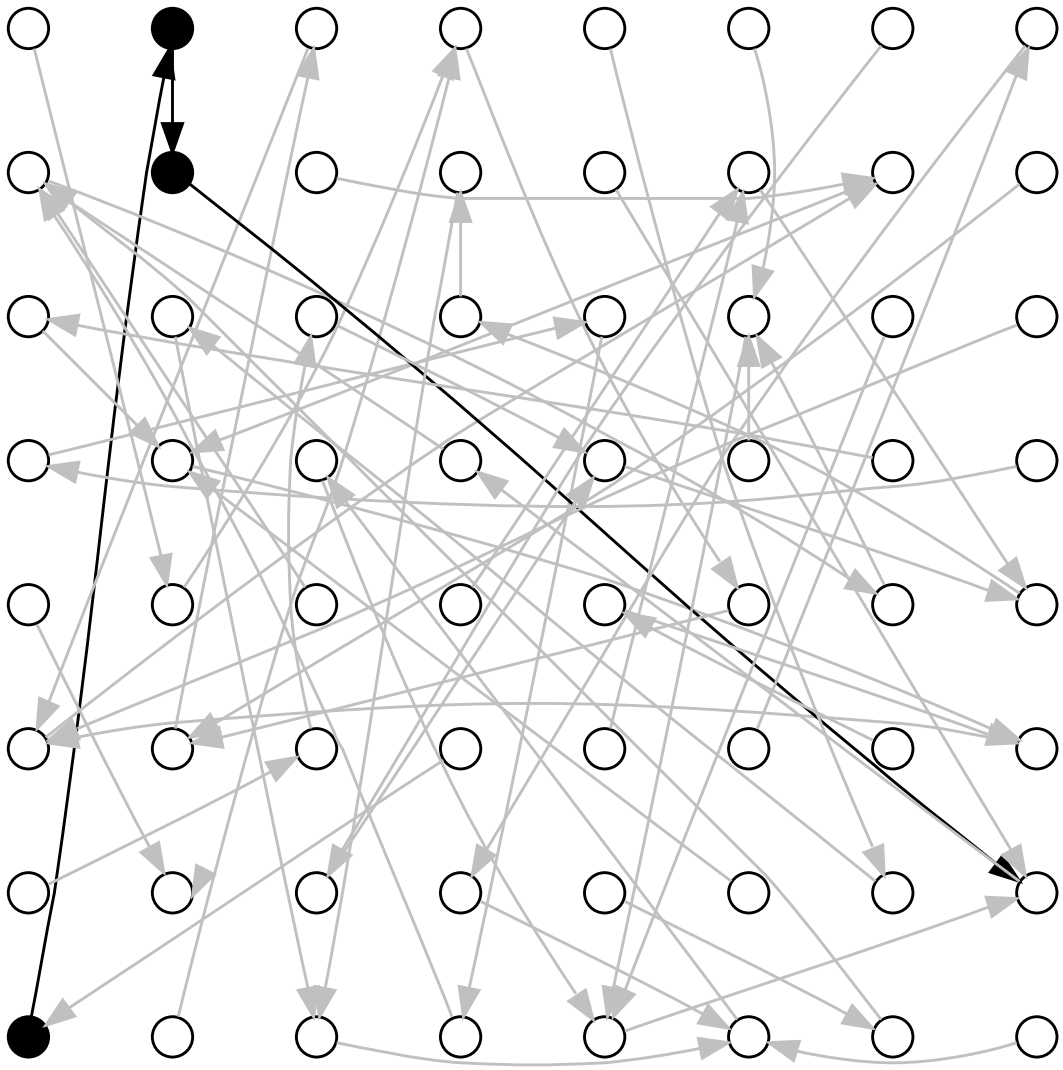
Make a pseudo-random walk
in the group $\langle P \rangle$,
where the next step depends
on current point: $W_{i+1} = f(W_i)$.

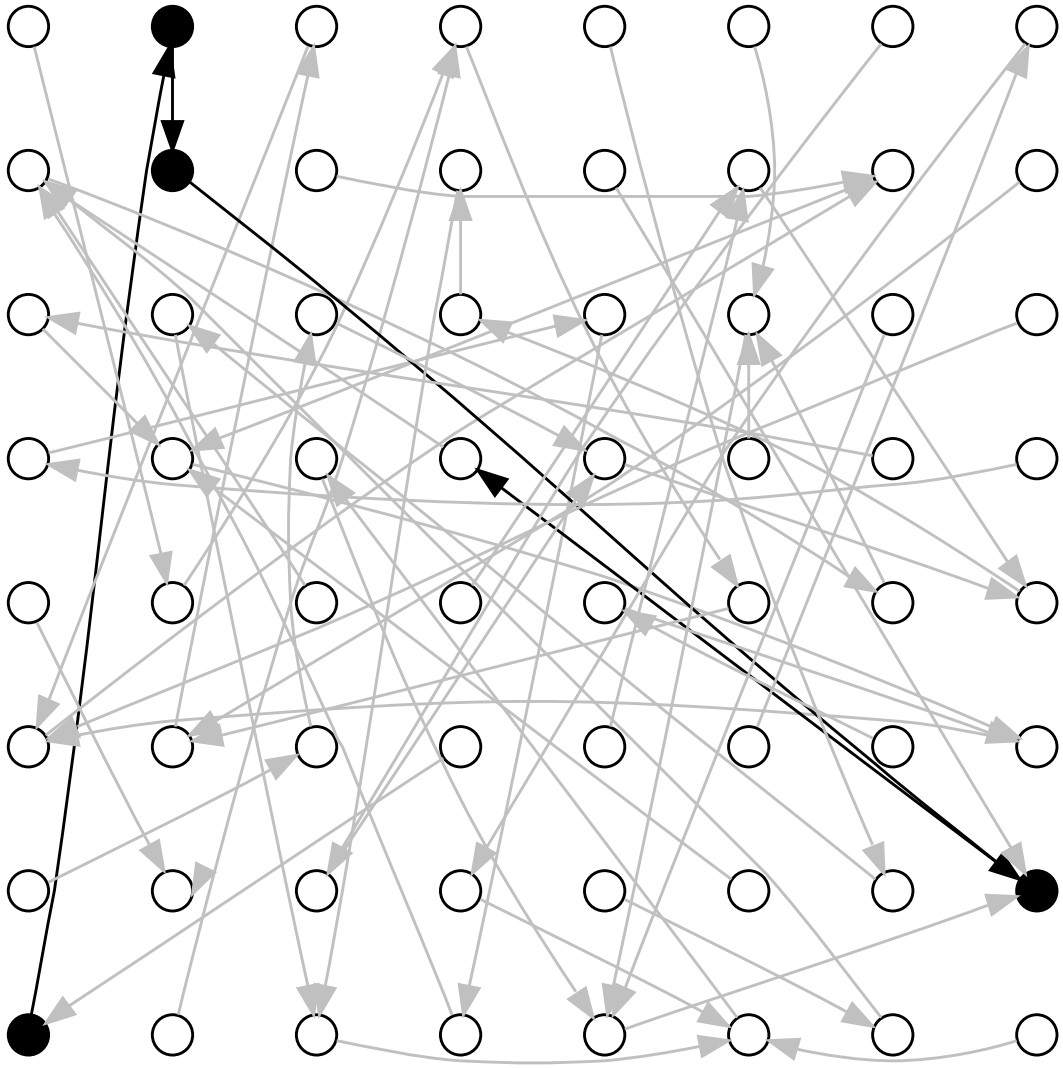
Birthday paradox:
Randomly choosing from ℓ
elements picks one element twice
after about $\sqrt{\pi \ell / 2}$ draws.

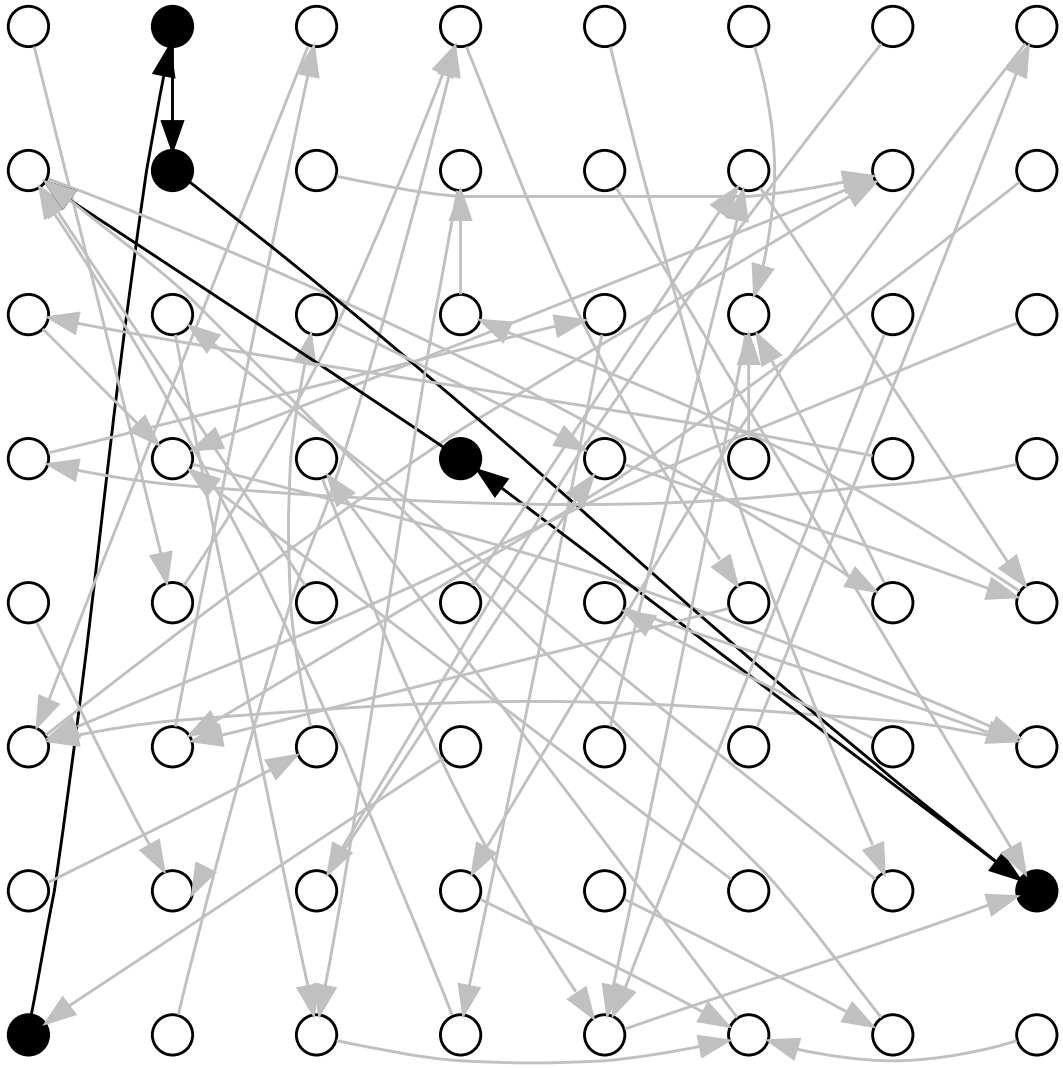
The walk now enters a cycle.
Cycle-finding algorithm
(e.g., Floyd) quickly detects this.

