Modeling the Security of Cryptography, Part 2: Public-Key Cryptography

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eprint.iacr.org/2012/318,
eprint.iacr.org/2012/458

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Prove, e.g., that bounds on insecurity of RSA-1024 imply similar bounds on insecurity of RSA-1024-PSS.

Conjecture bounds on insecurity of RSA-1024: e.g., "it takes time $Ce^{1.923(\log N)^{1/3}(\log \log N)^{2/3}}$ to invert RSA".

DL-based systems

E.g. forward security setting in TLS uses DH-key exchange on elliptic curve NIST P-256.

Break by solving ECDL in group of prime order $\ell \approx 2^{256}$.
ECDL input: points P, Q, where P is a standard generator.
ECDL output: $\log_P Q$.

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Break by solving ECDL in group of prime order $\ell \approx 2^{256}$. ECDL input: points P, Q, where P is a standard generator. ECDL output: $\log_P Q$.

Standard conjecture: For each $p \in [0, 1]$, each P-256 ECDL algorithm with success probability $\geq p$ takes "time" $\geq 2^{128} p^{1/2}$.

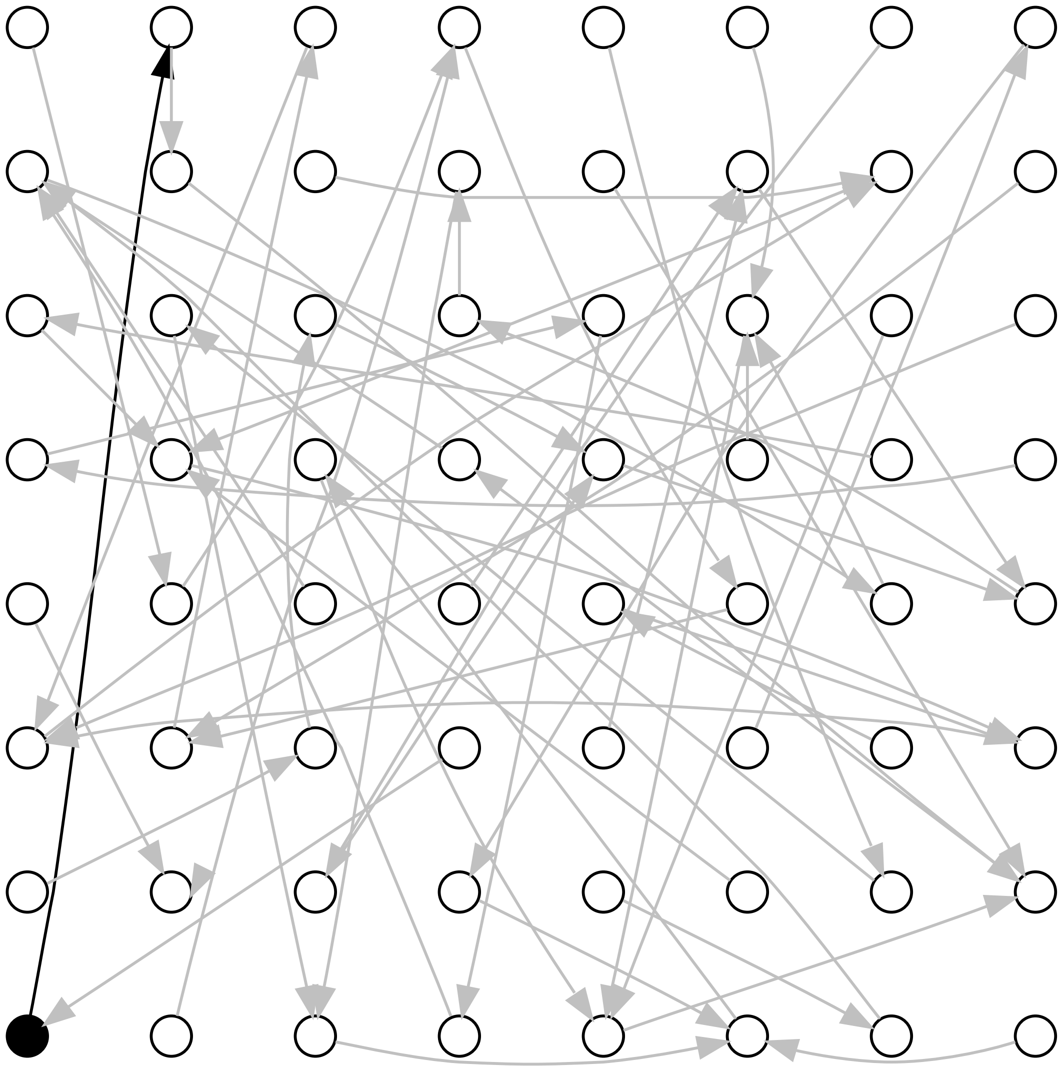
The rho method

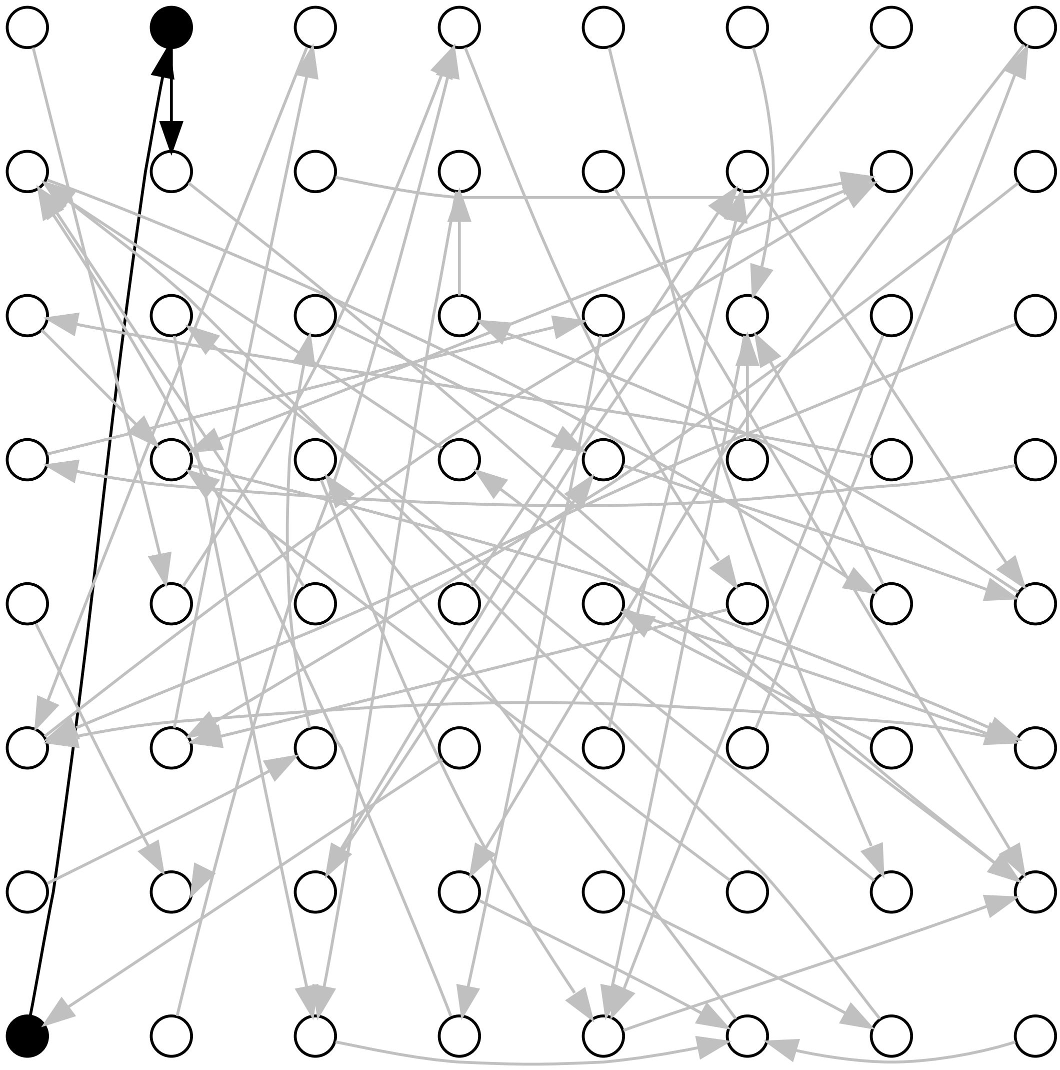
Simplified, non-parallel rho:

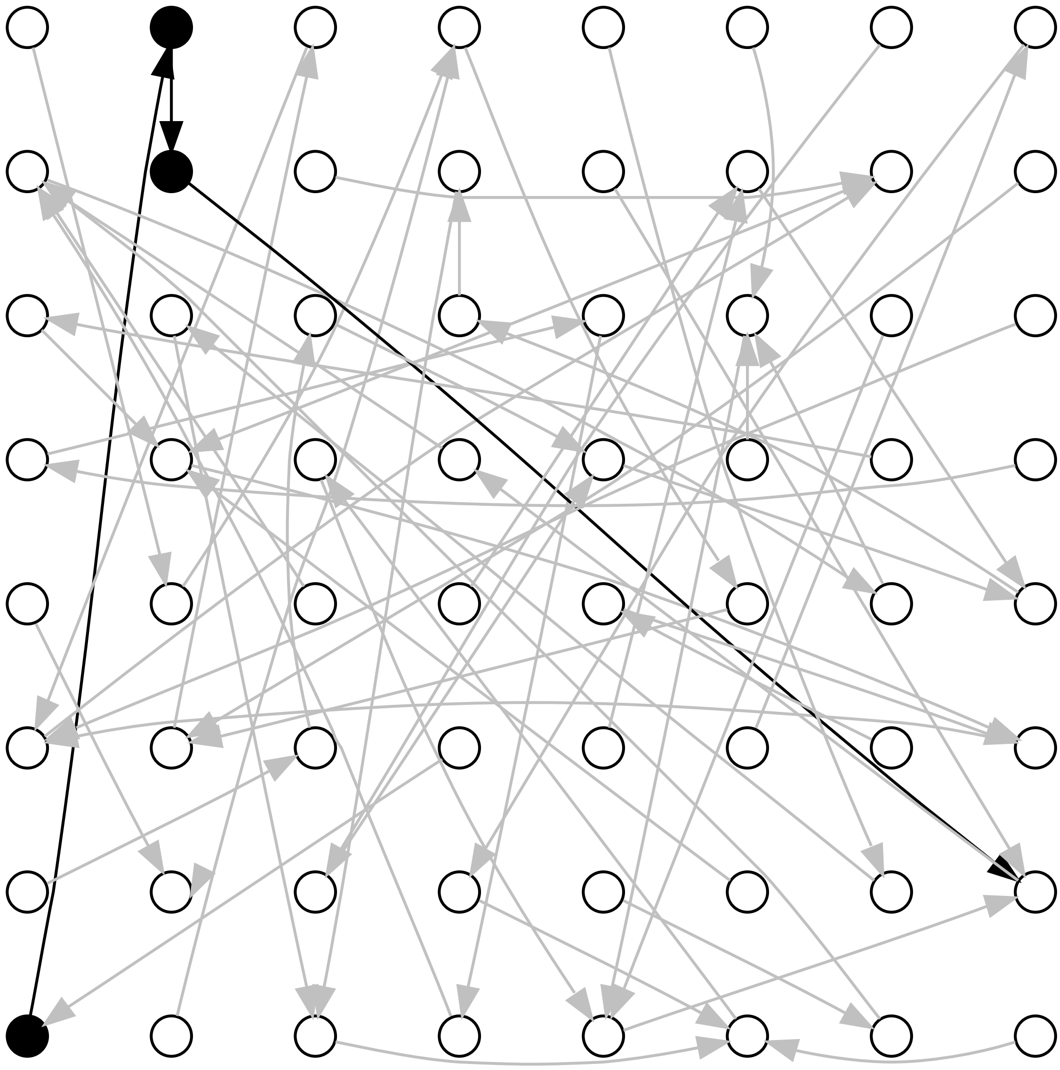
Make a pseudo-random walk $R_0, R_1, R_2, \ldots$ in the group $\langle P \rangle$, where current point determines the next point: $R_{i+1} = f(R_i)$.