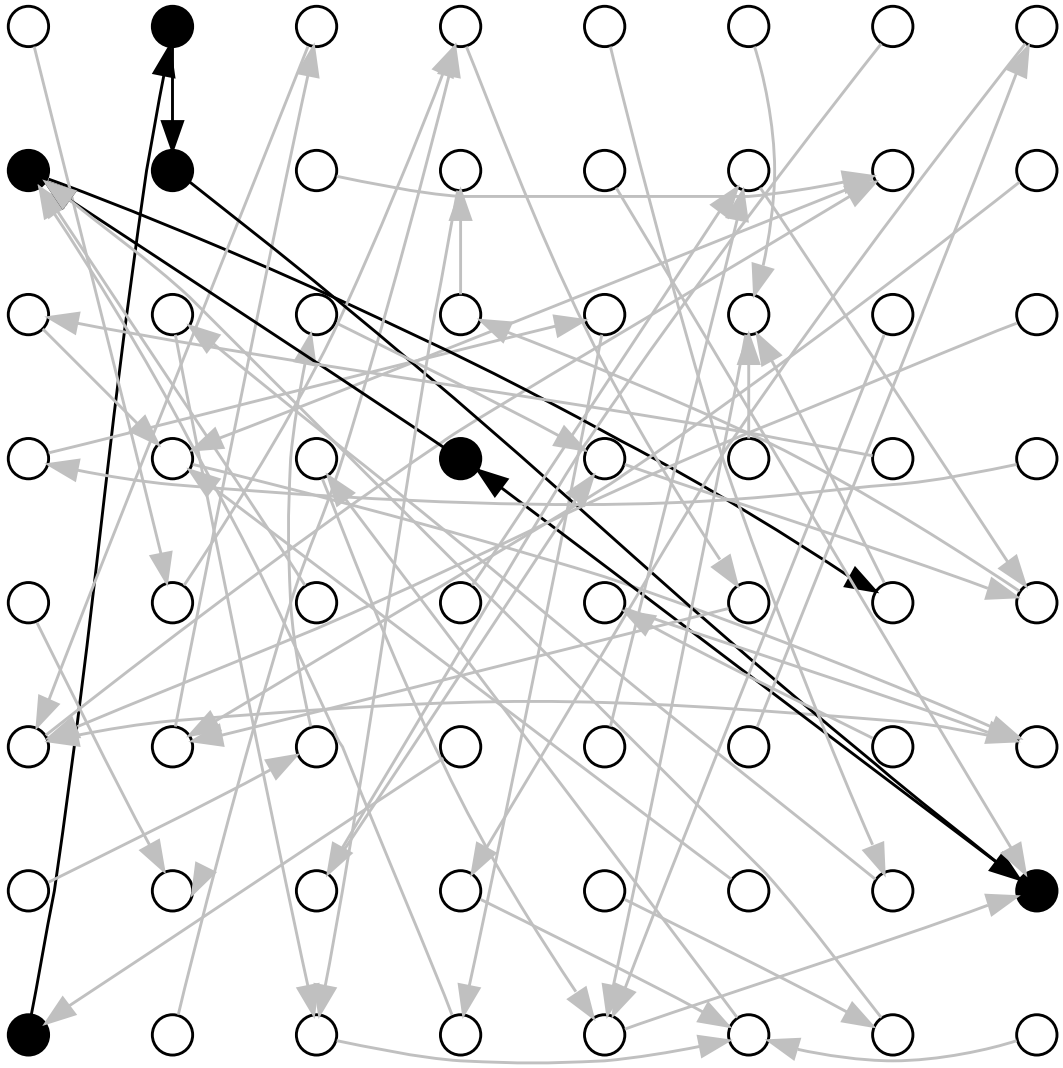


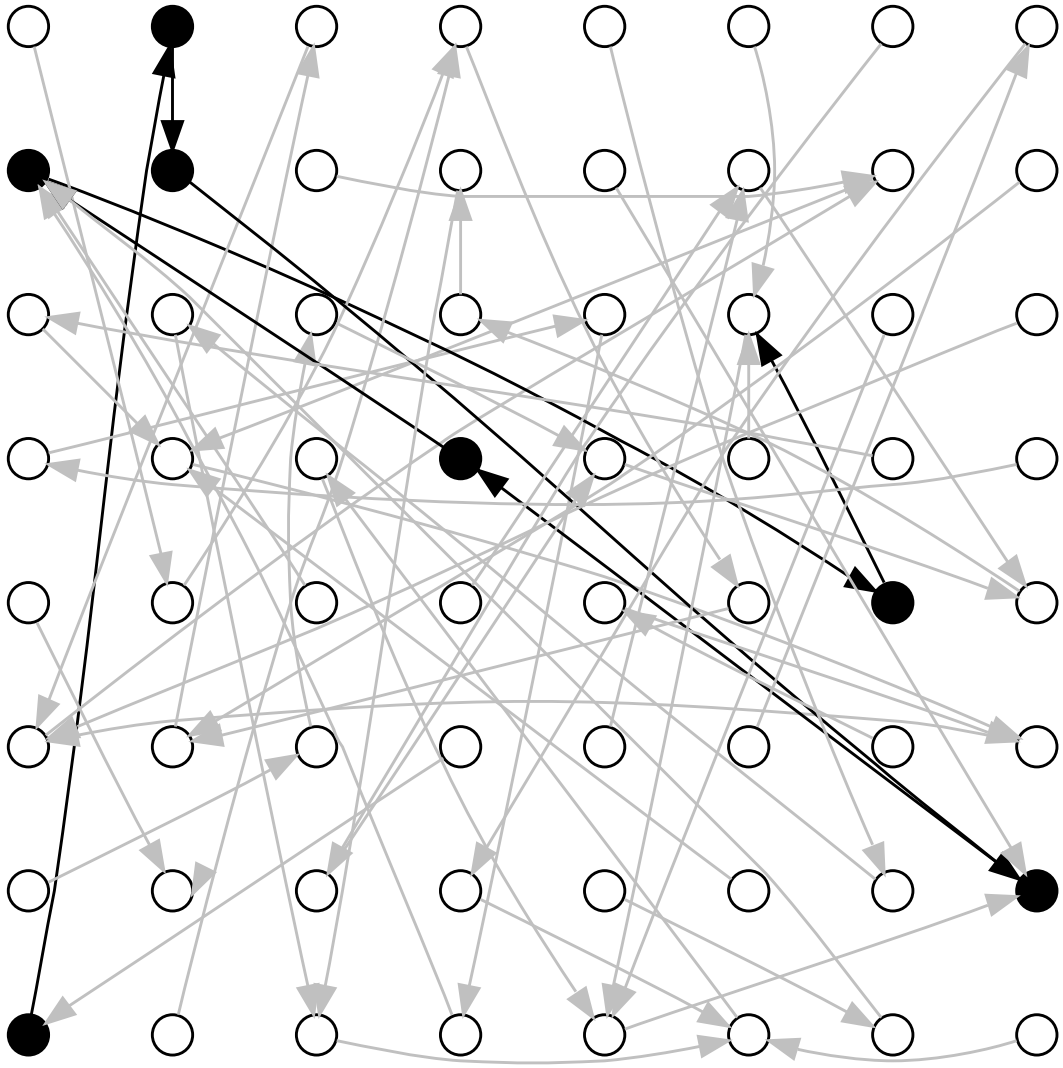


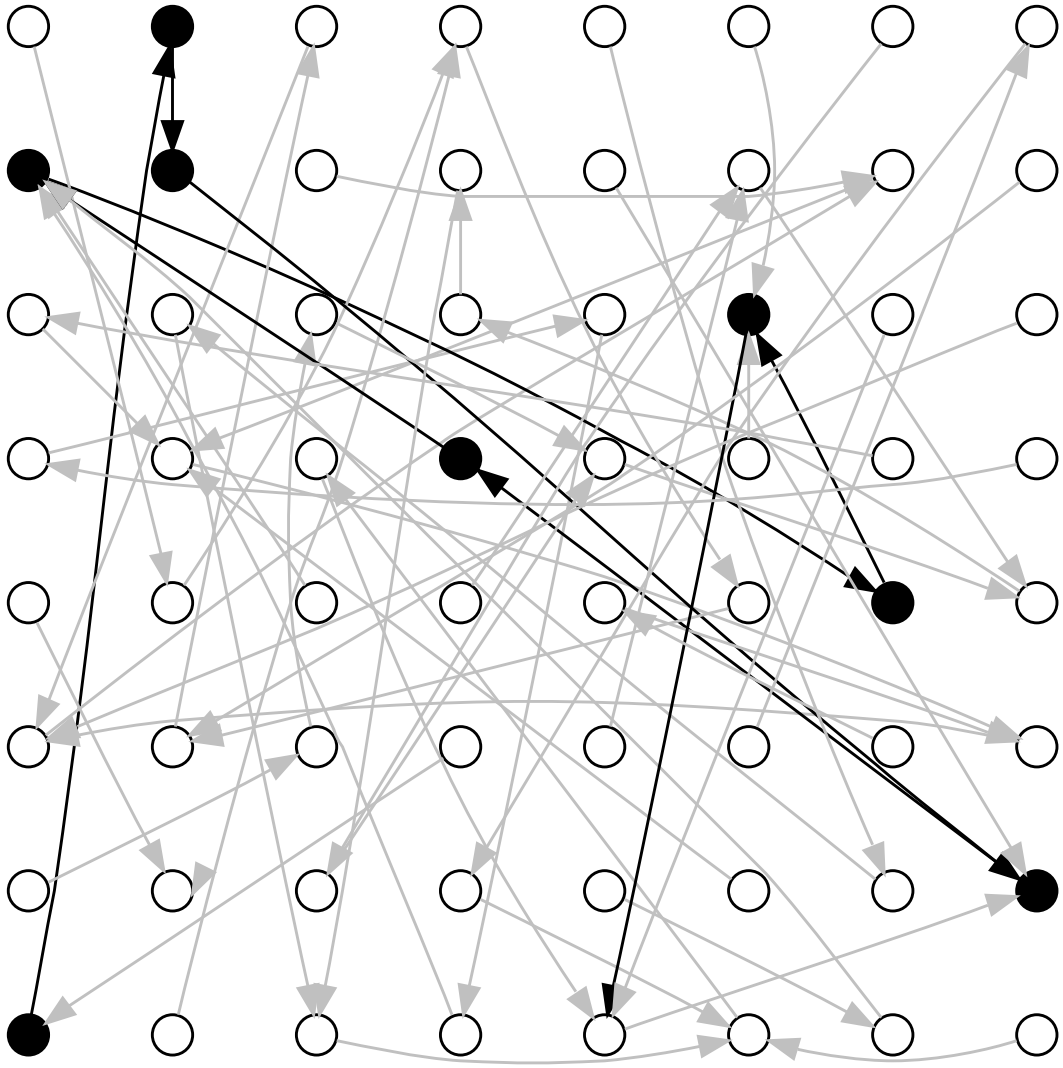


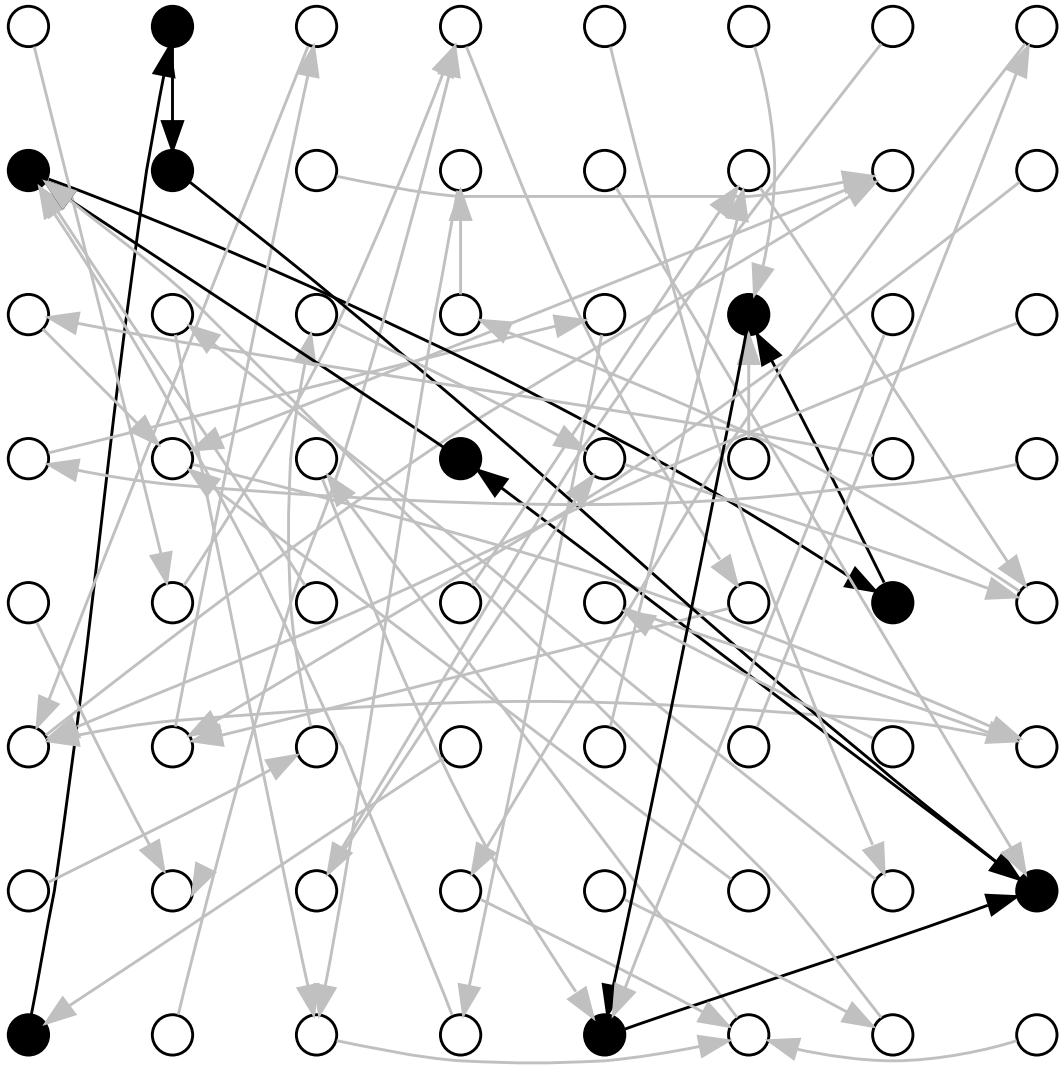


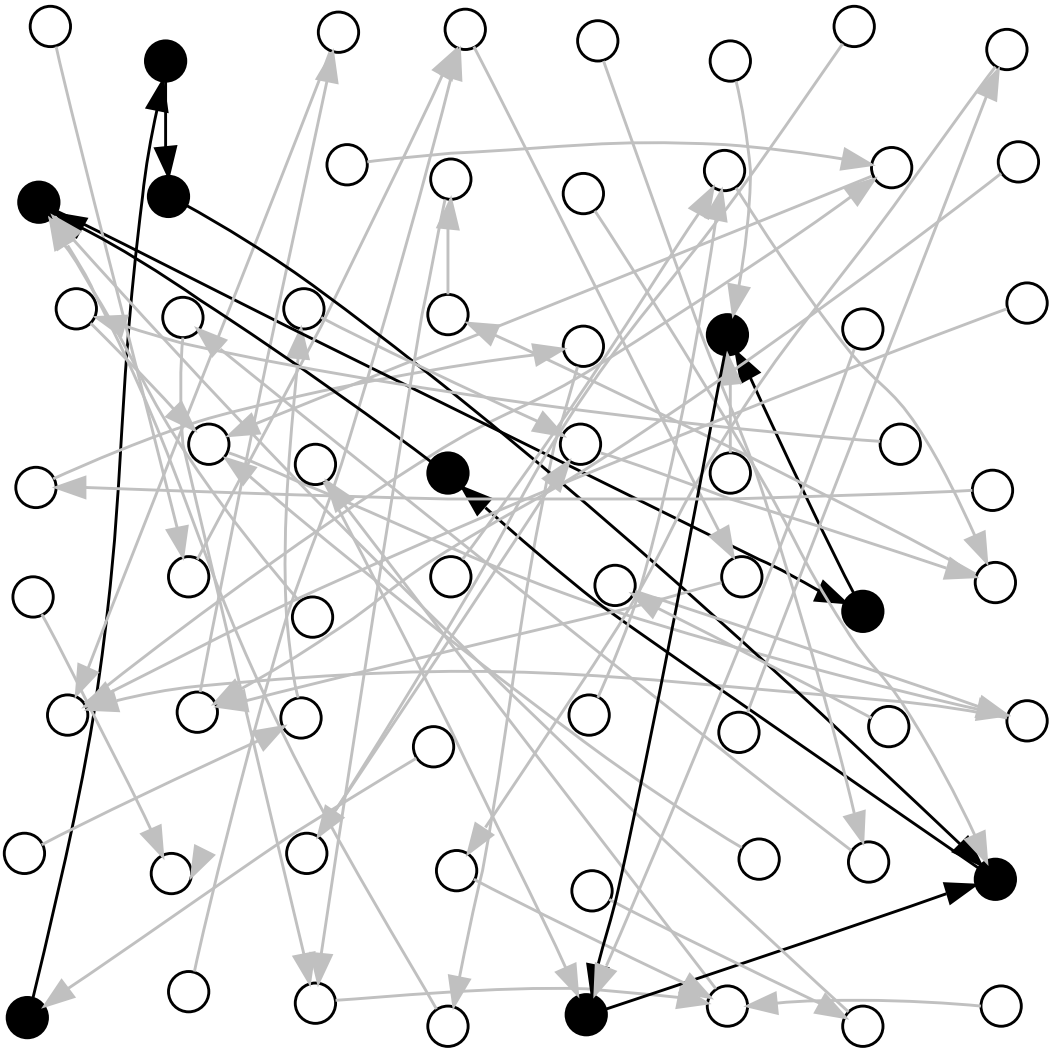


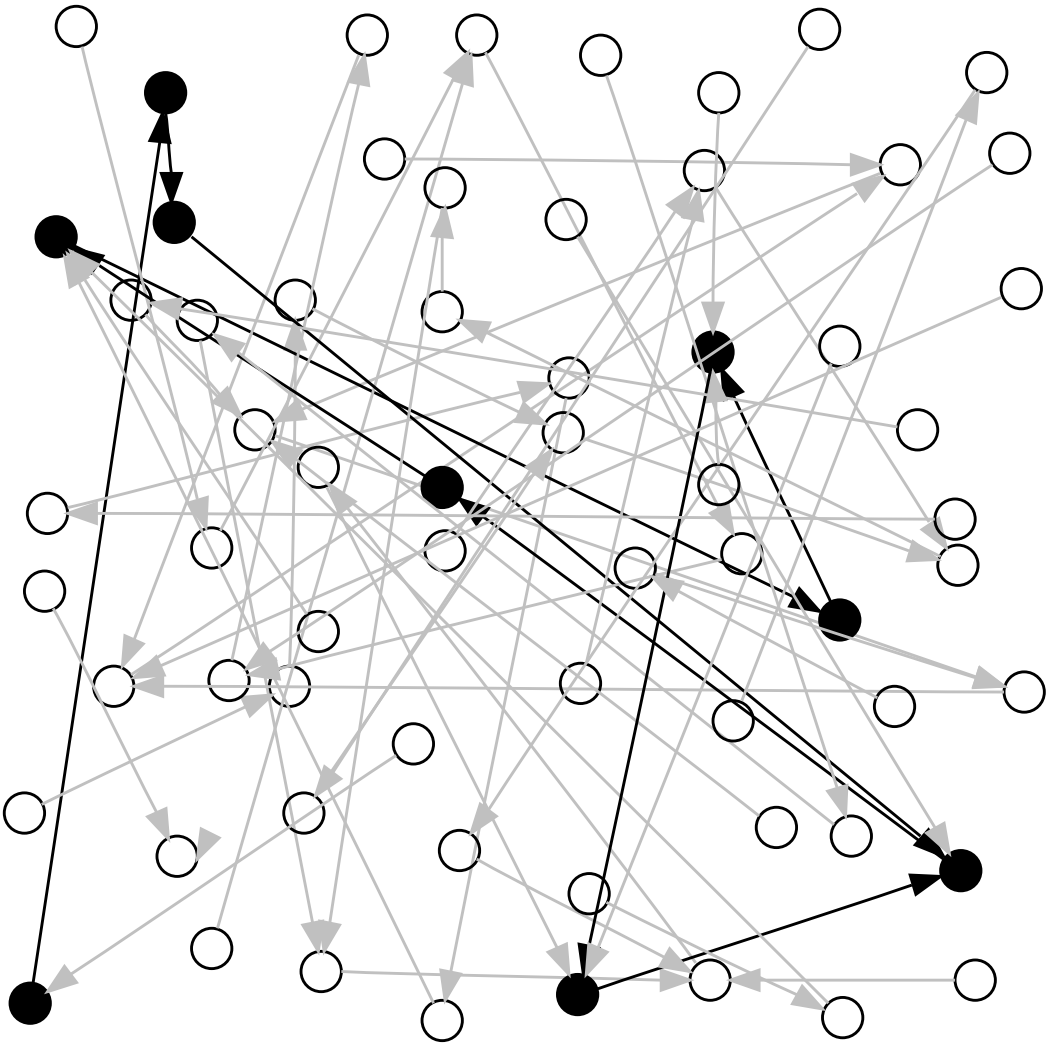


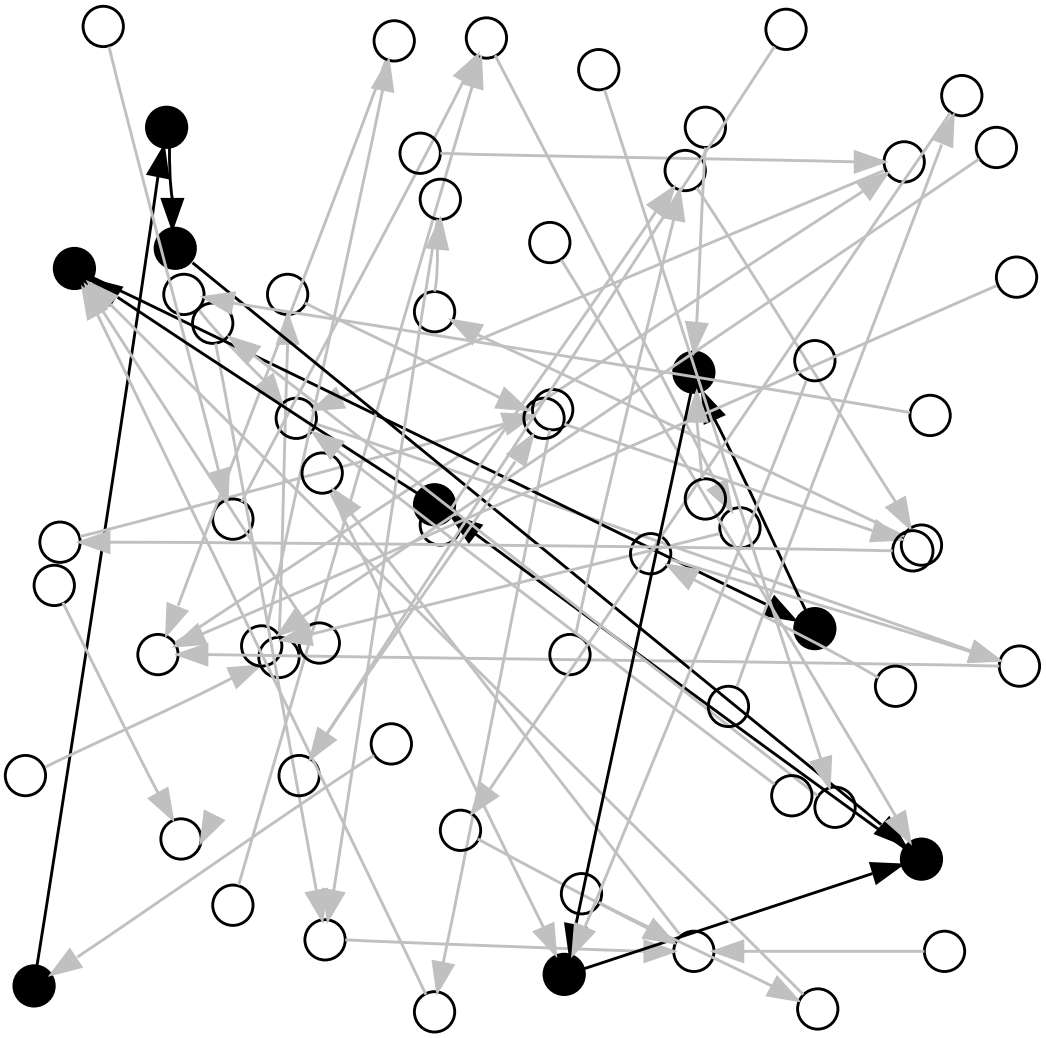


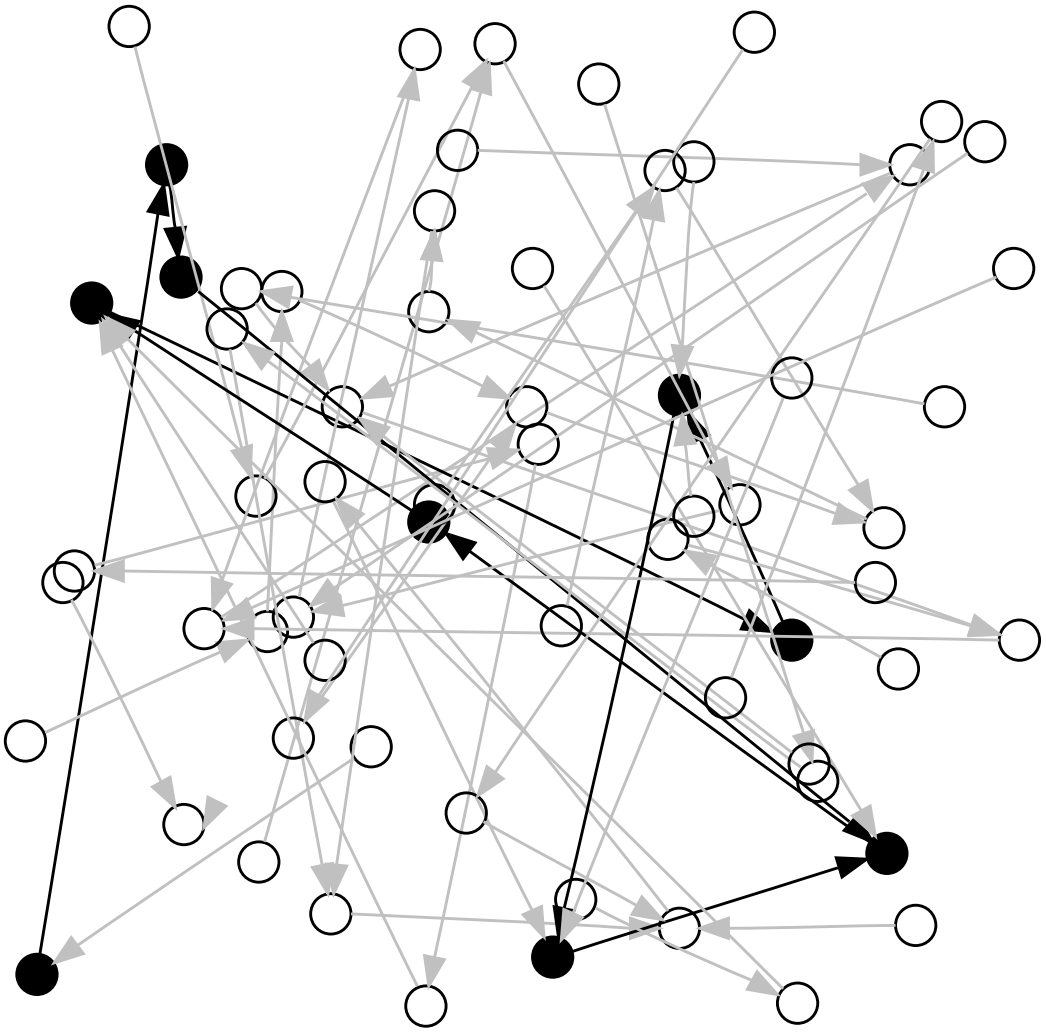


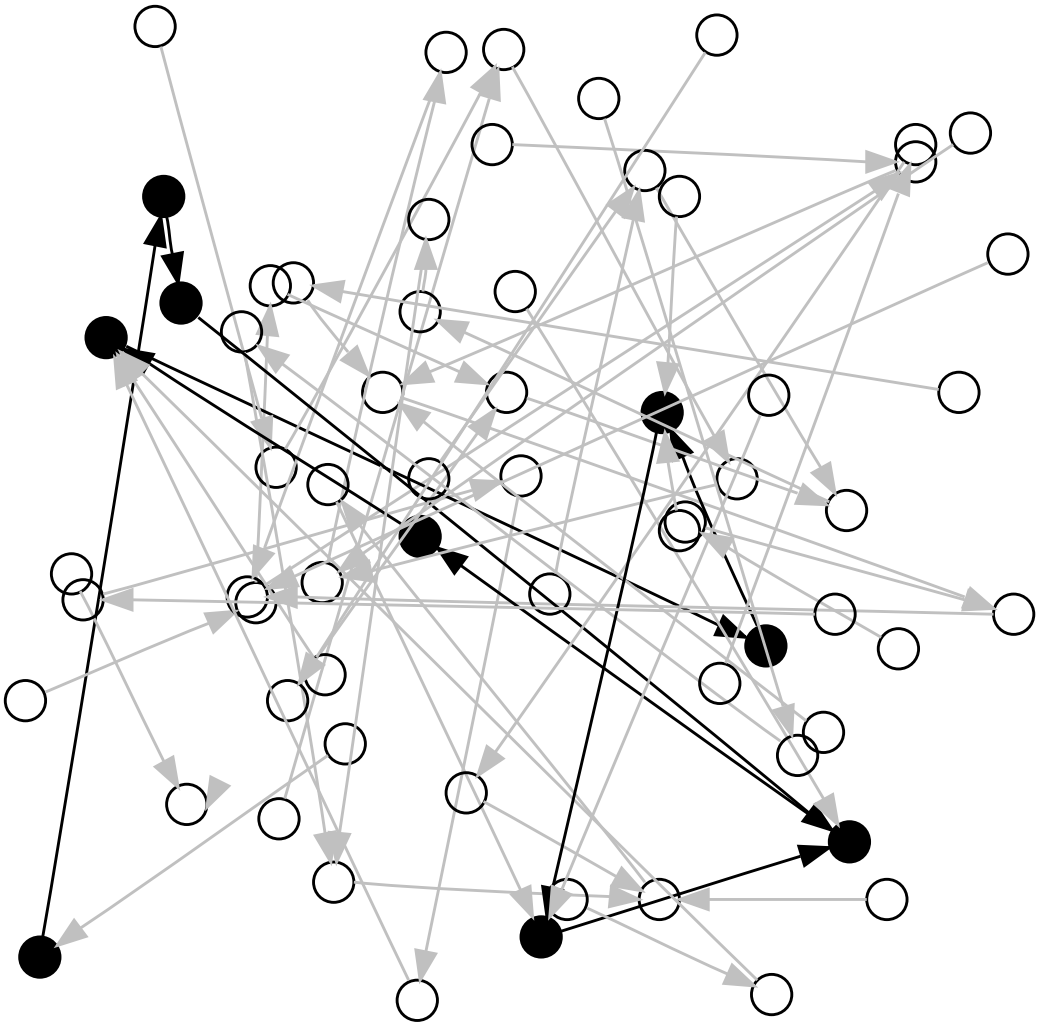


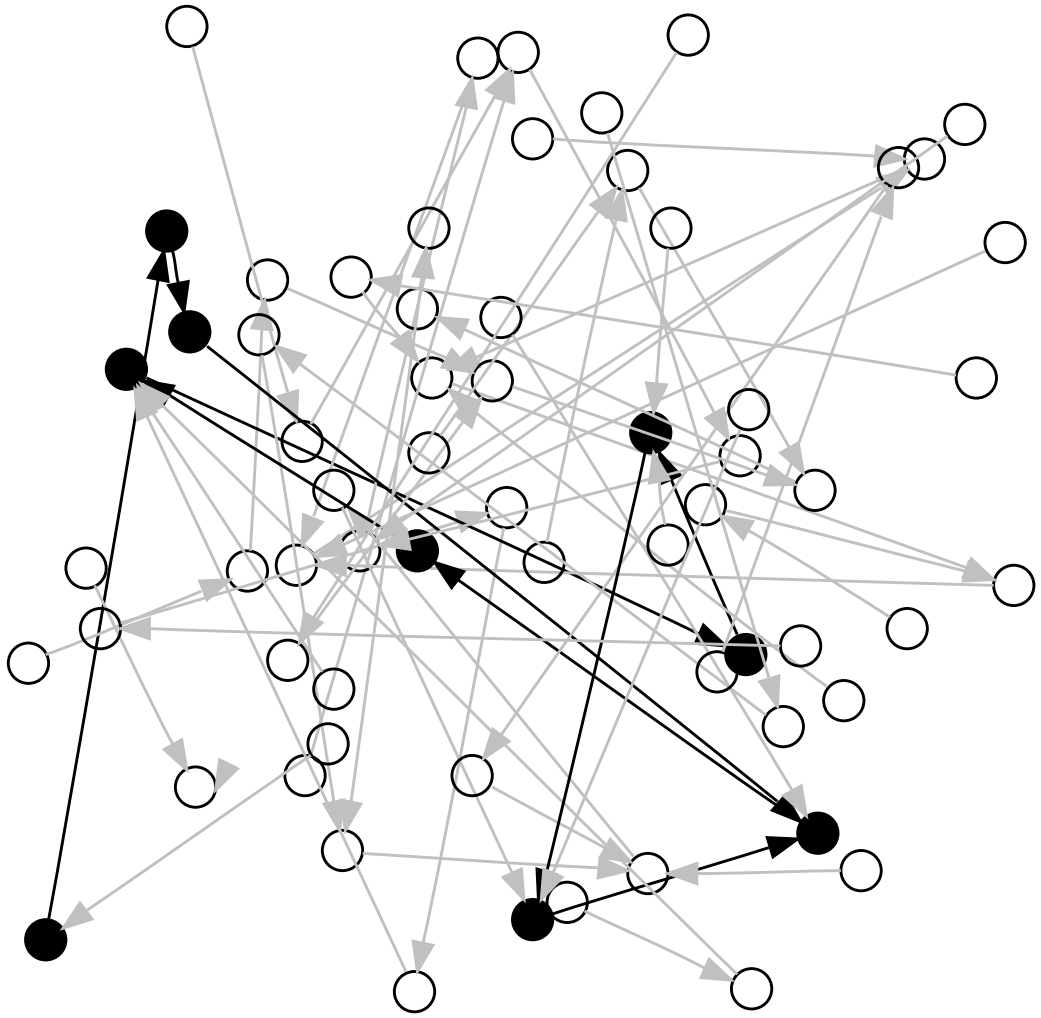


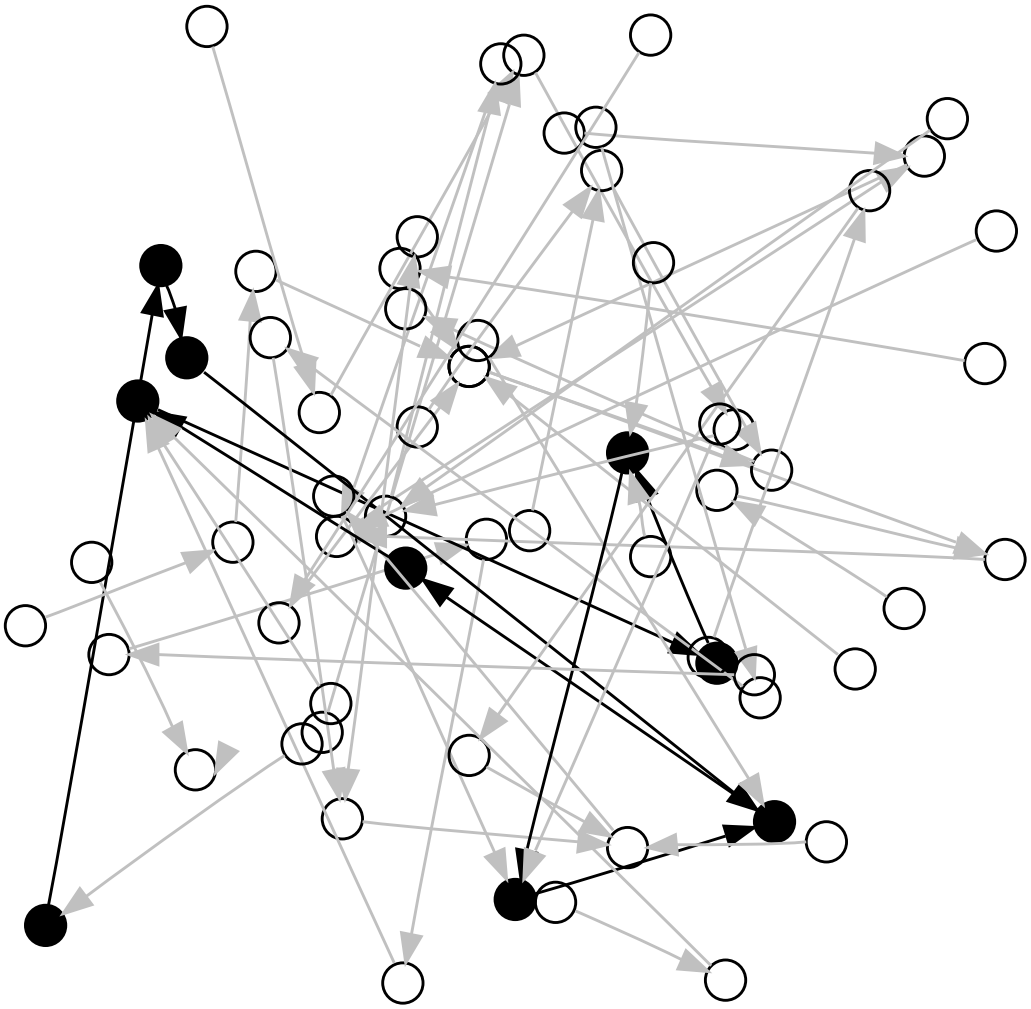


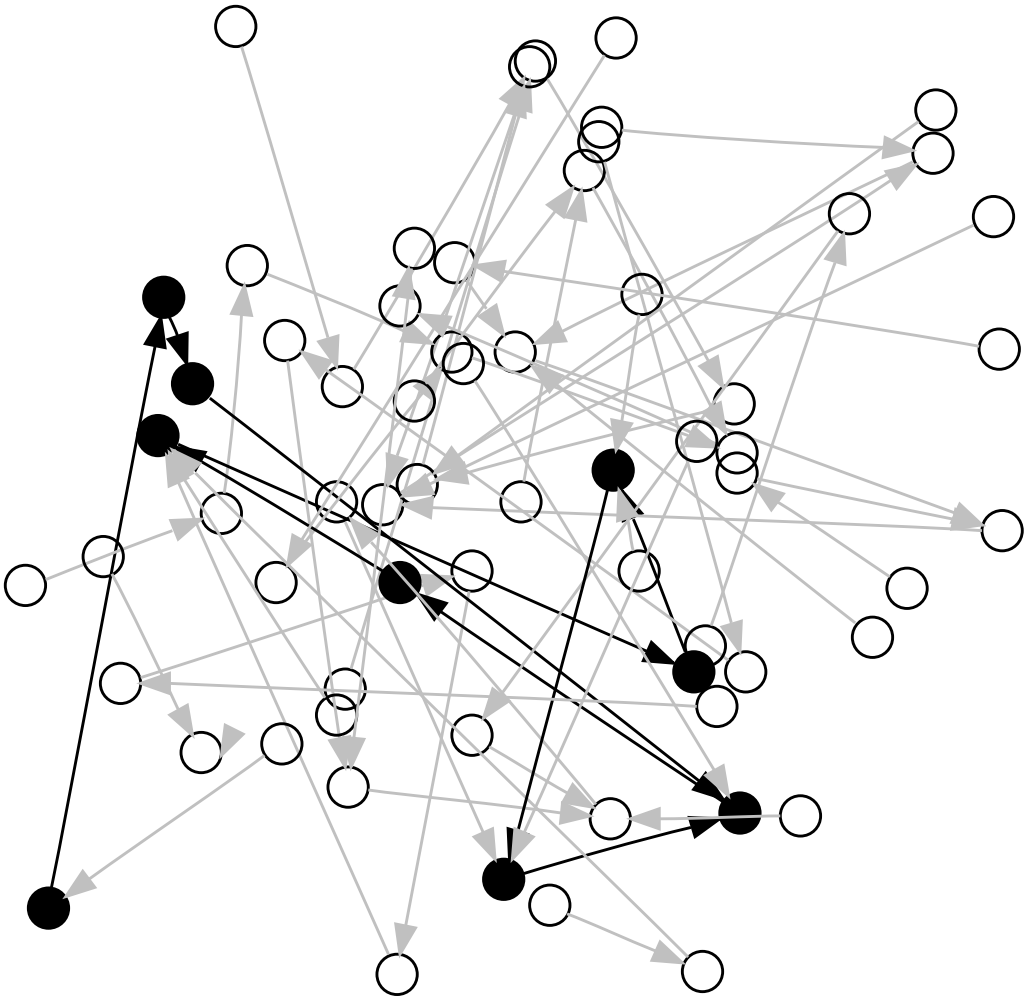


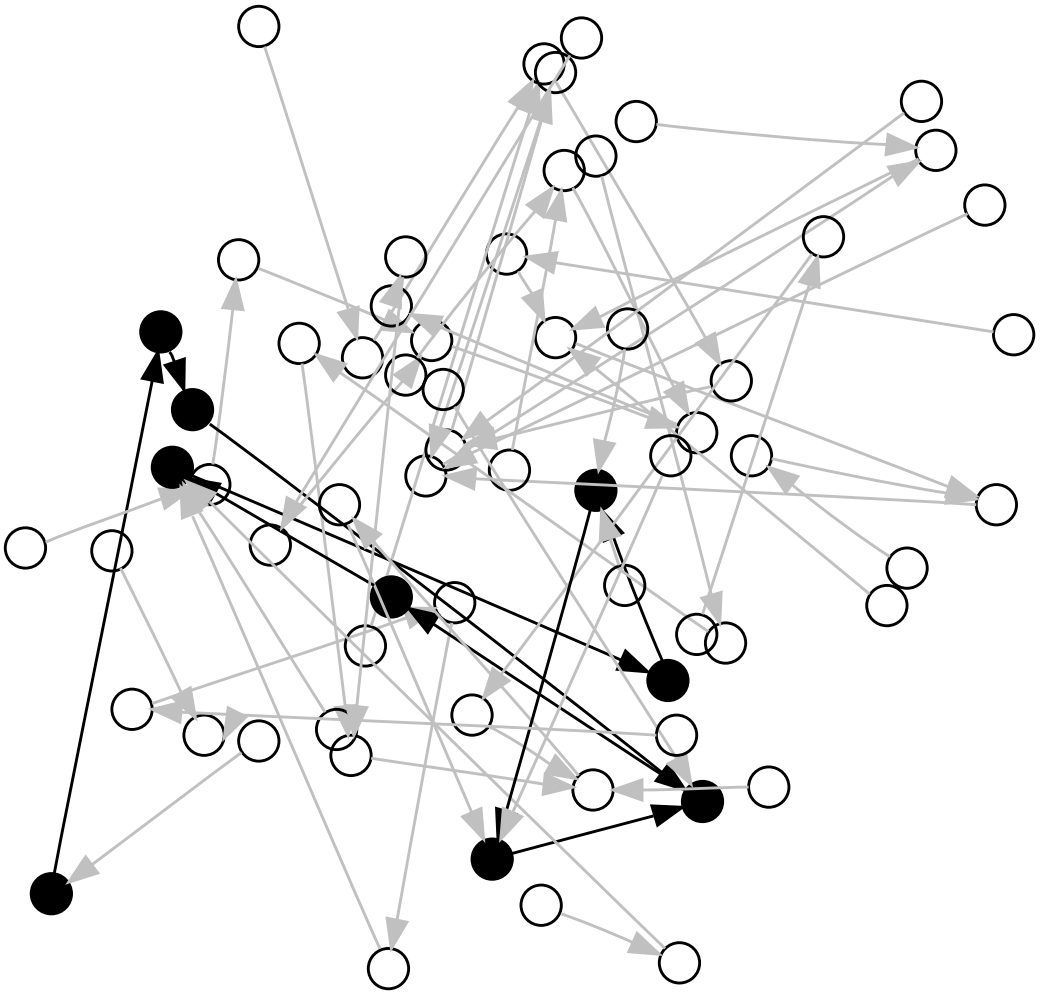


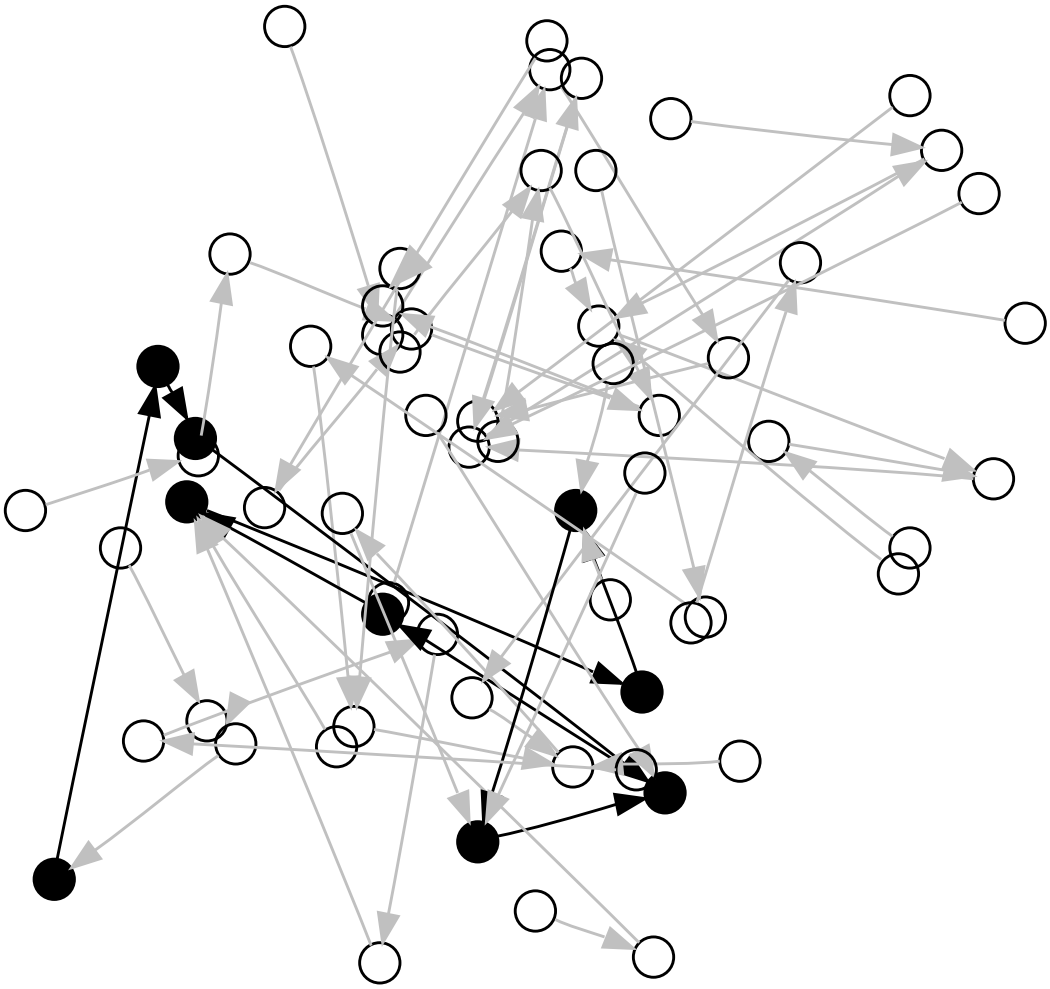


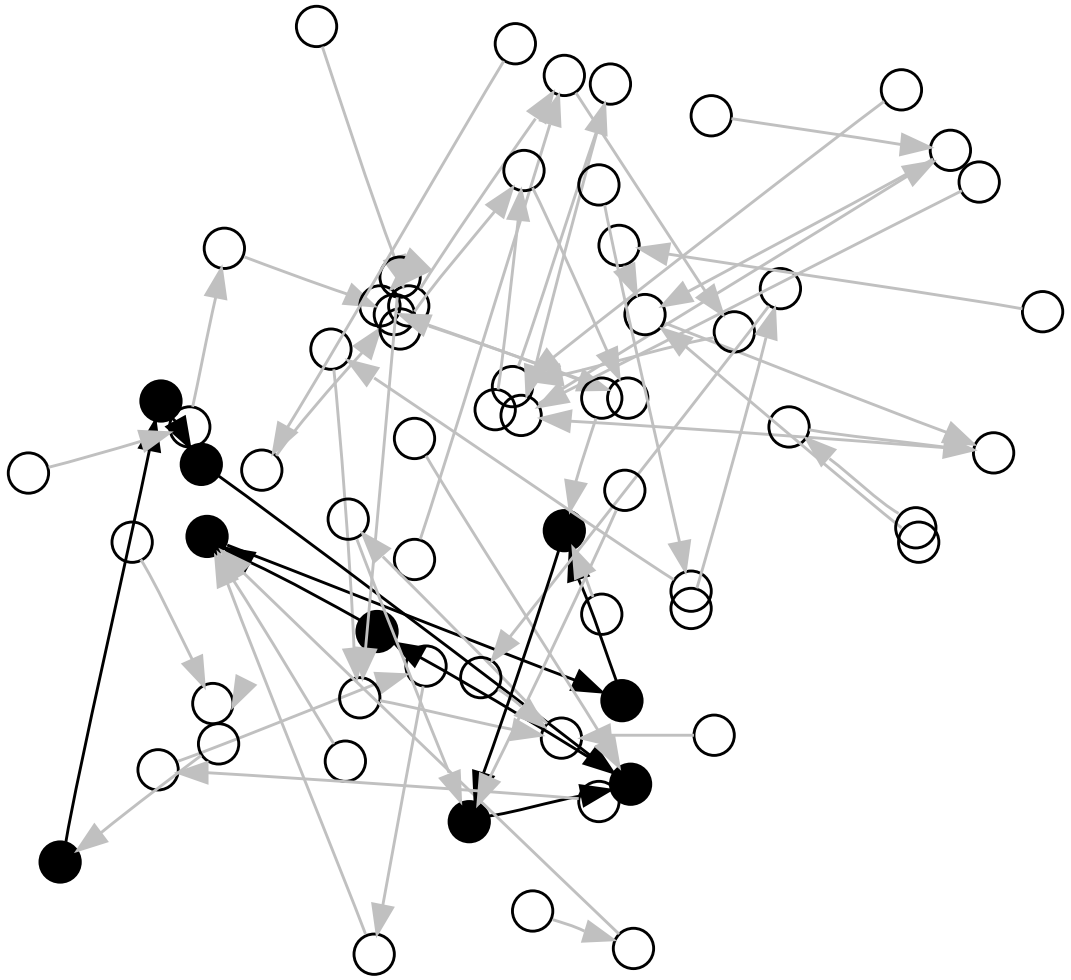


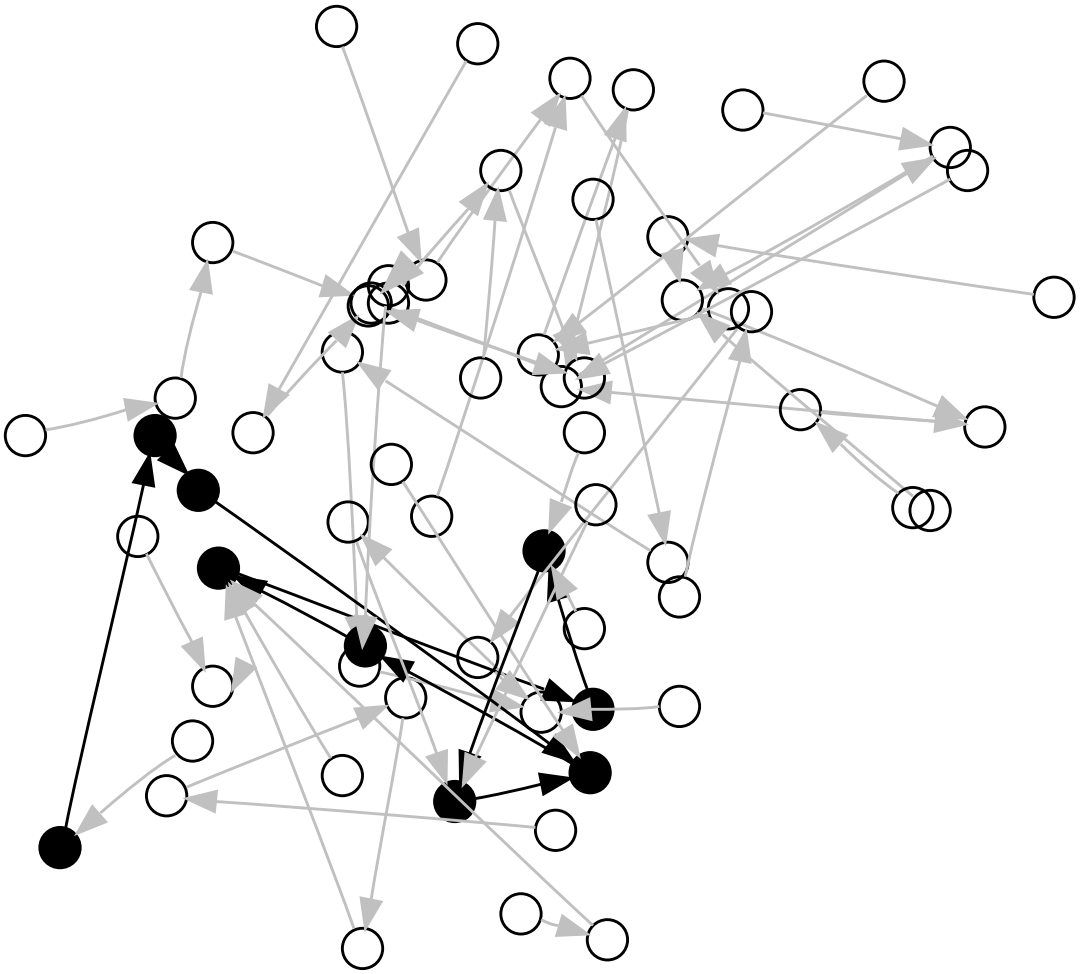


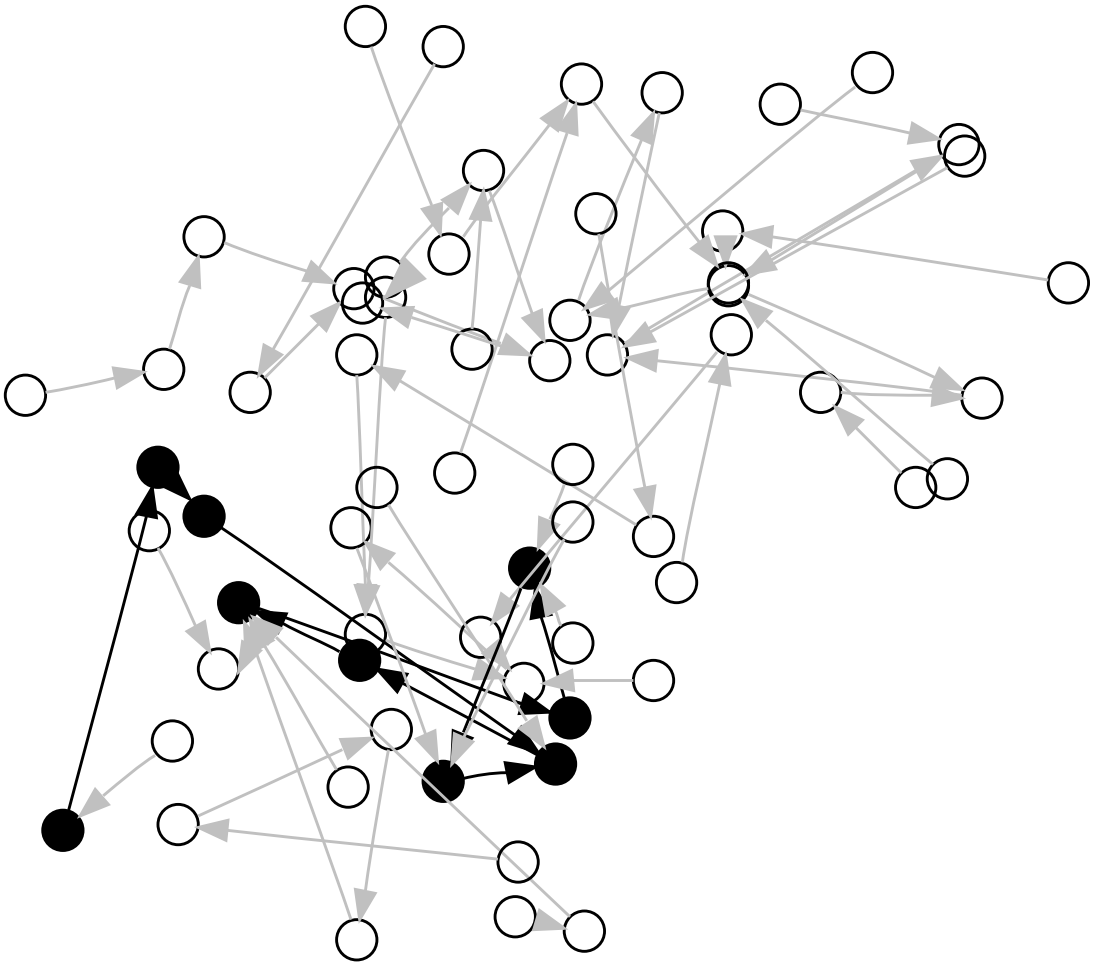


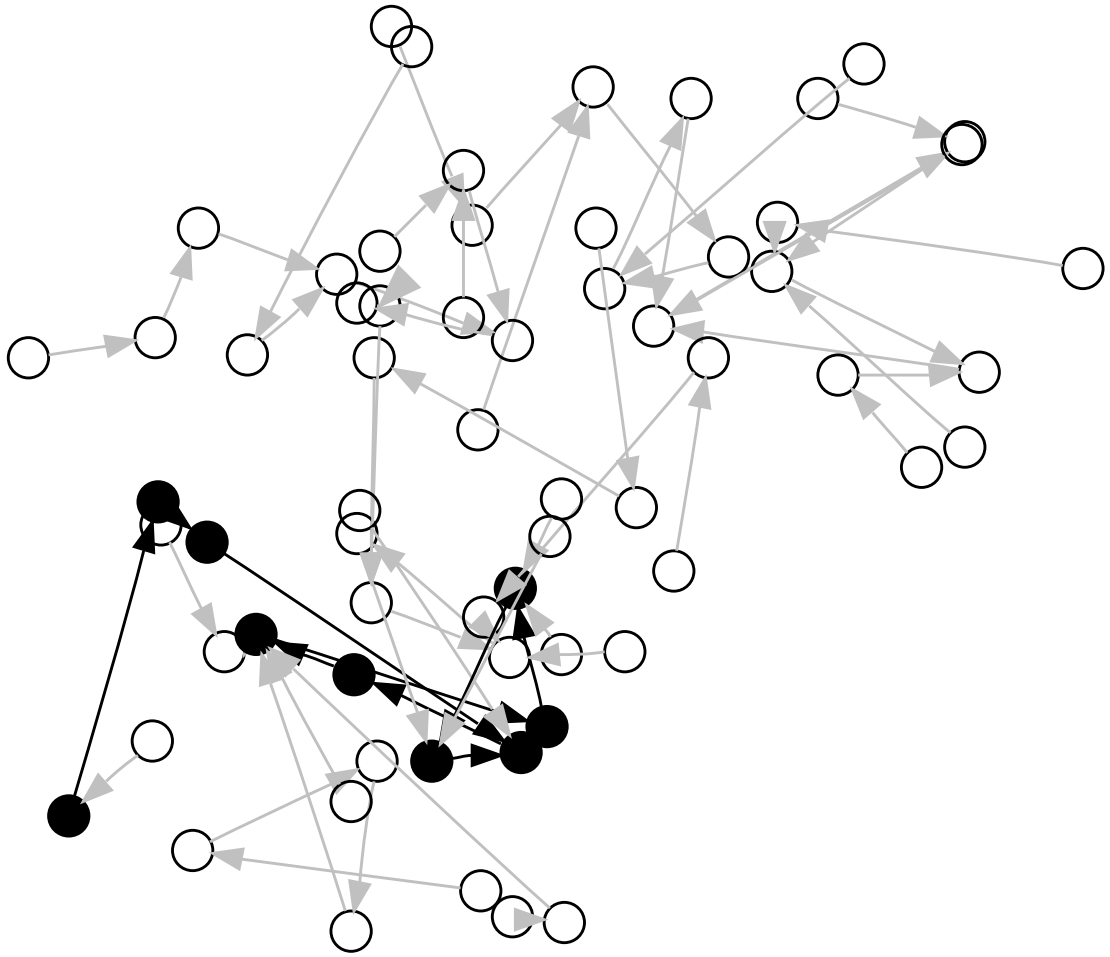


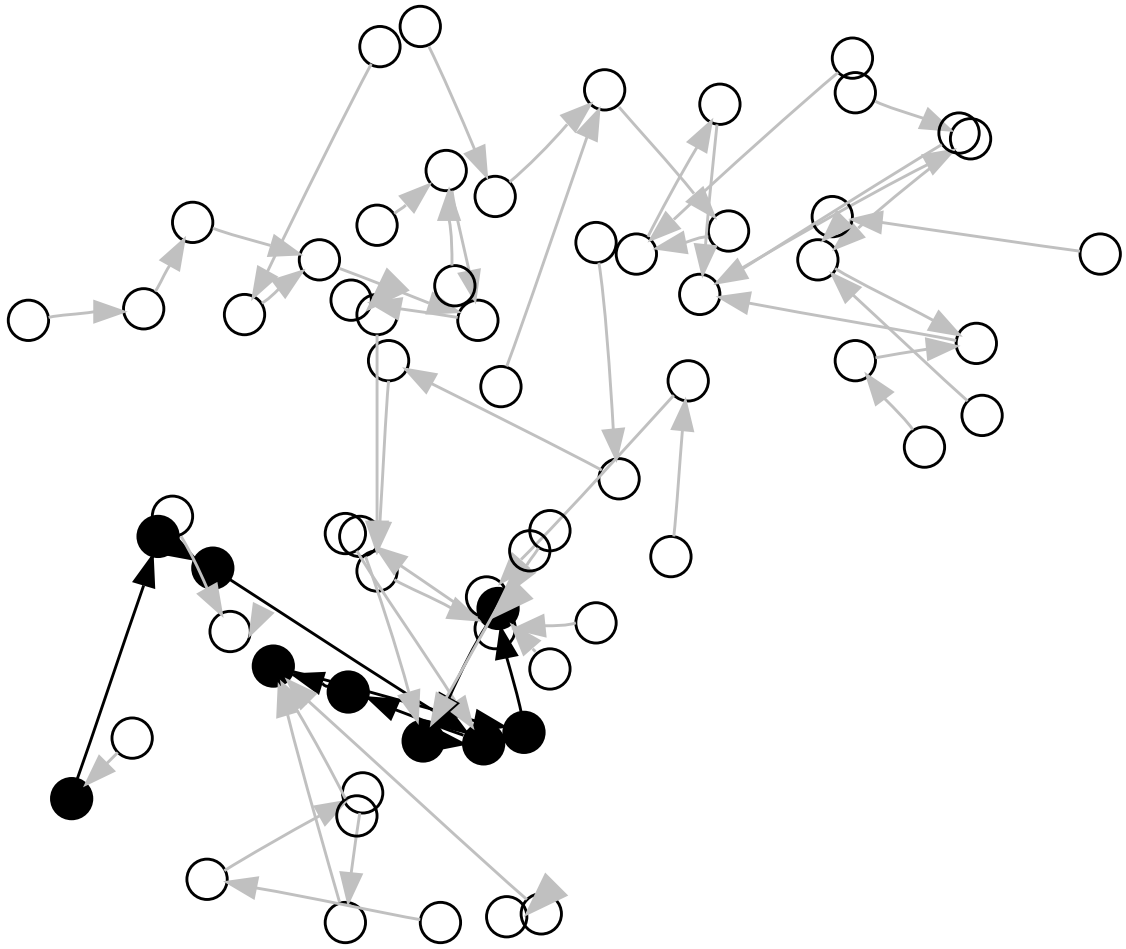