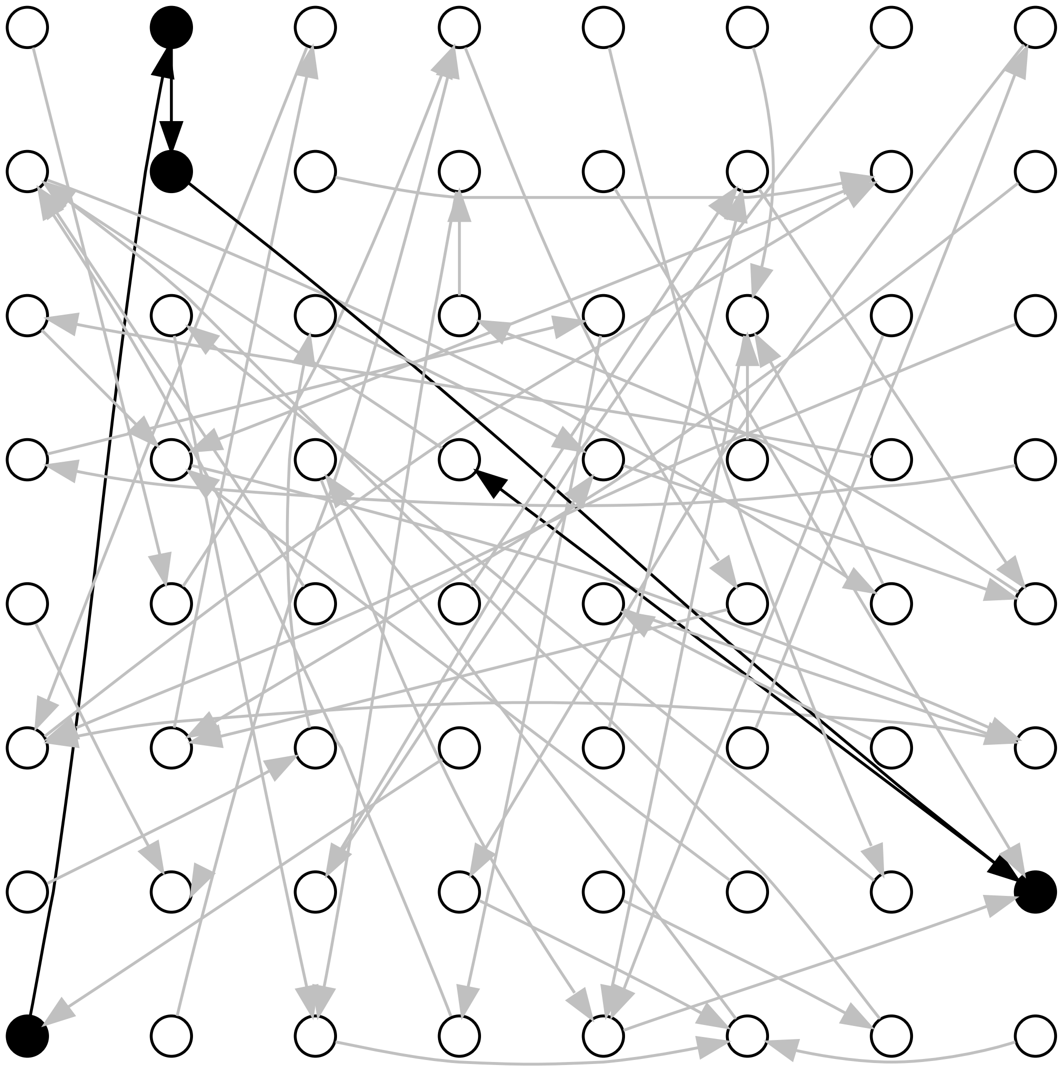
Birthday paradox:
Randomly choosing from ℓ elements picks one element twice after about $\sqrt{\pi\ell/2}$ draws.

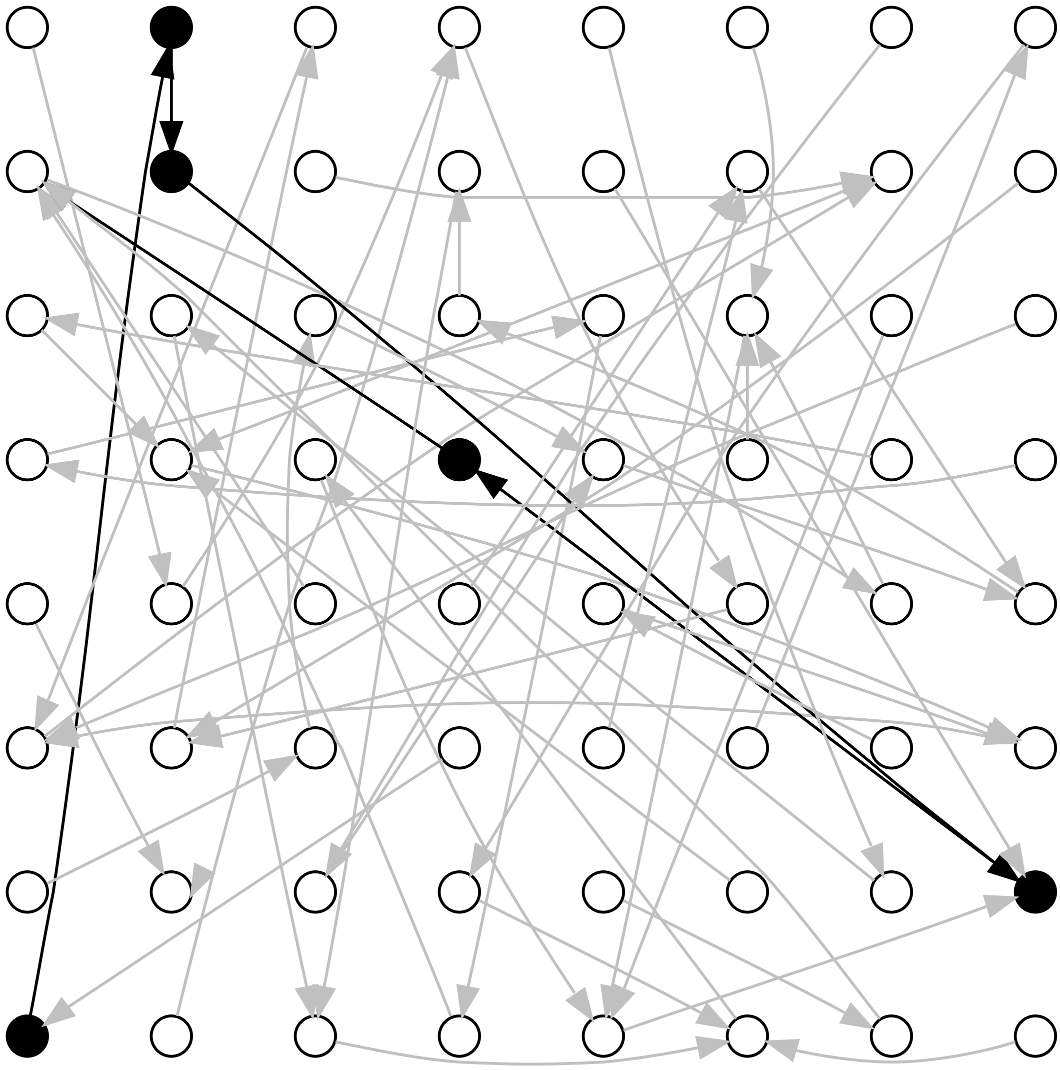
The walk now enters a cycle.
Cycle-finding algorithm (e.g., Floyd) quickly detects this.