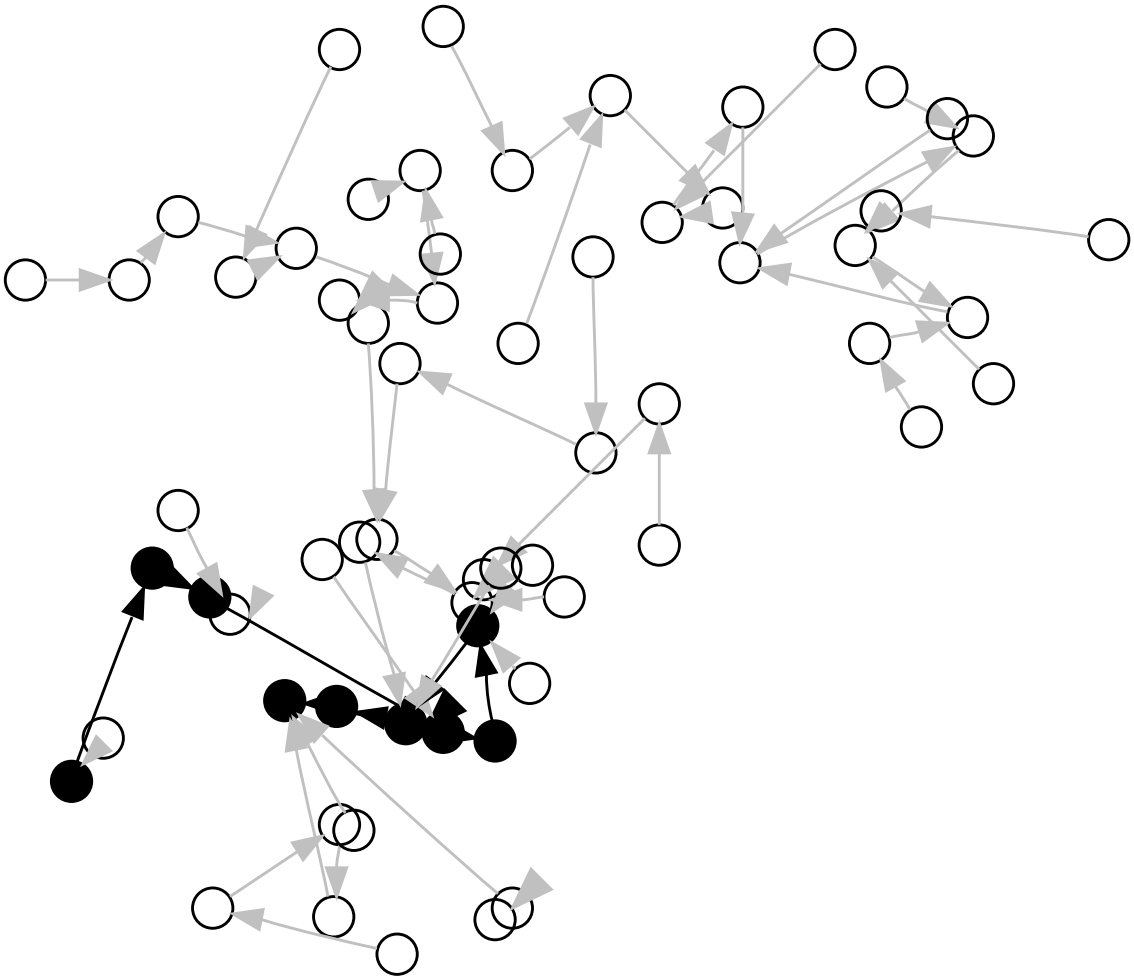


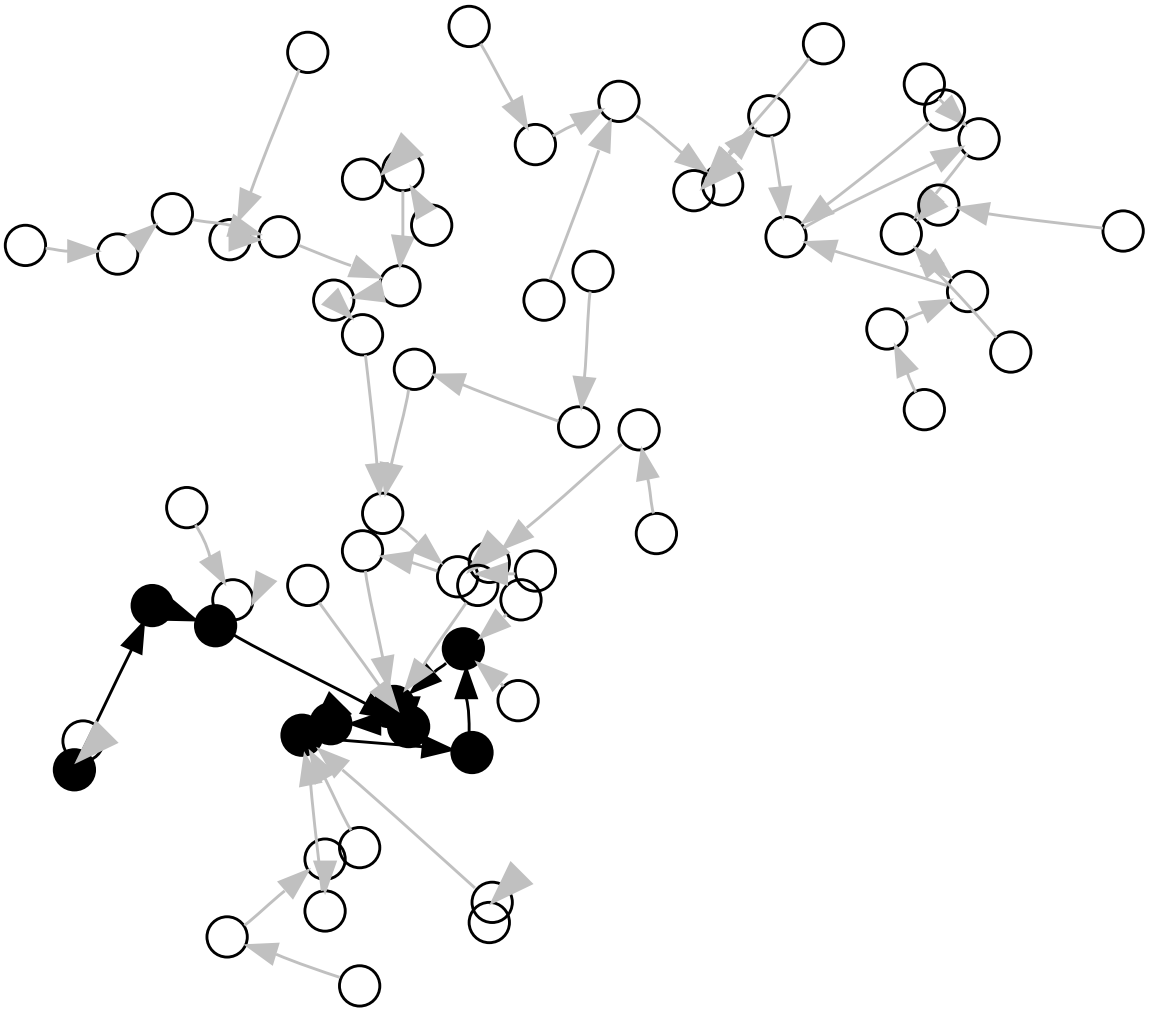


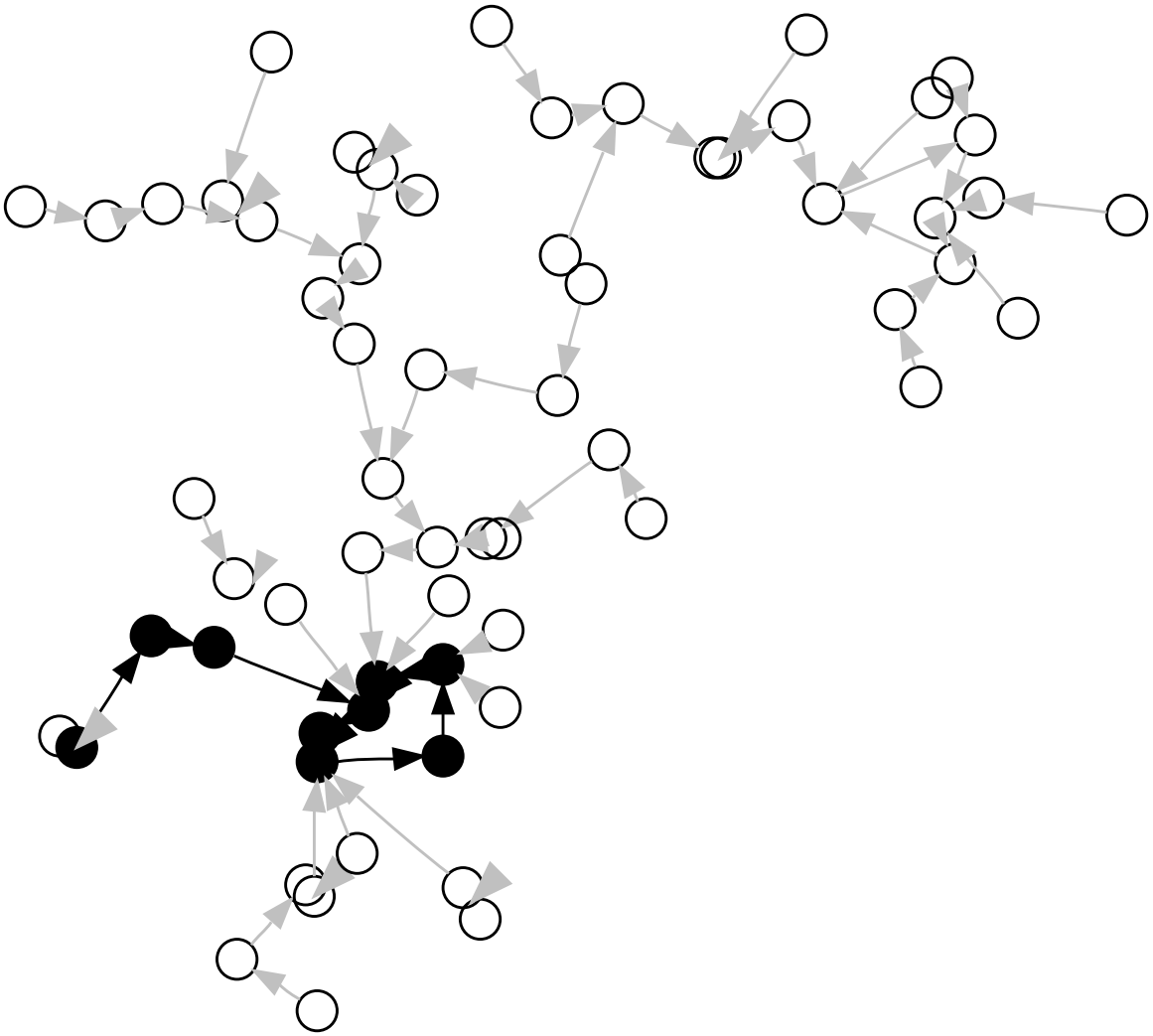


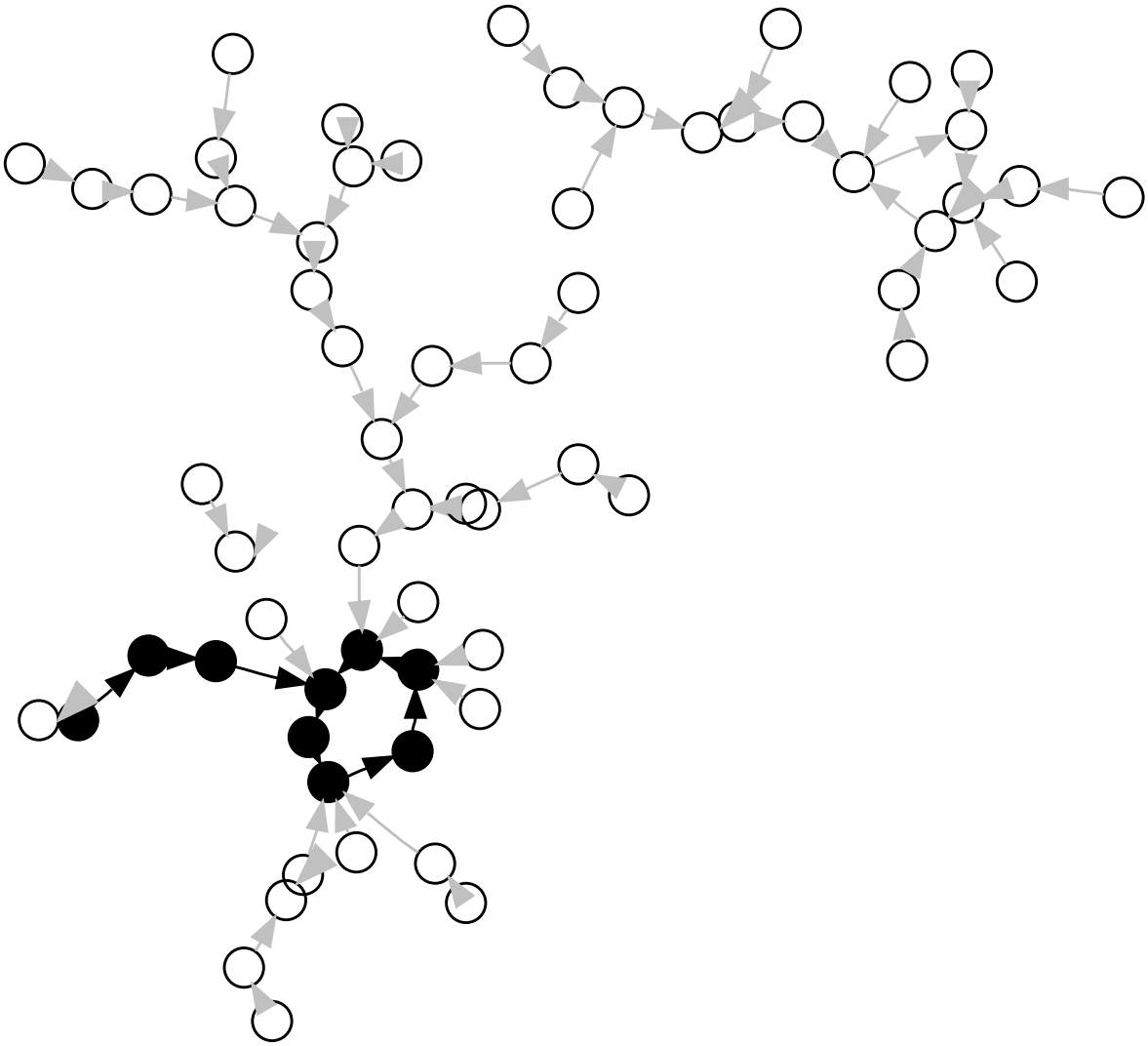


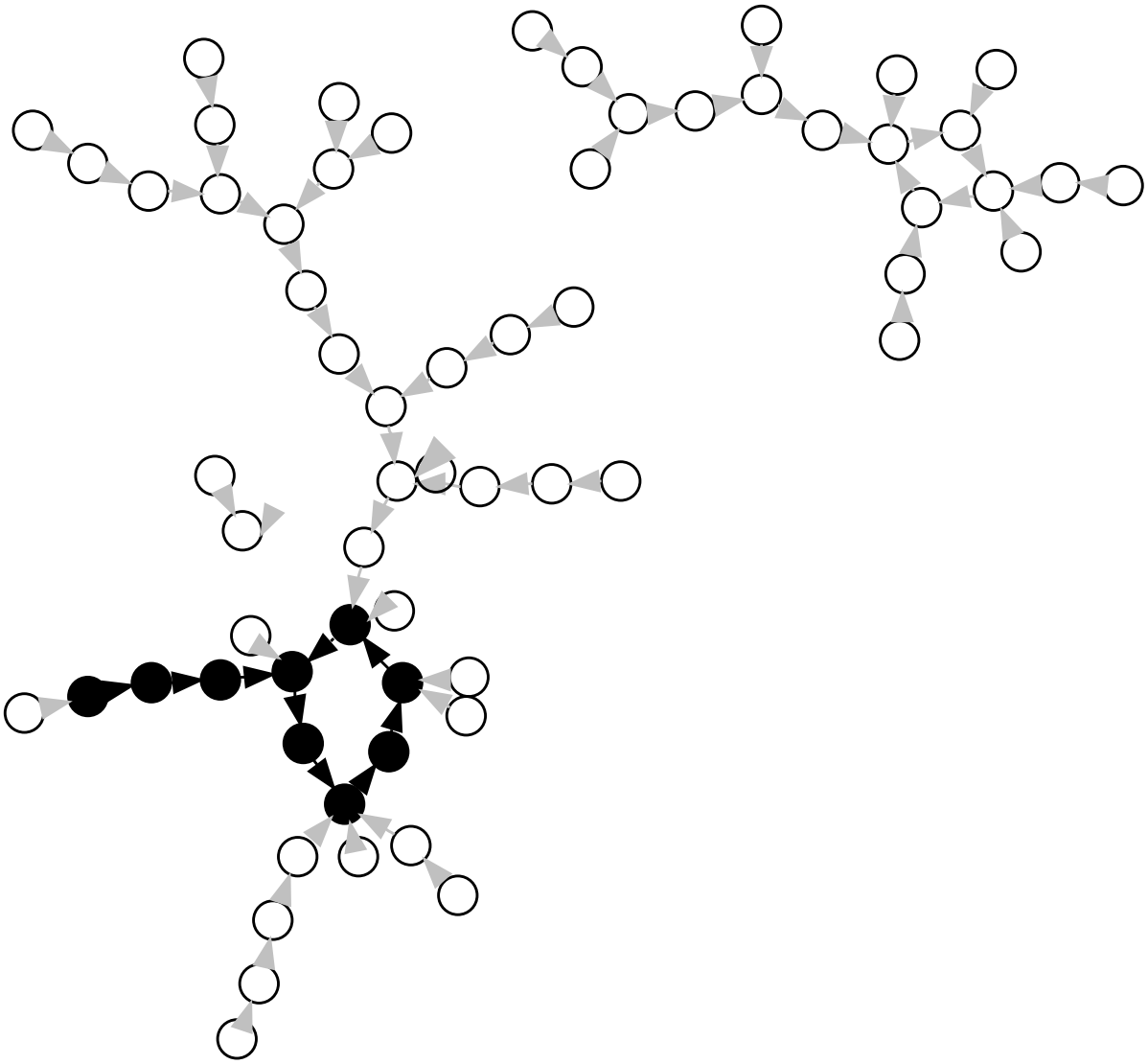












Assume that for each point
we know $a_i, b_i \in \mathbf{Z}/\ell\mathbf{Z}$
so that $W_i = a_i P + b_i Q$.

Then $W_i = W_j$ means that
$a_i P + b_i Q = a_j P + b_j Q$
so $(b_i - b_j)Q = (a_j - a_i)P$.
If $b_i \neq b_j$ the DLP is solved:
$n = (a_j - a_i)/(b_i - b_j)$.

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e.g. $f(W_i) = a(W_i)P + b(W_i)Q$,
starting from some initial
combination $W_0 = a_0 P + b_0 Q$.
If any W_i and W_j collide then
$W_{i+1} = W_{j+1}$, $W_{i+2} = W_{j+2}$,
etc.

If functions $a(W)$ and $b(W)$ are random modulo ℓ, iterations perform a random walk in $\langle P \rangle$.
If a and b are chosen such that $f(W_i) = f(-W_i)$ then the walk is defined on *equivalence classes* under $\pm$.
There are only $\lceil \ell/2 \rceil$ different classes. This reduces the average number of iterations by a factor of almost exactly $\sqrt{2}$.

In general, Pollard's rho method can be combined with any easily computed group automorphism of small order. More on that later.

Parallel collision search

Running Pollard's rho method on N computers gives speedup of $\approx \sqrt{N}$ from increased likelihood of finding collision.

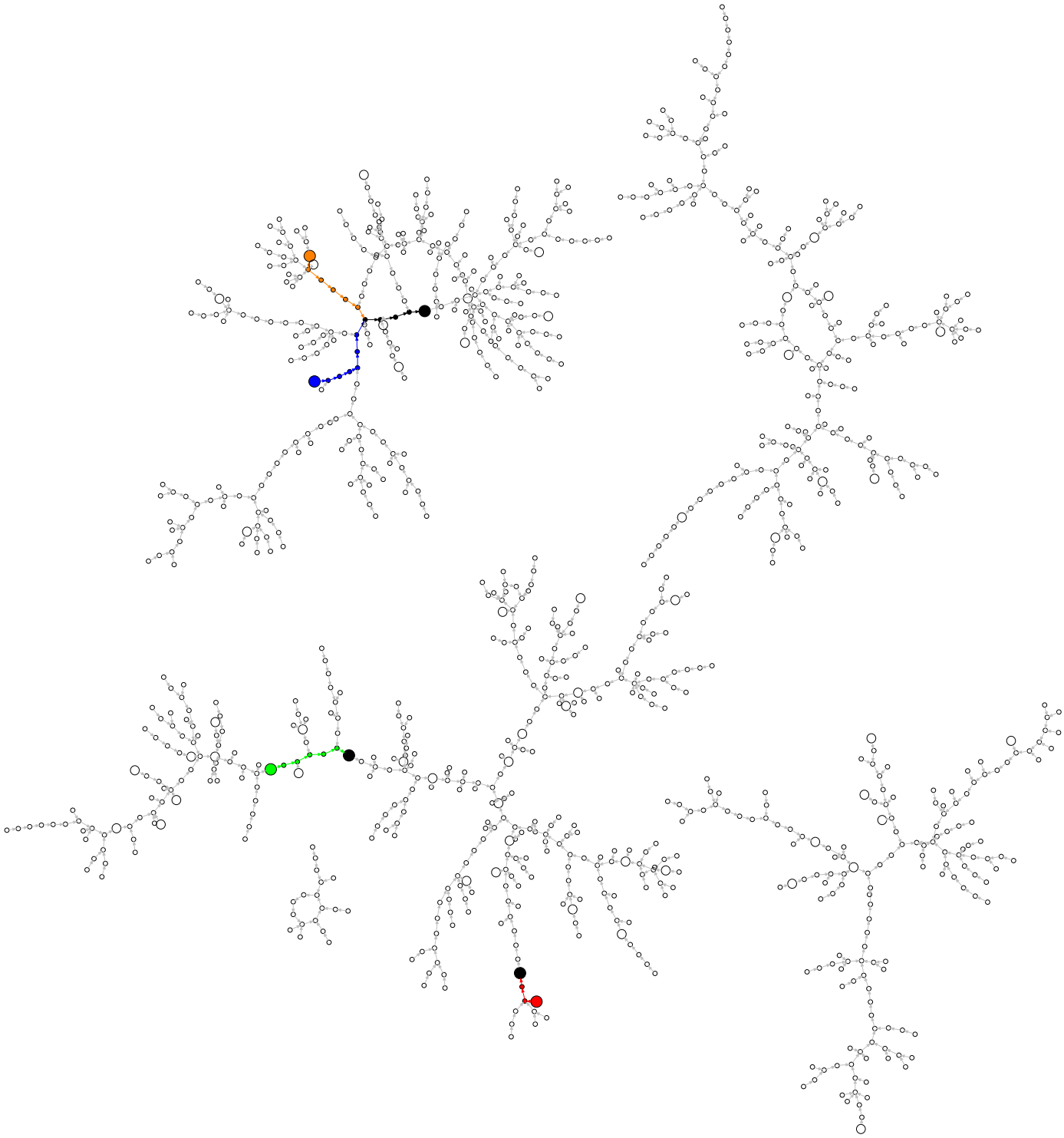
Want better way to spread computation across clients. Want to find collisions between walks on *different* machines, without frequent synchronization!