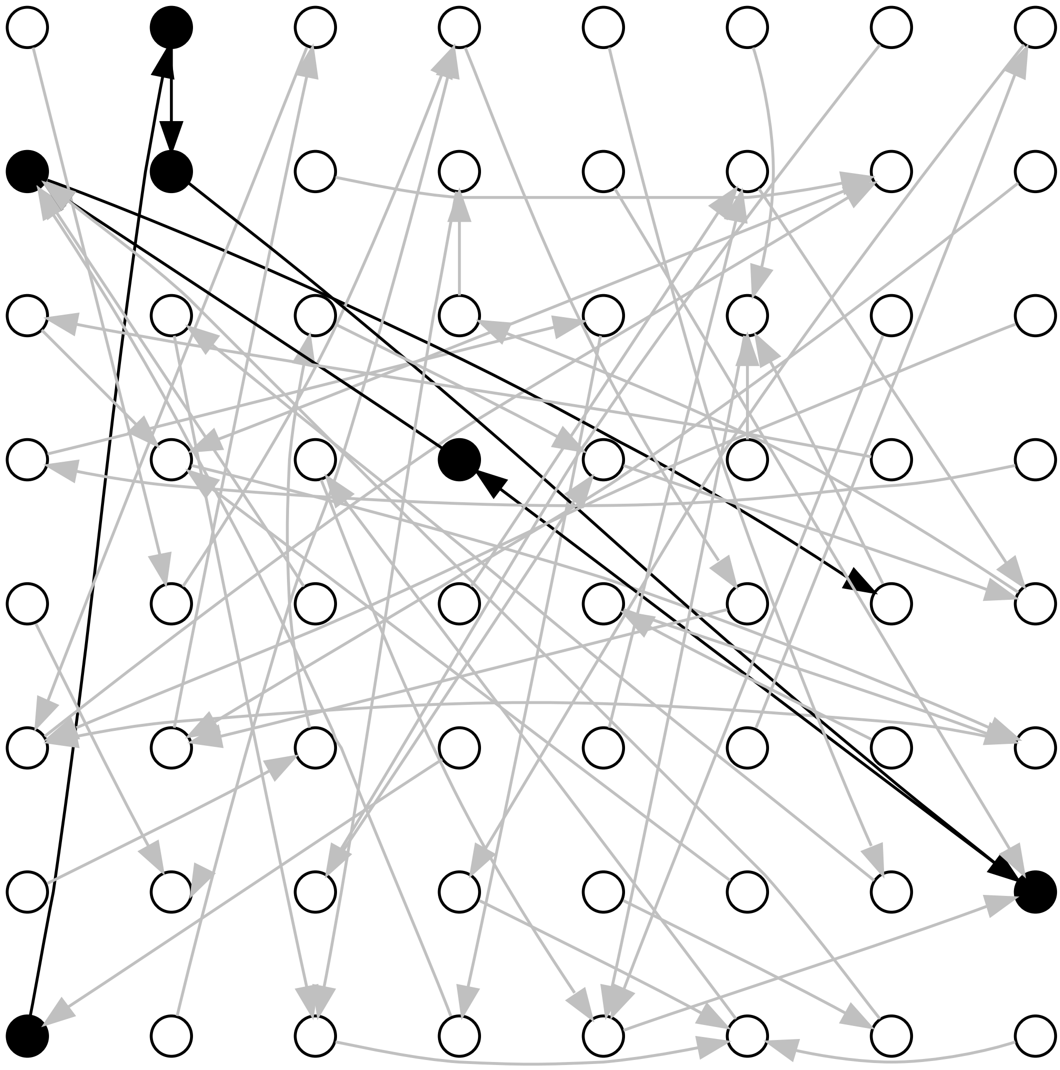


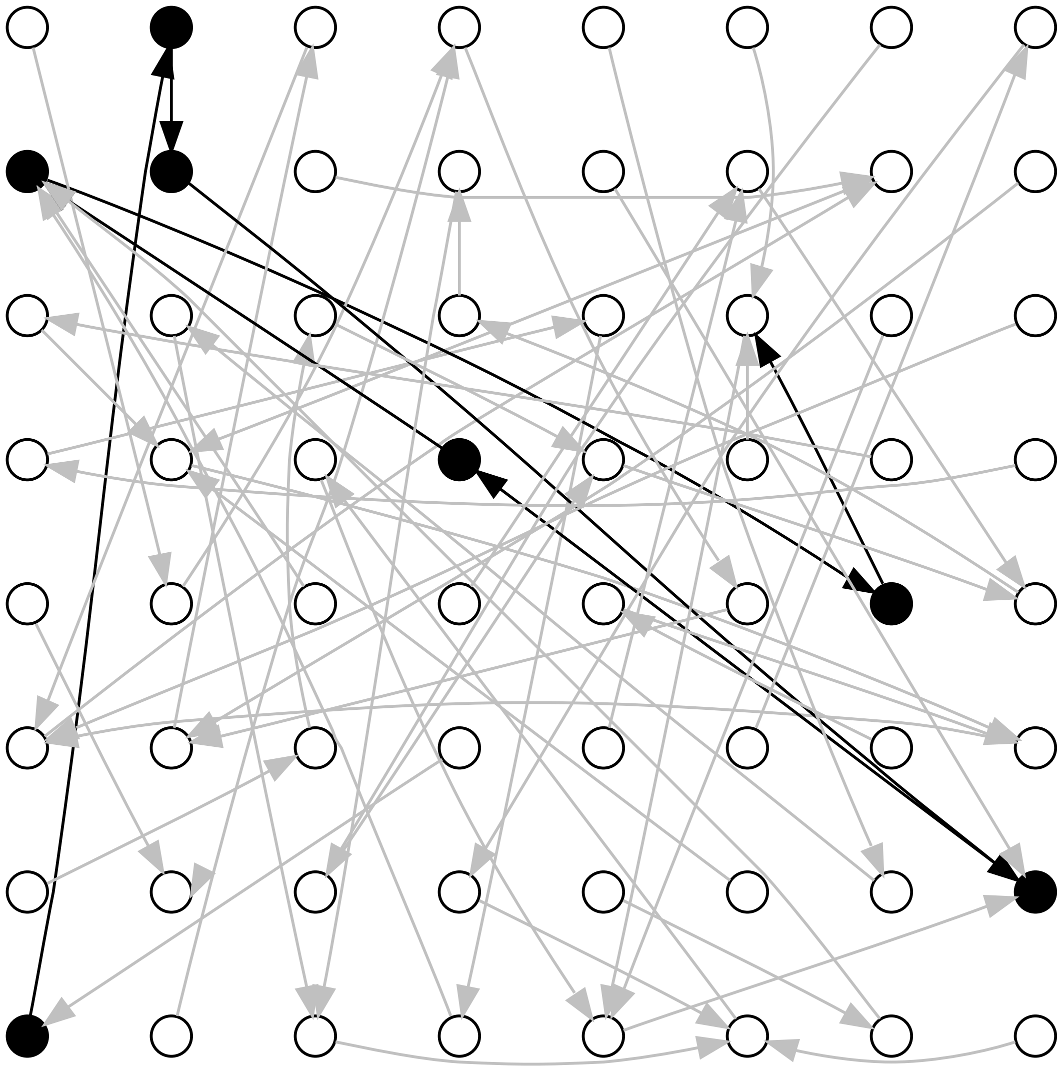


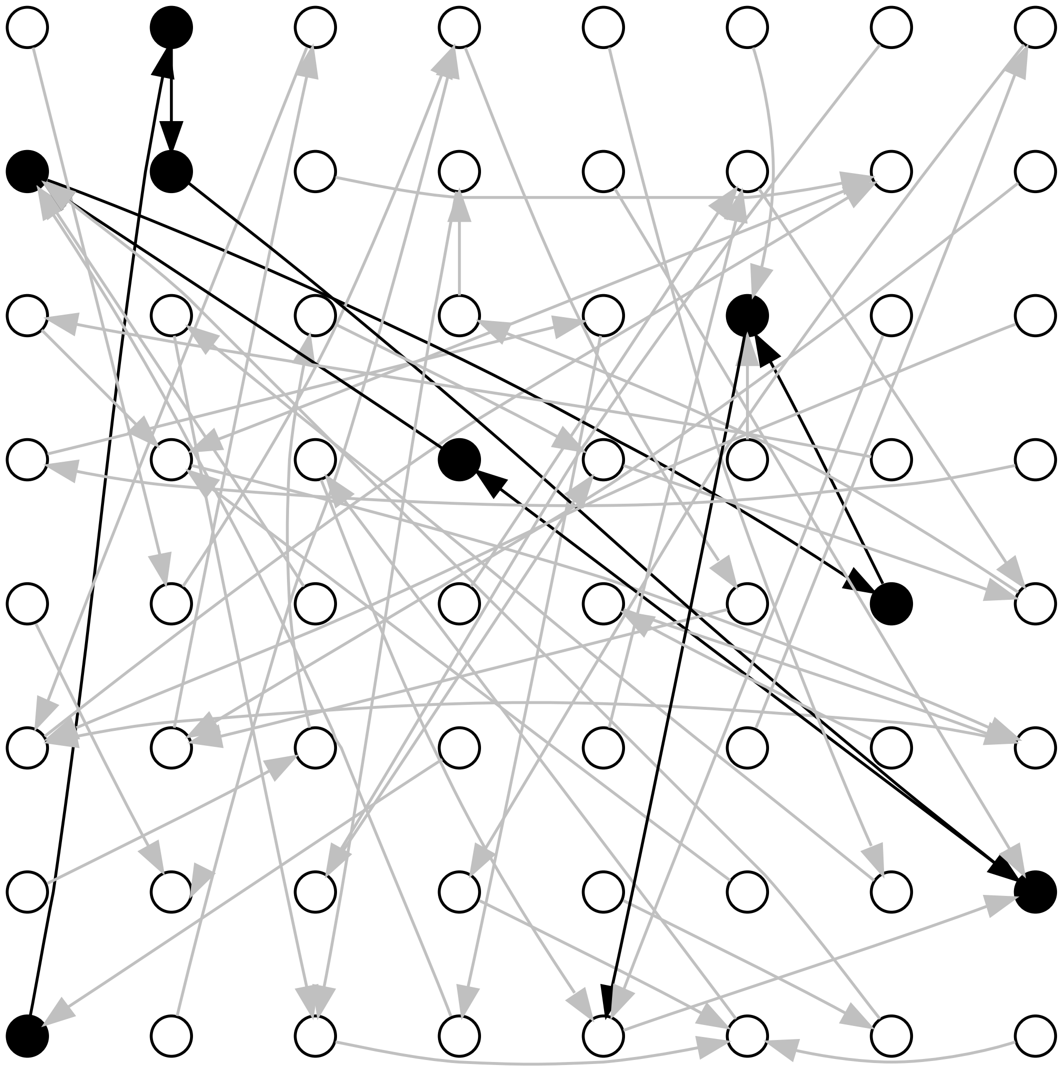


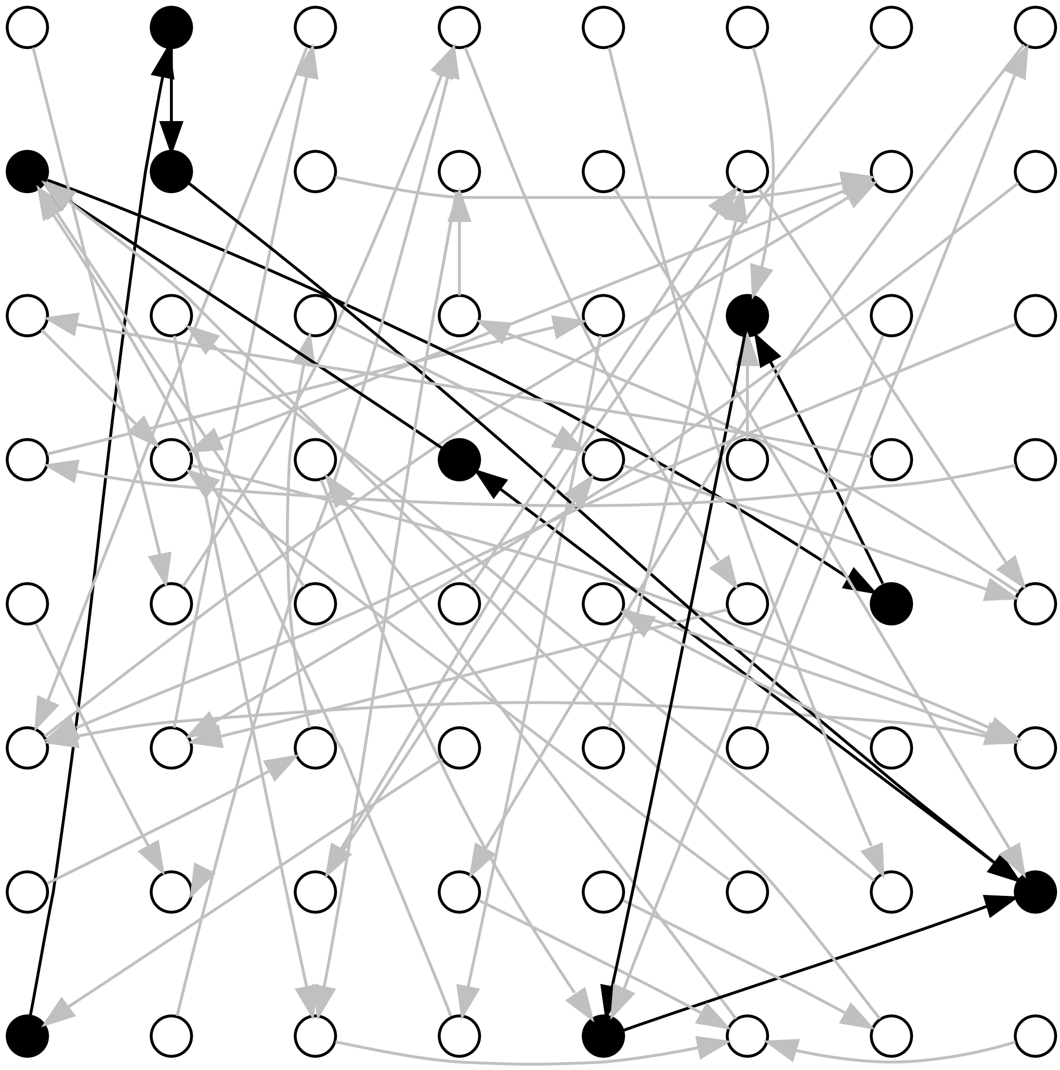


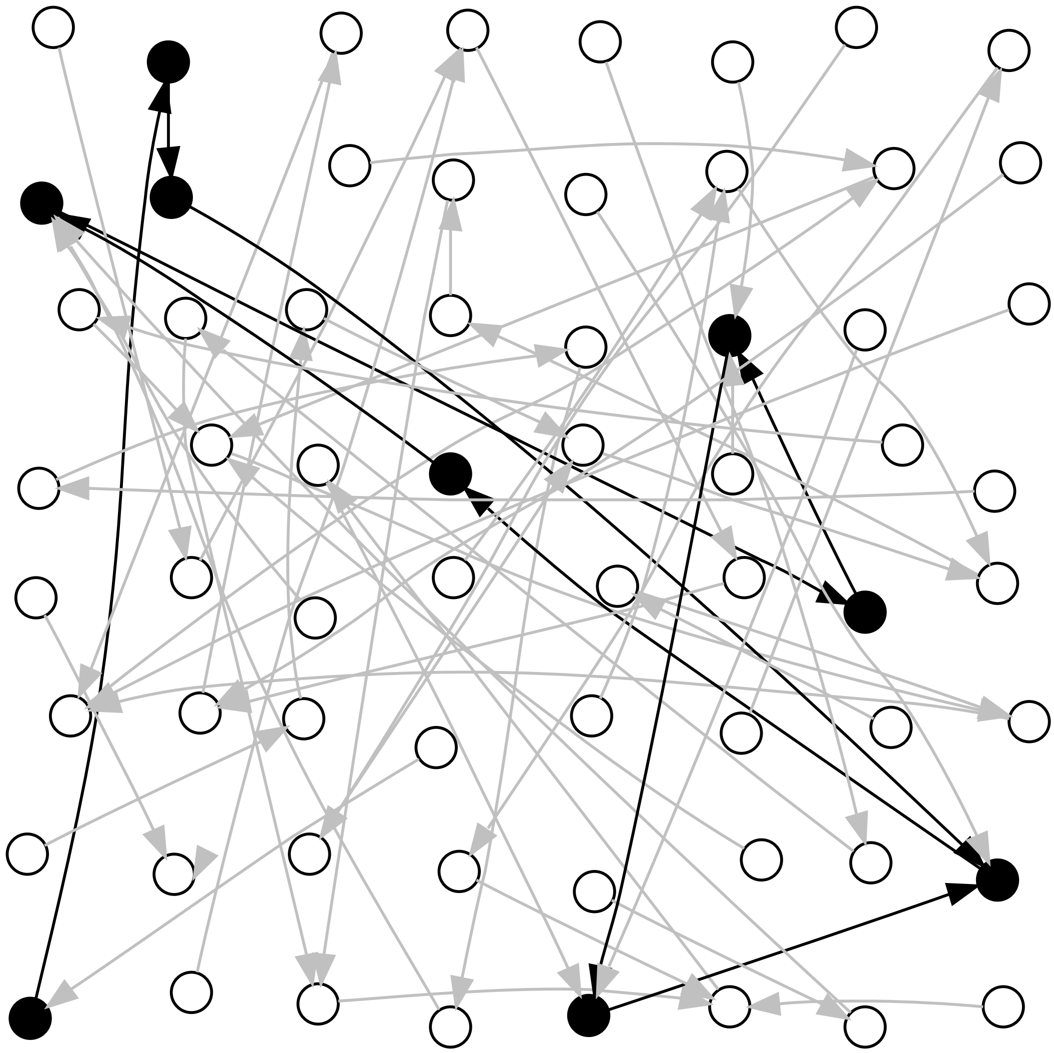


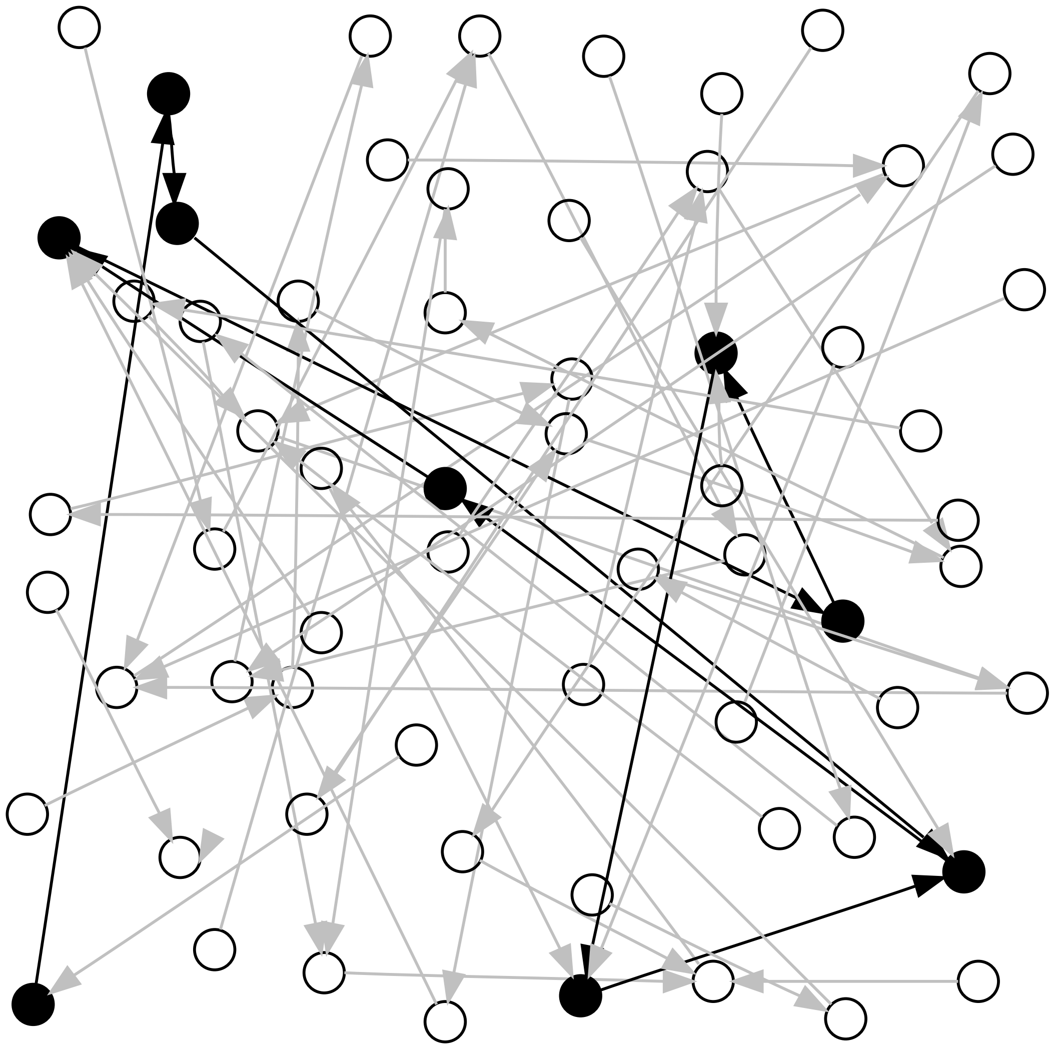


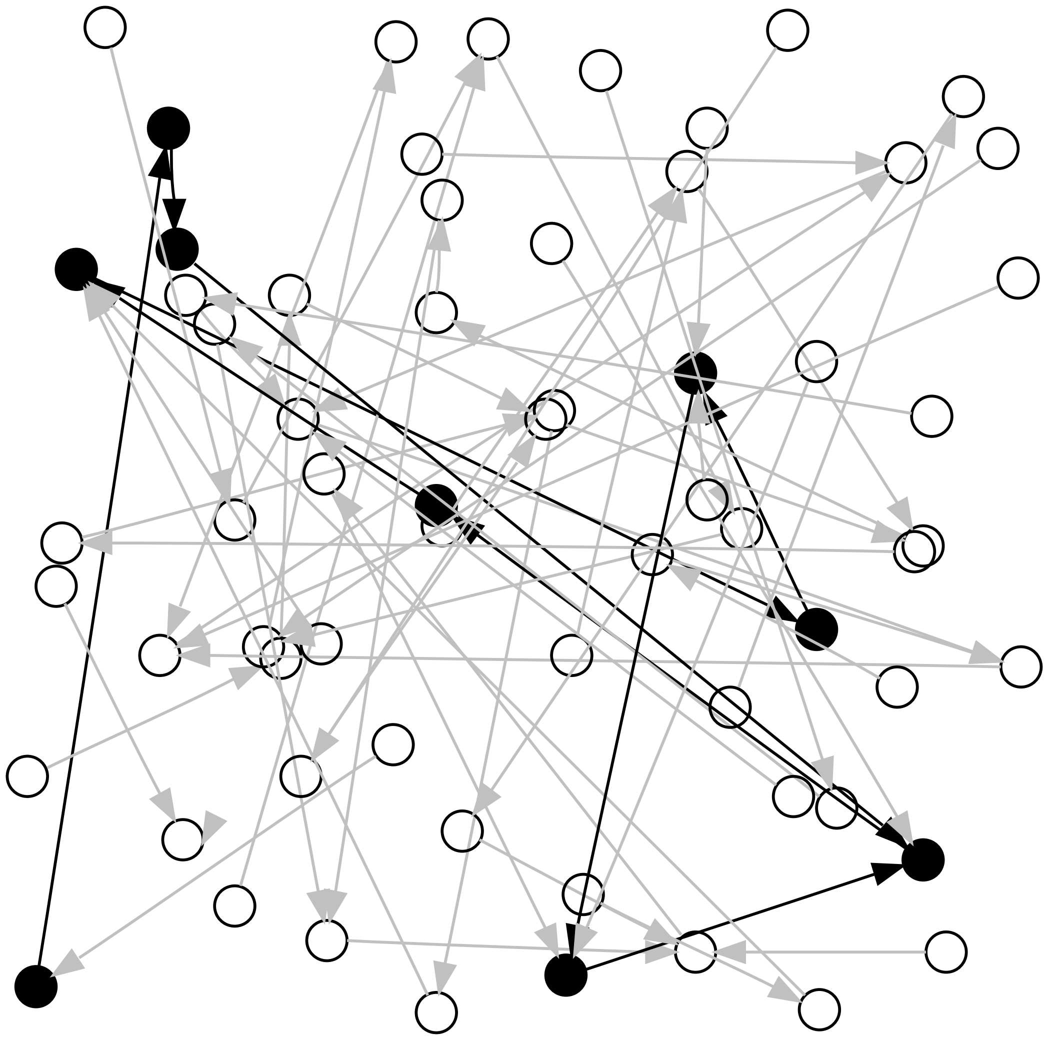


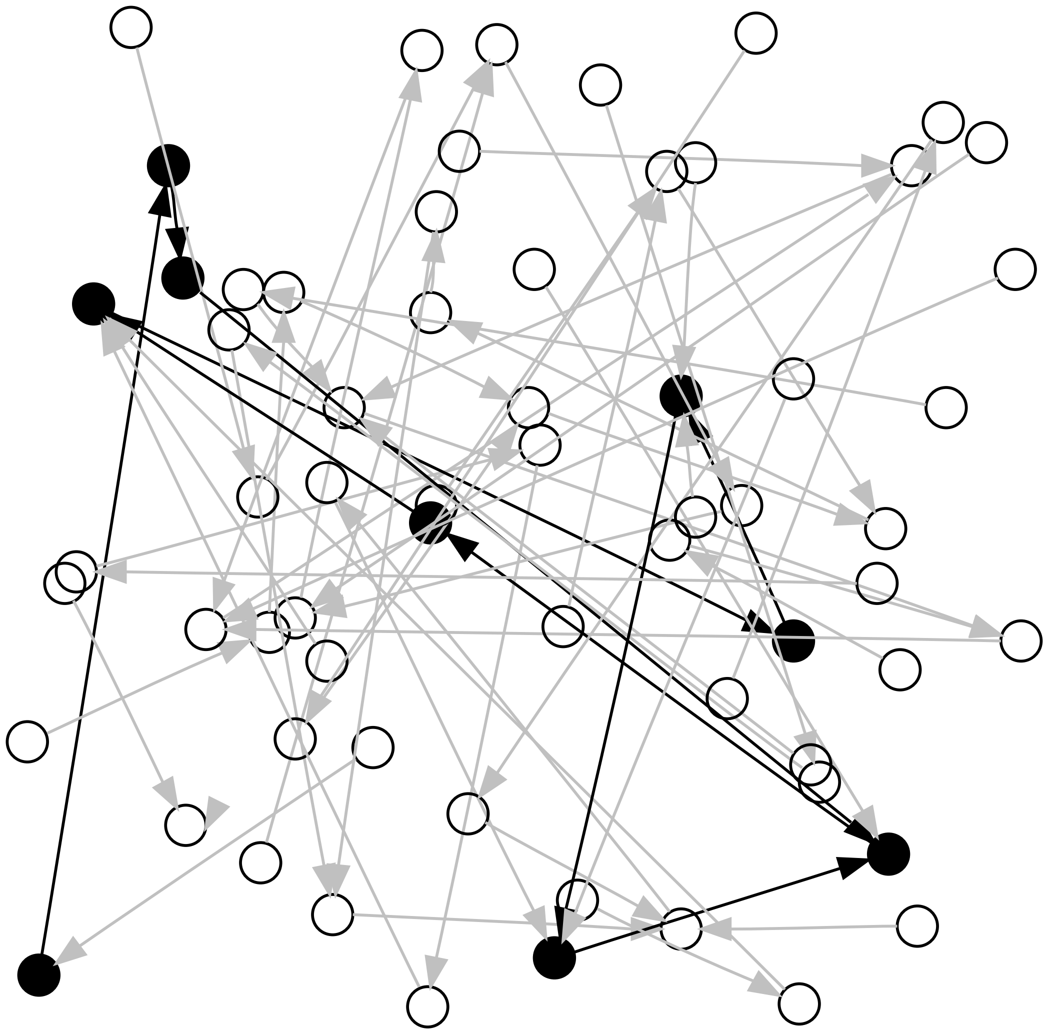


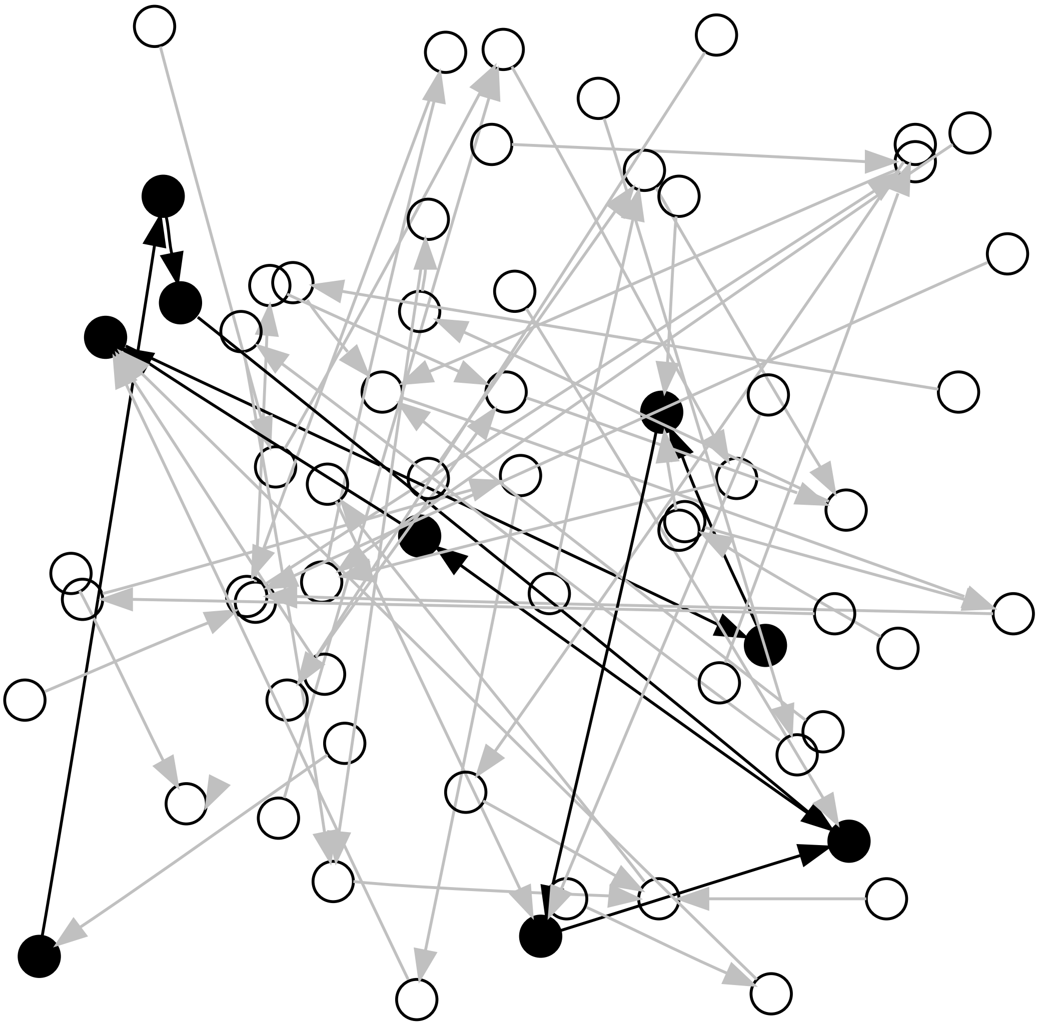


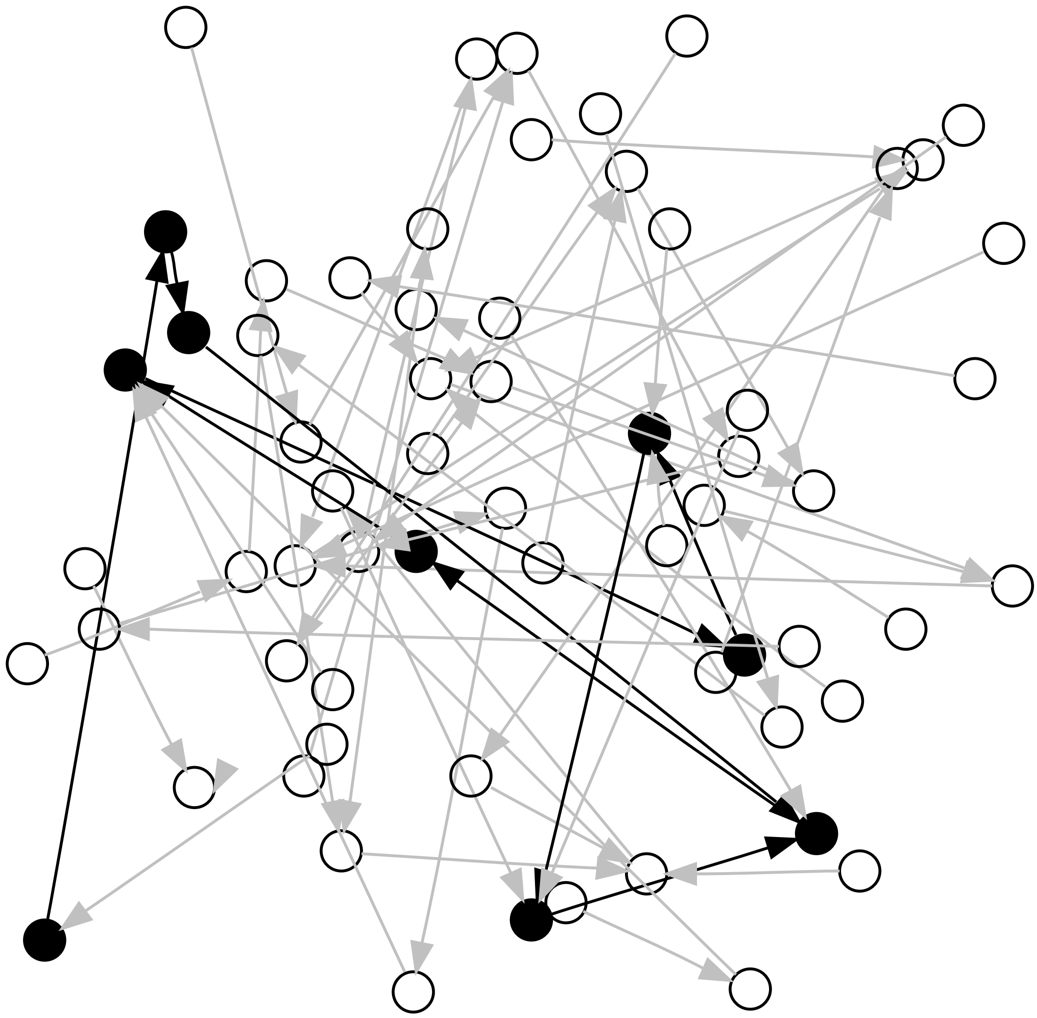


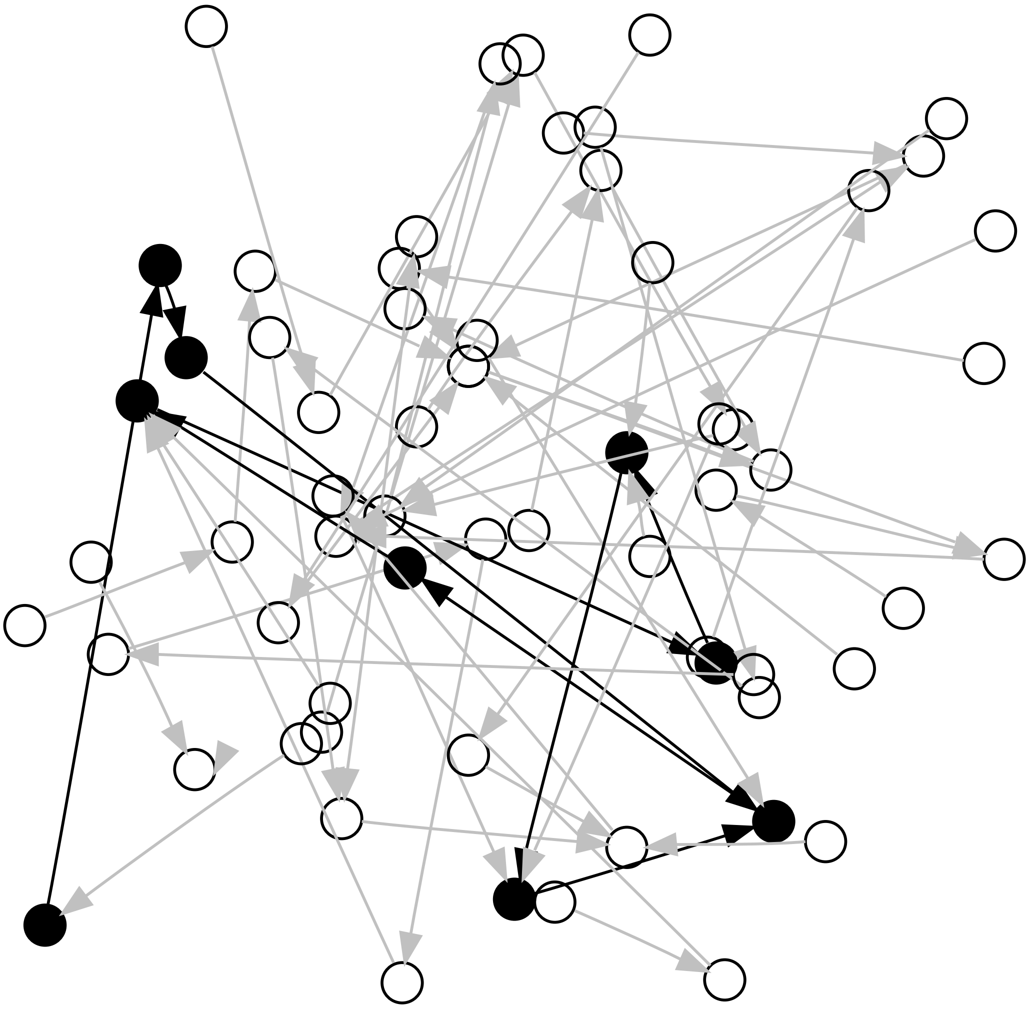


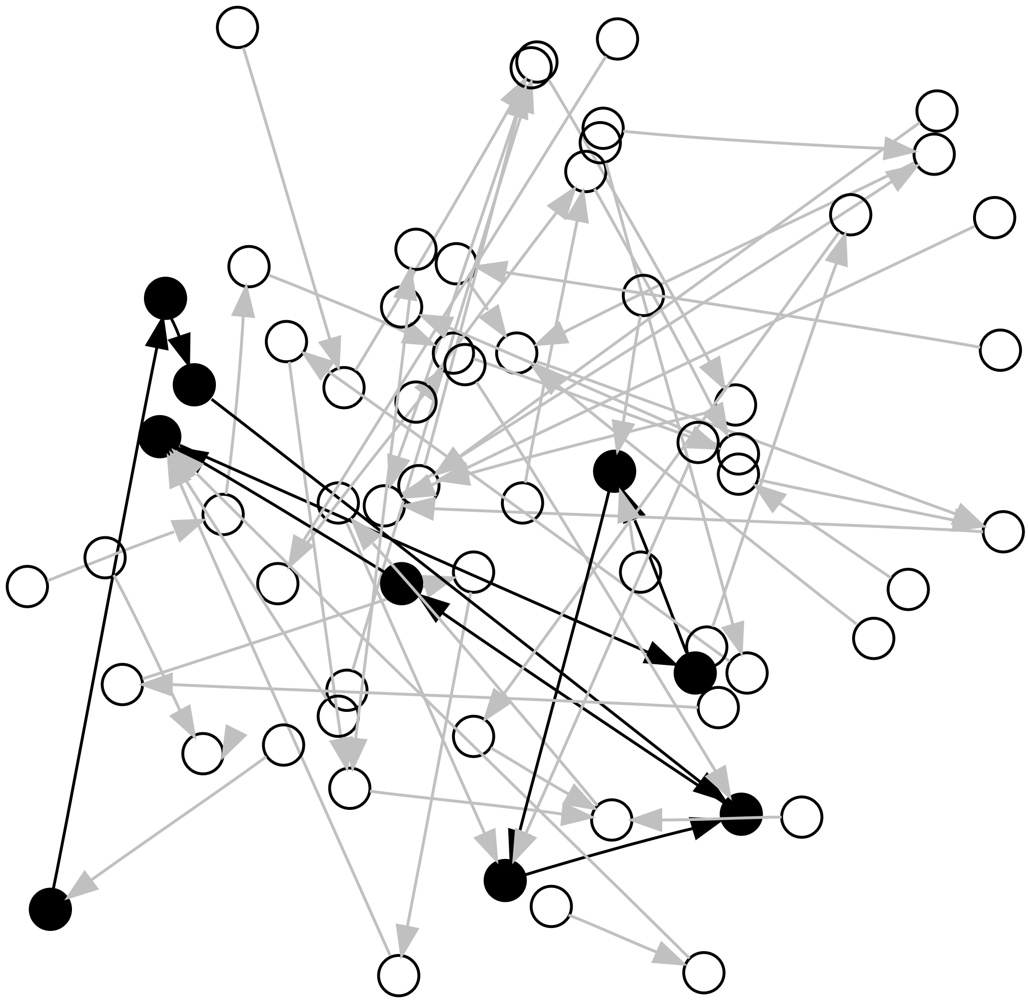