Better method due to van Oorschot and Wiener (1999). Declare some subset of $\langle P \rangle$ to be *distinguished points*.

Parallel rho: Perform many walks
with different starting points
but same update function f.
If two different walks
find the same point then
their subsequent steps will match.

Terminate each walk once it hits
a distinguished point and report
the point along with a_i and b_i to
server.
Server receives, stores, and sorts
all distinguished points.
Two walks reaching same
distinguished point give collision.
This collision solves the DLP.

Attacker chooses frequency and definition of distinguished points.
Tradeoffs are possible:
If distinguished points are rare, a small number of very long walks will be performed. This reduces the number of distinguished points sent to the server but increases the delay before a collision is recognized.
If distinguished points are frequent, many shorter walks will be performed.
In any case do not wait for cycle.
Total # of iterations unchanged.



Additive walks

Generic rho method requires two scalar multiplications for each iteration.
Could replace by double-scalar multiplication; could further merge the 2-scalar multiplications across several parallel iterations.

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More efficient: use *additive walk*:
Start with $W_0 = a_0 P$ and put
$f(W_i) = W_i + c_j P + d_j Q$
where $j = h(W_i)$.

Pollard's initial proposal:

Use $x(W_i) \bmod 3$ as h

and update:

$$W_{i+1} = \begin{cases} W_i + P \text{ for } x(W_i) \bmod 3 = 0 \\ 2W_i \quad \text{for } x(W_i) \bmod 3 = 1 \\ W_i + Q \text{ for } x(W_i) \bmod 3 = 2 \end{cases}$$

Easy to update a_i and b_i.

$$(a_{i+1}, b_{i+1}) = \begin{cases} (a_i + 1, b_i) \text{ for } x(W_i) \bmod 3 = 0 \\ (2a_i, 2b_i) \quad \text{for } x(W_i) \bmod 3 = 1 \\ (a_i, b_i + 1) \text{ for } x(W_i) \bmod 3 = 2 \end{cases}$$

Additive walk requires only one addition per iteration.
h maps from $\langle P \rangle$ to $\{0, 1, \ldots, r-1\}$, and $R_j = c_j P + d_j Q$ are precomputed for each $j \in \{0, 1, \ldots, r-1\}$.

Easy coefficient update:
$W_i = a_i P + b_i Q$,
where a_i and b_i are defined recursively as follows:
$a_{i+1} = a_i + c_{h(W_i)}$ and
$b_{i+1} = b_i + d_{h(W_i)}$.

Additive walks have disadvantages:
The walks are noticeably nonrandom; this means they need more iterations than the generic rho method to find a collision.

This effect disappears as r grows, but but then the precomputed table $R_0, \ldots, R_{r-1}$ does not fit into fast memory. This depends on the platform, e.g. trouble for GPUs.
More trouble with adding walks later.

Randomness of adding walks

Let $h(W) = i$ with probability p_i.
Fix a point T, and let W and W' be two independent uniform random points.
Let $W \neq W'$ both map to T.
This event occurs if there are $i \neq j$ such that simultaneously:
$T = W + R_i = W' + R_j$;
$h(W) = i$; $h(W') = j$.

These conditions have probability $1/\ell^2$, p_i, and p_j respectively.

Summing over all (i, j)
gives the overall probability
$\left(\sum_{i \neq j} p_i p_j\right) / \ell^2 \quad =$
$\left(\sum_{i,j} p_i p_j - \sum_i p_i^2\right) / \ell^2 \; =$
$\left(1 - \sum_i p_i^2\right) / \ell^2$.

This means that the probability of an immediate collision from W and W' is $\left(1 - \sum_i p_i^2\right) / \ell$, where we added over the ℓ choices of T. In the simple case that all the p_i are $1/r$, the difference from the optimal $\sqrt{\pi \ell / 2}$ iterations is a factor of $1/\sqrt{1 - 1/r} \approx 1 + 1/(2r)$.

Various heuristics leading to standard $\sqrt{1-1/r}$ formula in different ways:

1981 Brent–Pollard;

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2009 ECC2K-130 paper, eprint 2009/541.

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2010 Bernstein–Lange:
Standard formula is wrong!
There is a further slowdown
from higher-order anti-collisions:
e.g. $W + R_i + R_k \neq W' + R_j + R_l$
if $R_i + R_k = R_j + R_l$.
$\approx 1\%$ slowdown for ECC2K-130.

Eliminating storage

Usual description: each walk
keeps track of a_i and b_i
with $W_i = a_i P + b_i Q$.

This requires each client to
implement arithmetic modulo ℓ
or at least keep track of
how often each R_j is used.

For distinguished points
these values are
transmitted to server (bandwidth)
which stores them as
e.g. (W_i, a_i, b_i) (space).

2009 ECC2K-130 paper:
Remember where you started.
If $W_i = W_j$ is the collision of distinguished points,
can recompute these walks with a_i, b_i, a_j, and b_j;
walk is deterministic!
Server stores 2^{45} distinguished points; only needs to know coefficients for 2 of them.

Our setup: Each walk remembers seed; server stores distinguished point and seed.
Saves time, bandwidth, space.

Negation and rho

$W = (x, y)$ and $-W = (x, -y)$
have same x-coordinate.
Search for x-coordinate collision.

Search space for collisions is
only $\lceil \ell/2 \rceil$; this gives factor $\sqrt{2}$
speedup . . . if $f(W_i) = f(-W_i)$.

To ensure $f(W_i) = f(-W_i)$:
Define $j = h(|W_i|)$ and
$f(W_i) = |W_i| + c_j P + d_j Q$.
Define $|W_i|$ as, e.g., lexicographic
minimum of $W_i, -W_i$.
This negation speedup
is textbook material.

Problem: this walk can
run into fruitless cycles!
Example: If $|W_{i+1}| = -W_{i+1}$
and $h(|W_{i+1}|) = j = h(|W_i|)$
then $W_{i+2} = f(W_{i+1}) =$
$-W_{i+1} + c_j P + d_j Q =$
$-(|W_i| + c_j P + d_j Q) + c_j P + d_j Q =$
$-|W_i|$ so $|W_{i+2}| = |W_i|$
so $W_{i+3} = W_{i+1}$
so $W_{i+4} = W_{i+2}$ etc.

If h maps to r different values
then expect this example to occur
with probability $1/(2r)$
at each step.
Known issue, not quite textbook.

Eliminating fruitless cycles

Issue of fruitless cycles is known and several fixes are proposed. See appendix of full version ePrint 2011/003 for even more details and historical comments.

Summary: most of them got it wrong.

Eliminating fruitless cycles

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Summary: most of them got it wrong.

So what to do?
Choose a big r, e.g. $r = 2048$.
$1/(2r) = 1/4096$ small;
cycles infrequent.

Define $|(x, y)|$ to mean
(x, y) for $y \in \{0, 2, 4, \ldots, p-1\}$
or
$(x, -y)$ for $y \in \{1, 3, 5, \ldots, p-2\}$.

Precompute points
$R_0, R_1, \ldots, R_{r-1}$ as known
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Precompute points
$R_0, R_1, \ldots, R_{r-1}$ as known random multiples of P. Here you can do full scalar multiplication in inversion-free coordinates!
Start each walk at a point $W_0 = |b_0 Q|$, where b_0 is chosen randomly.
Compute $W_1, W_2, \ldots$ as $W_{i+1} = |W_i + R_{h(W_i)}|$.

Occasionally, every w iterations,
check for fruitless cycles
of length 2.
For those cases change the
definition of W_i as follows:
Compute W_{i-1} and check
whether $W_{i-1} = W_{i-3}$.
If $W_{i-1} \neq W_{i-3}$, put $W_i = W_{i-1}$.
If $W_{i-1} = W_{i-3}$, put
$W_i = |2 \min\{W_{i-1}, W_{i-2}\}|$,
where min means
lexicographic minimum.
Doubling the point
makes it escape the cycle.

Cycles of length 4, 6, or 12
occur far less frequently.
Cycles of length 4, or 6
are detected when checking
for cycles of length 12;
so skip individual ones.

Same way of escape:
define $W_i =$
$|2\min\{W_{i-1}, W_{i-2}, W_{i-3}, W_{i-4},$
$W_{i-5}, W_{i-6}, W_{i-7}, W_{i-8},$
$W_{i-9}, W_{i-10}, W_{i-11}, W_{i-12}\}|$
if trapped
and $W_i = W_{i-1}$ otherwise.

Do not store all these points!

When checking for cycle,
store only potential entry point
W_{i-13} (one coordinate, for
comparison) and the
smallest point encountered since
(to escape).

For large DLP
look for larger cycles;
in general, look for
fruitless cycles of even lengths
up to $\approx (\log \ell)/(\log r)$.

How to choose w?

Fruitless cycles of length 2 appear
with probability $\approx 1/(2r)$.
These cycles persist
until detected.
After w iterations,
probability of cycle $\approx w/(2r)$,
wastes $\approx w/2$ iterations
(on average) if it does appear.

Do not choose w
as small as possible!
If a cycle has *not* appeared then
the check wastes an iteration.

The overall loss is approximately
$1 + w^2/(4r)$ iterations out of w.
To minimize the quotient
$1/w + w/(4r)$ we take $w \approx 2\sqrt{r}$.

Cycles of length $2c$ appear with
probability $\approx 1/r^c$,
optimal checking frequency is
$\approx 1/r^{c/2}$.
Loss rapidly disappears
as c increases.
Can use lcm of cycle lengths
to check.

Concrete example: 112-bit DLP

Use $r = 2048$. Check for 2-cycles every 48 iterations.
Check for larger cycles much less frequently.
Unify the checks for 4-cycles and 6-cycles into a check for 12-cycles every 49152 iterations.

Choice of r has big impact!
$r = 512$ calls for checking
for 2-cycles every 24 iterations.
In general, negation overhead
$\approx$ doubles when table size
is reduced by factor of 4.

Bernstein, Lange, Schwabe
(PKC 2011):

Our software solves
random ECDL on the same curve
(with no precomputation)
in 35.6 PS3 years on average.

For comparison:
Bos–Kaihara–Kleinjung–Lenstra–
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uses 65 PS3 years on average.

First big speedup:
We use the negation map.
Second speedup: Fast arithmetic.

Why are we confident this works?

We only have one PlayStation 3, not 200 used in the current record, & don't want to wait for 36 years to show that we actually compute the right thing.

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Can produced scaled versions:
Use *same* prime field
(so that we can compare the field arithmetic) and same curve shape
$y^2 = x^3 - 3x + b$
but vary b to get curves with small subgroups.

This produces other curves, and many of those have smaller order subgroups.
Specify DLP in subgroup of size 2^{50}, or 2^{55}, or 2^{60} and show that the actual running time matches the expectation.
And that DLP is correct.

We used same property for a point to be distinguished as in big attack; probability is 2^{-20}.
Need to watch out that walks do not run into rho-type cycles (artefact of small group order).
We aborted overlong walks.

Recall: p has 112 bits.
28 bytes for table entry (x, y).

We expand to 36 bytes
to accelerate arithmetic.
We compress to 32 bytes
by insisting on small x, y;
very fast initial computation.

Only 64KB for table.
Our Cell table-load cost: 0,
overlapping loads with arithmetic.
No "cache inefficiencies."

What about fruitless cycles?

We run 45 iterations.
We then save s;
run 2 slightly slower iterations
tracking minimum (s, x, y);
then double tracked (x, y)
if new s equals saved s.

(Occasionally replace 2 by 12
to detect 4-cycles, 6-cycles.
Such cycles are almost
too rare to worry about,
but detecting them has a
completely negligible cost.)

Maybe fruitless cycles waste
some of the 47 iterations.
. . . but this is infrequent.
Lose $\approx 0.6\%$ of all iterations.

Tracking minimum isn't free,
but most iterations skip it!
Same for final s comparison.
Still no conditional branches.
Overall cost $\approx 1.3\%$.

Doubling occurs for only
$\approx 1/4096$ of all iterations.
We use SIMD quite lazily here;
overall cost $\approx 0.6\%$.
Can reduce this cost further.

To confirm iteration effectiveness
we have run many experiments
on $y^2 = x^3 - 3x + 9$
over the same $\mathbf{F}_p$,
using smaller-order P.
Matched DL cost predictions.

Final conclusions:
Sensible use of negation,
with or without SIMD,
has negligible impact
on cost of each iteration.
Impact on number of iterations
is almost exactly $\sqrt{2}$.
Overall benefit is
extremely close to $\sqrt{2}$.