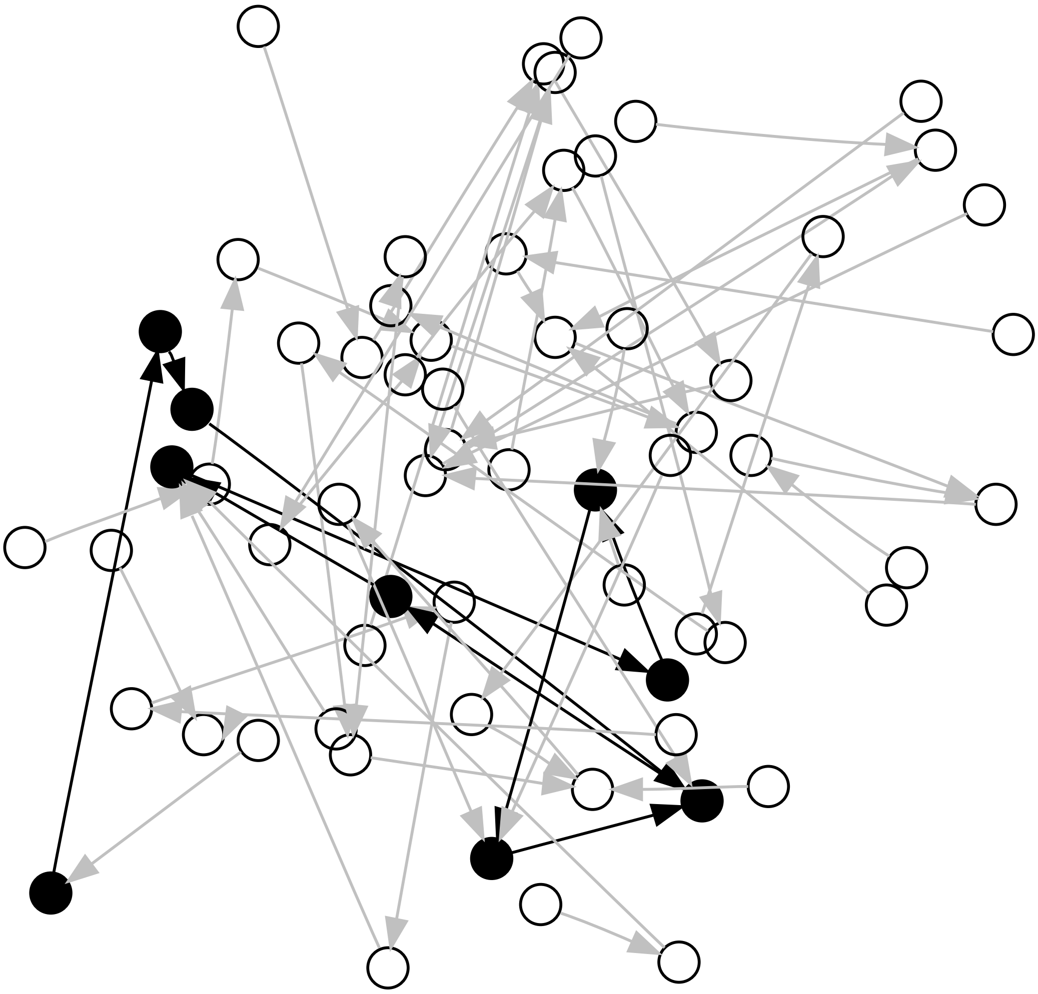


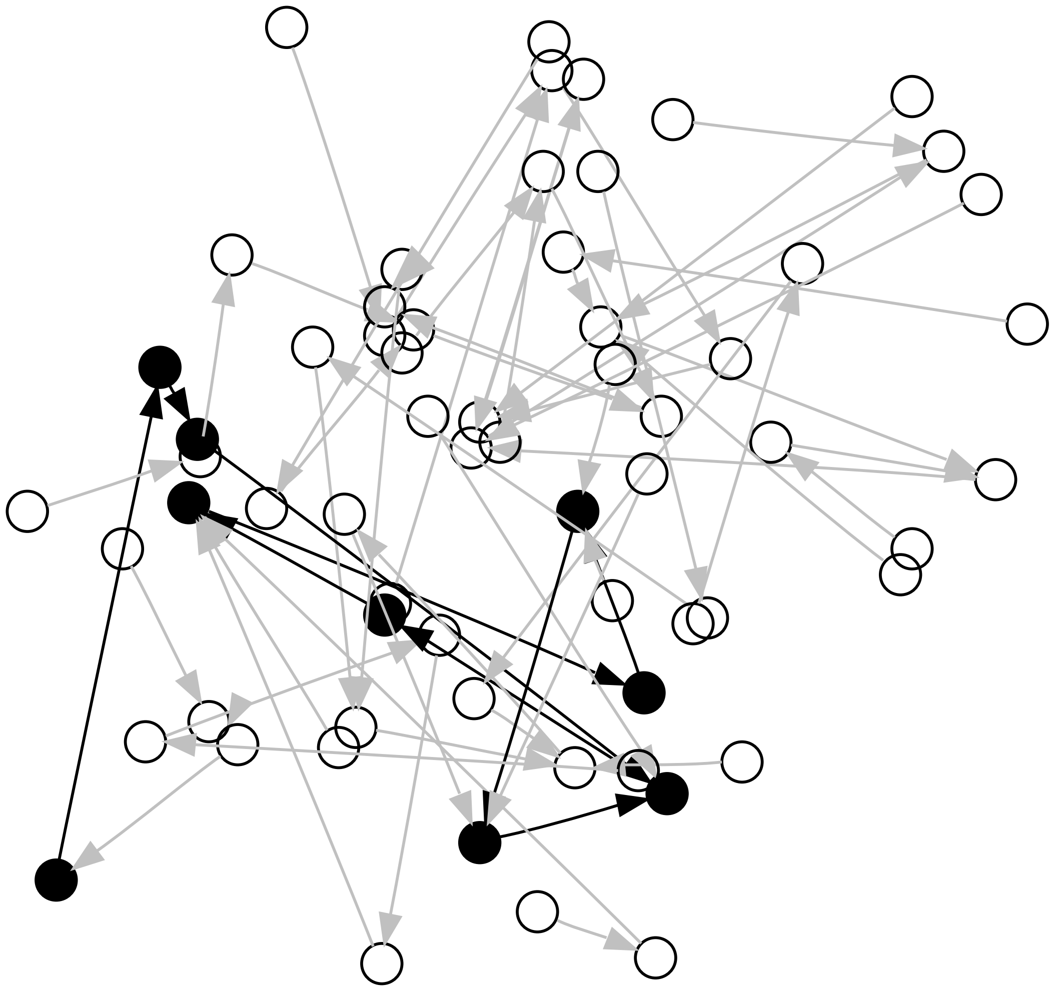


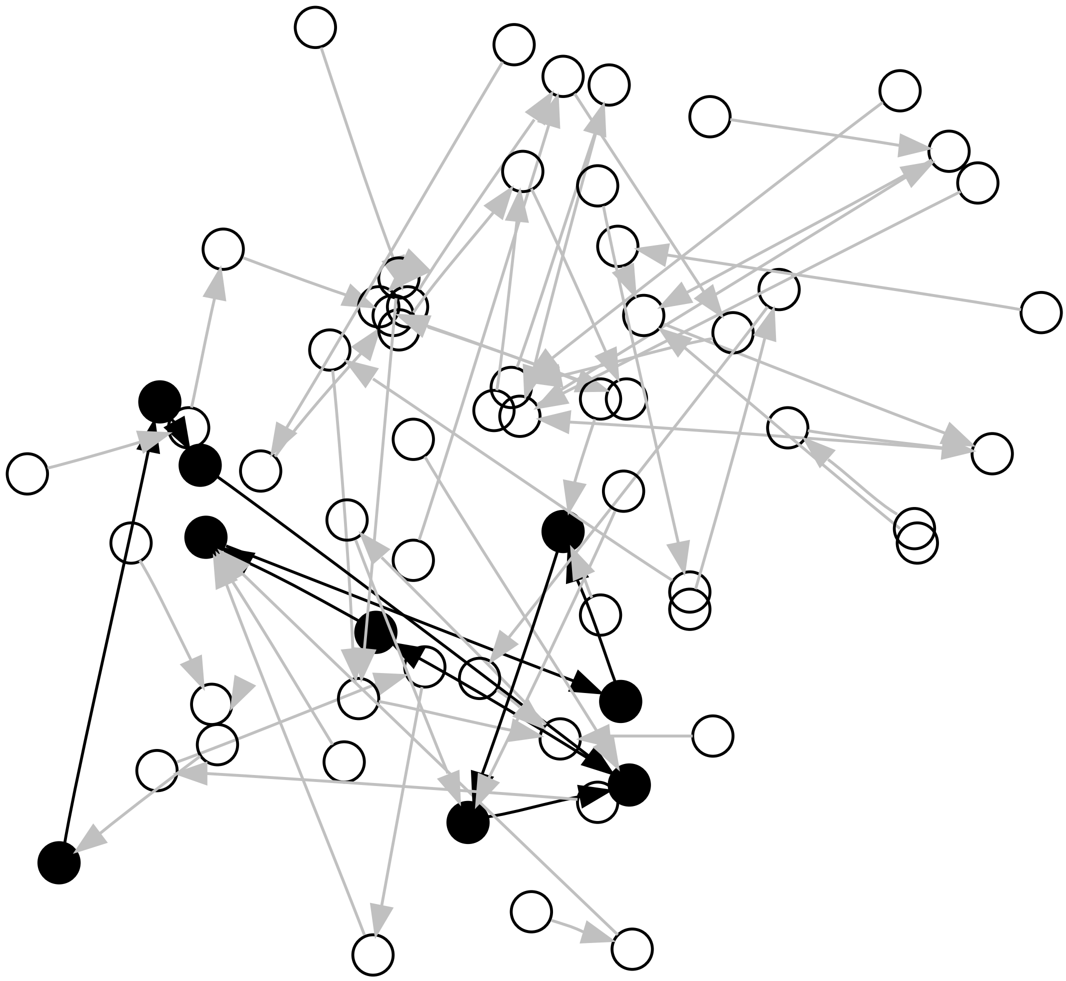


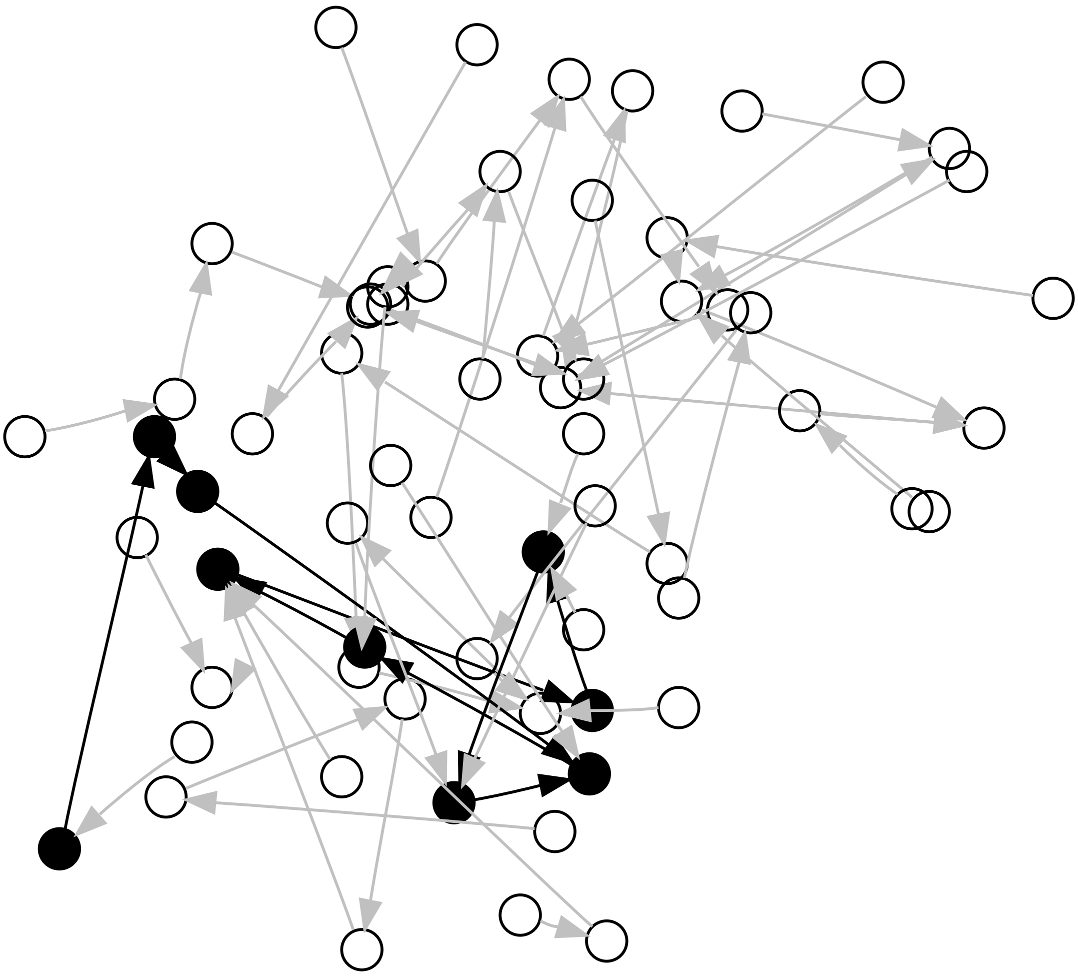


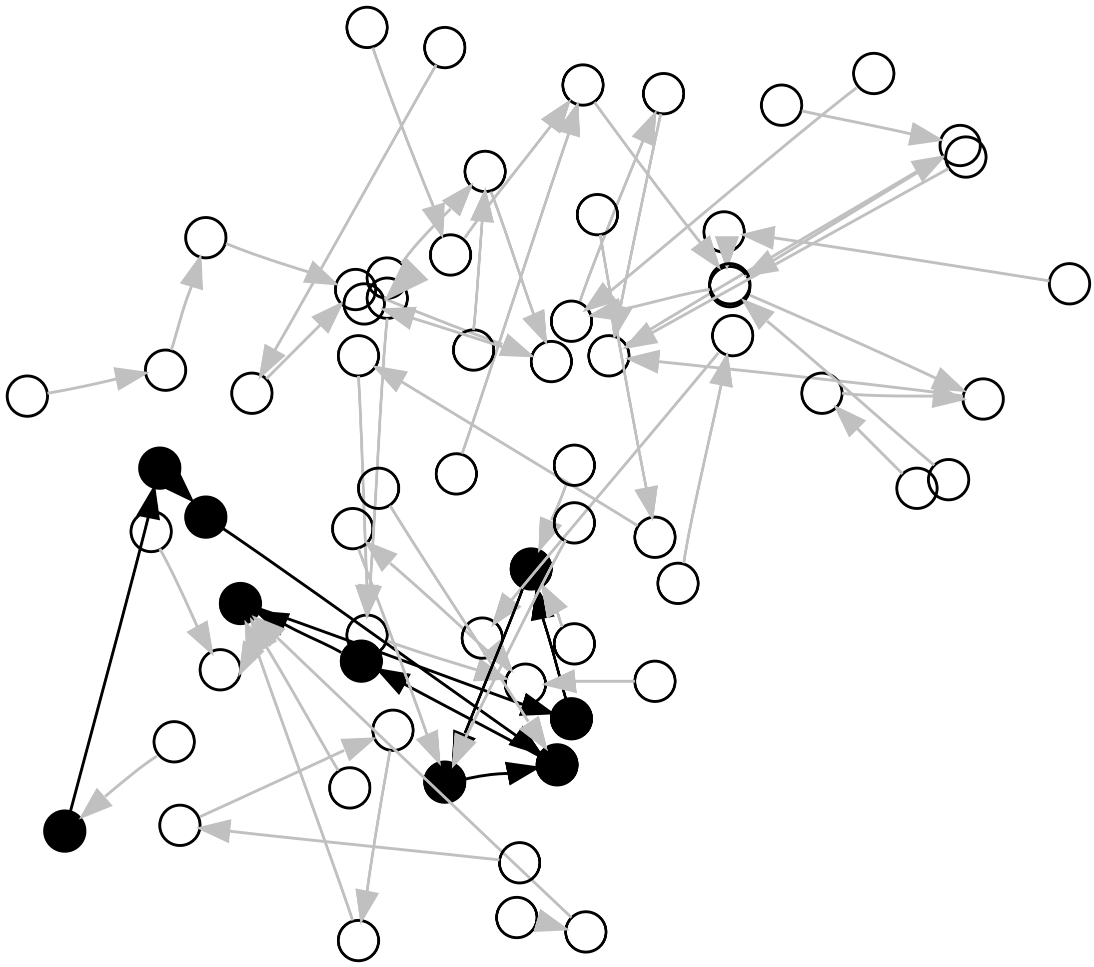


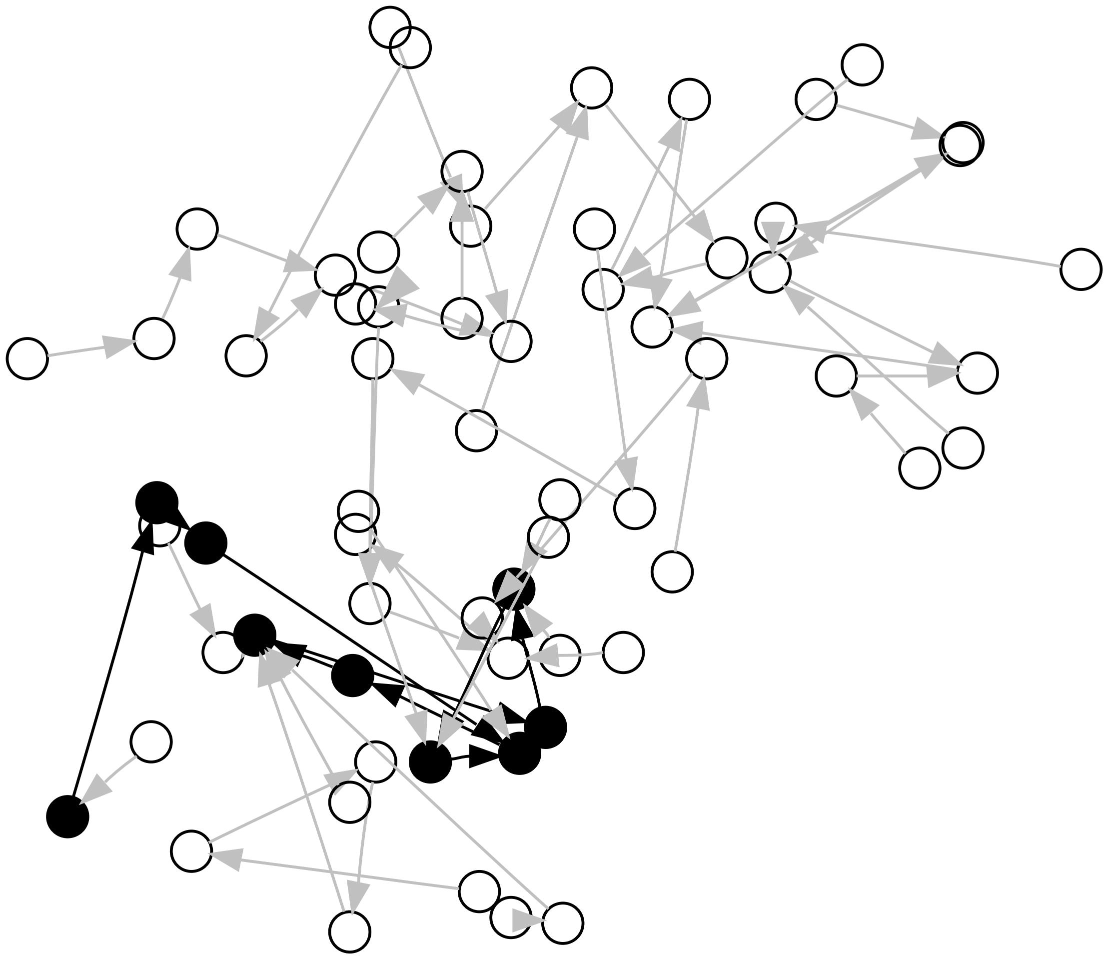


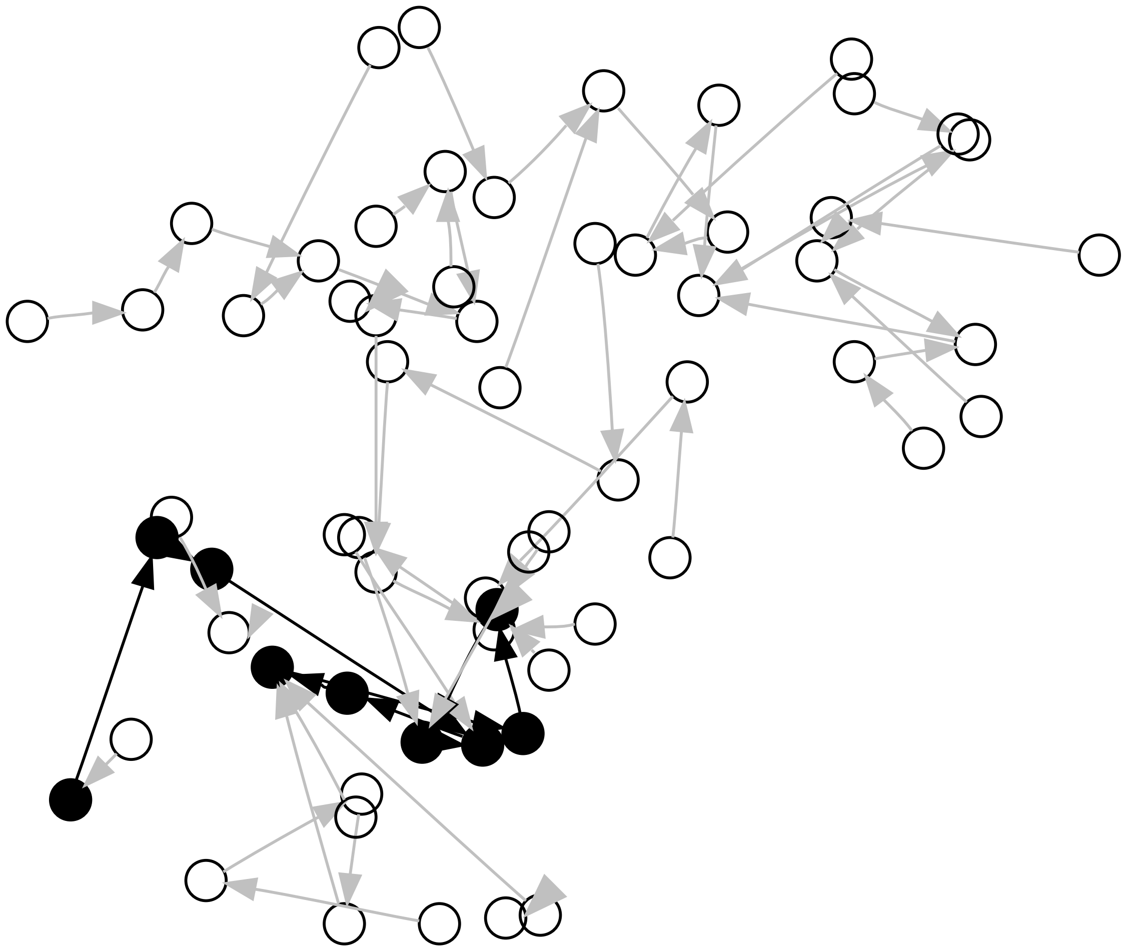


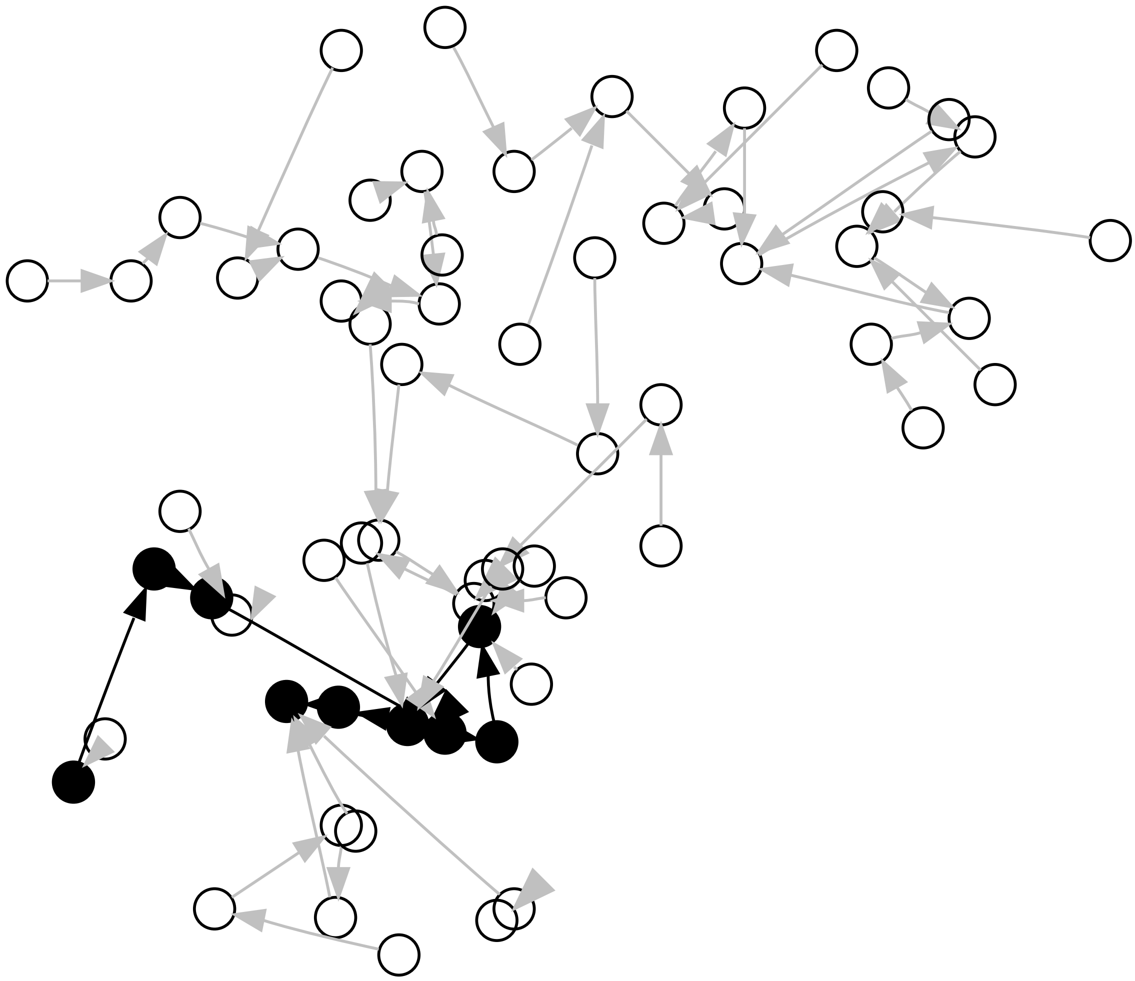


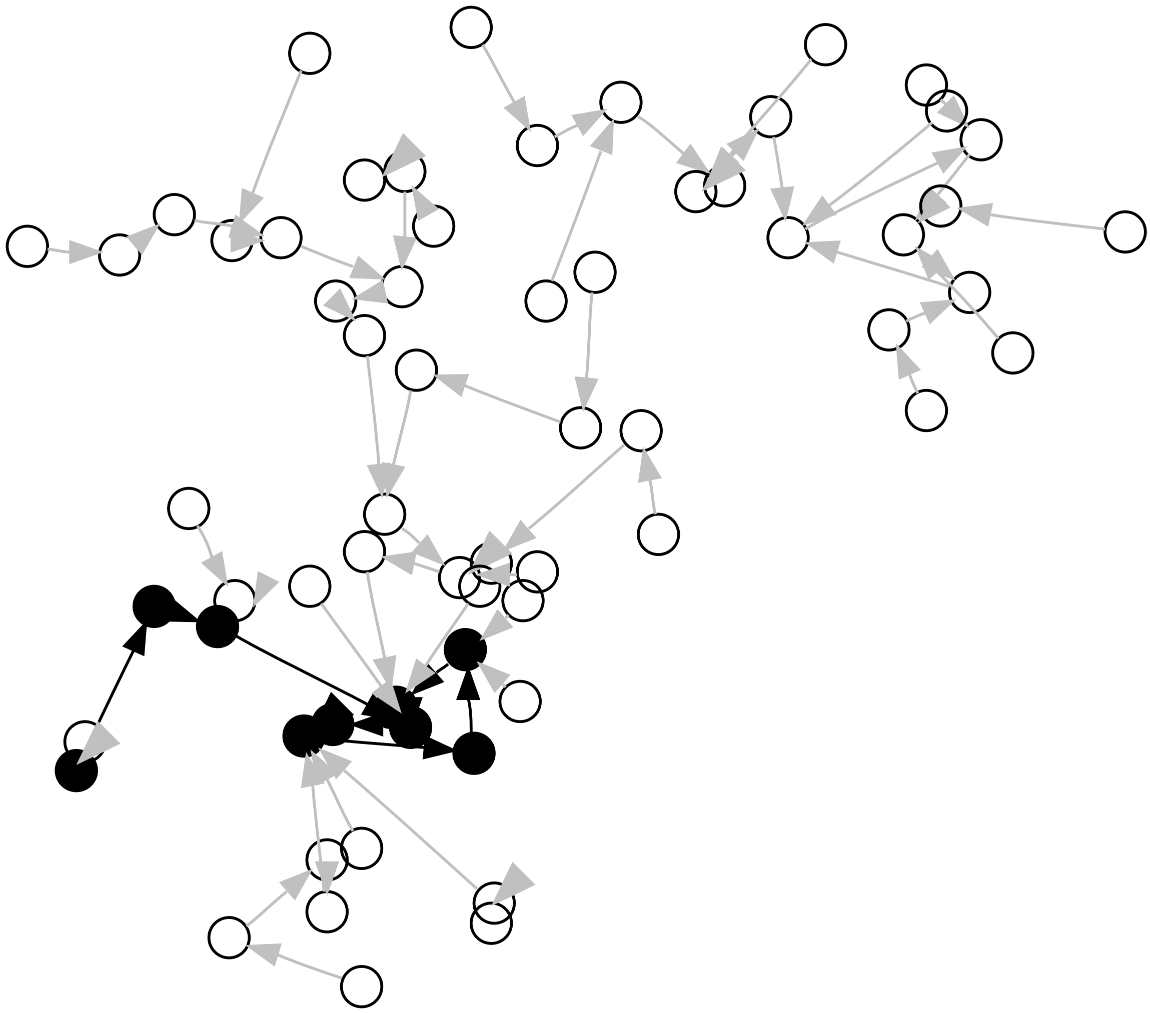


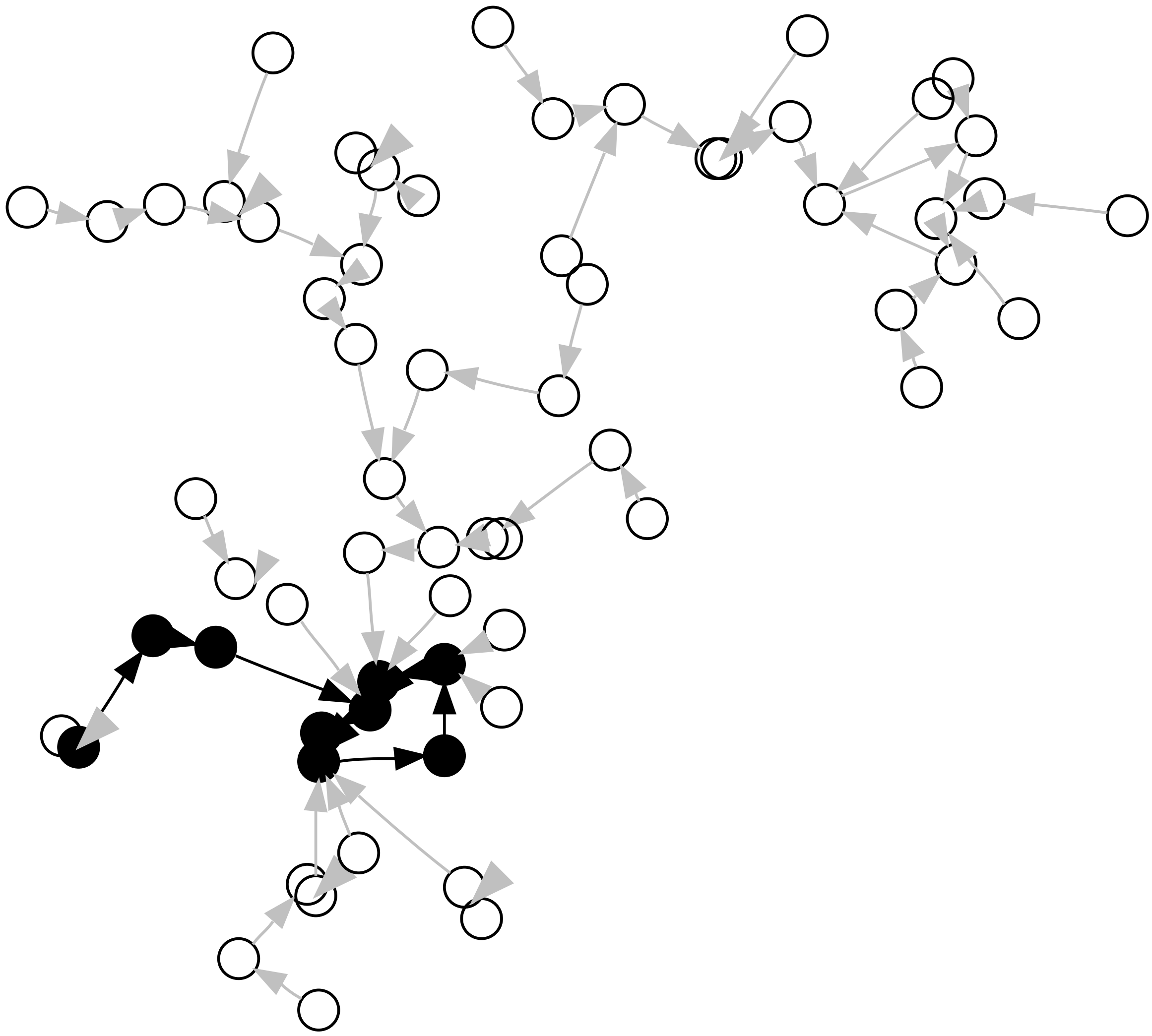


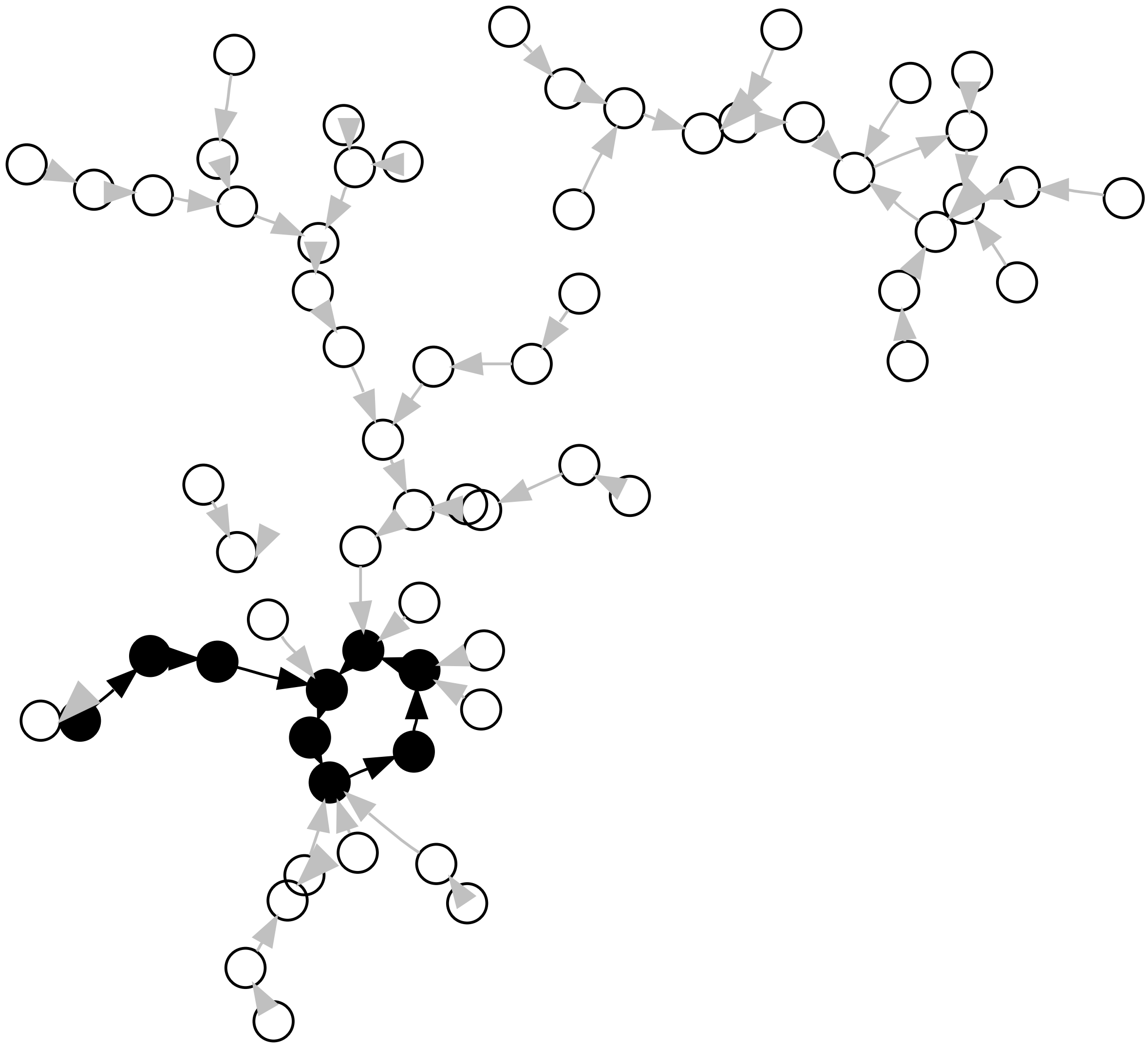


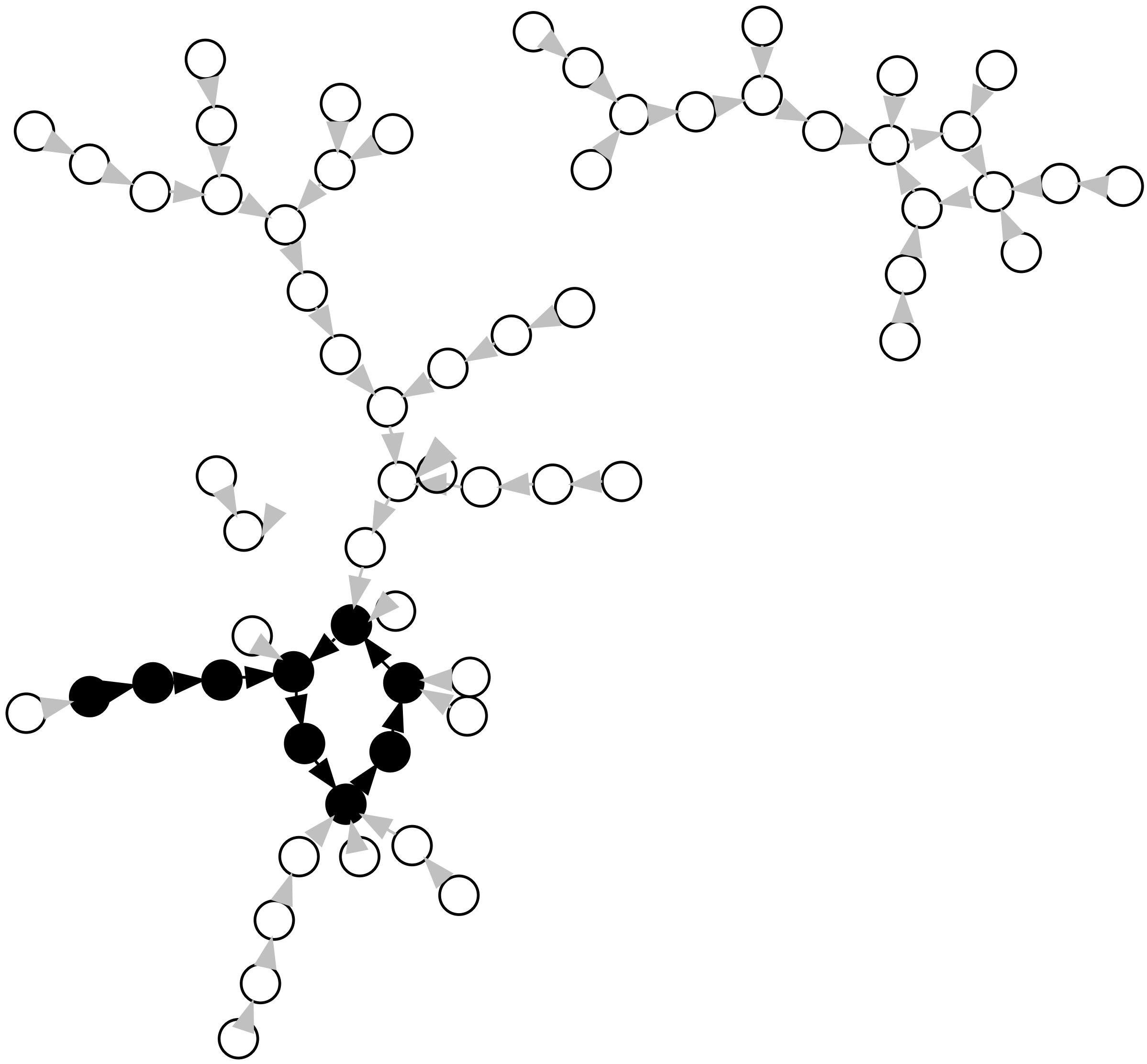












Goal: Compute $\log_P Q$.

Assume that for each i
we know $x_i, y_i \in \mathbf{Z}/\ell\mathbf{Z}$
so that $R_i = y_i P + x_i Q$.

Then $R_i = R_j$ means that
$y_i P + x_i Q = y_j P + x_j Q$
so $(y_i - y_j)P = (x_j - x_i)Q$.
If $x_i \neq x_j$ the DLP is solved:
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e.g. "base-(P, Q) r-adding walk":
precompute $S_1, S_2, \ldots, S_r$
as random combinations $aP + bQ$;
define $f(R) = R + S_{H(R)}$
where H hashes to $\{1, 2, \ldots, r\}$.

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2010 Bernstein–Lange (ANTS 2012): actually more complicated; higher-degree anticollisions.

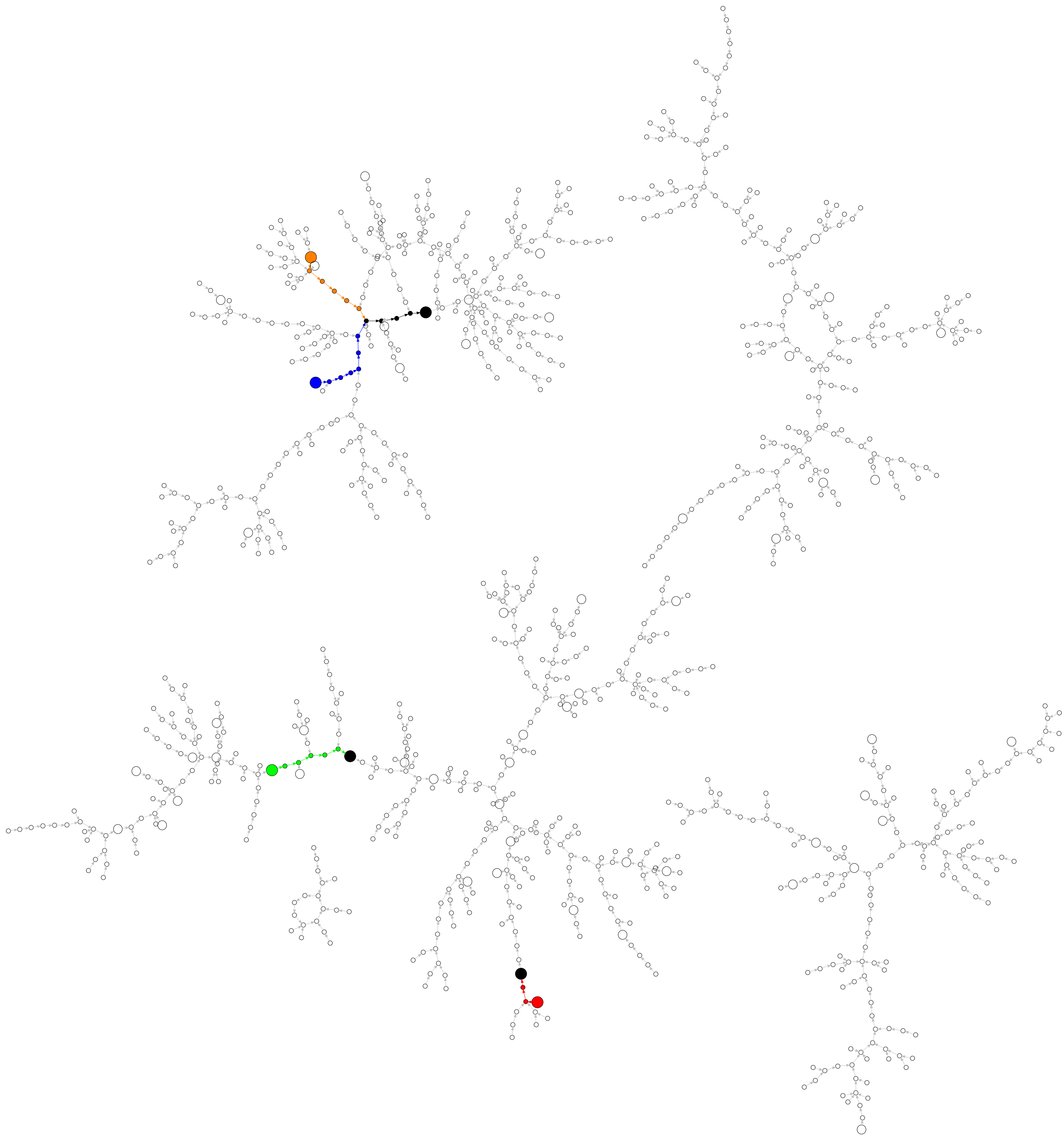
Parallel rho

1994 van Oorschot–Wiener:

Declare some subset of $\langle P \rangle$ to be the set of *distinguished points*: e.g., all $R \in \langle P \rangle$ where last 20 bits of representation of R are 0.

Perform, in parallel, walks for different starting points $Q+yP$ but same update function f.

Terminate each walk once it hits a distinguished point. Report point to central server. Server receives, stores, and sorts all distinguished points.



Two colliding walks will reach
the same distinguished point.
Server sees collision, finds DL.

State of the art

Can break DLP in group of order ℓ in $\sqrt{\pi\ell/2}$ group operations.

For elliptic curves can gain factor of $\sqrt{2}$ by using negation map.

Solving DLP on NIST P-256 takes $\approx 2^{128}$ group operations.

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Let's see what free precomputation does to this . . .

Cube-root ECDL algorithms

Assuming plausible heuristics, overwhelmingly verified by computer experiment:

There exists a P-256 ECDL algorithm that takes "time" $\approx 2^{85}$ and has success probability ≈ 1.

"Time" includes algorithm length.

Inescapable conclusion: **The standard conjectures** (regarding P-256 ECDL hardness, P-256 ECDSA security, etc.) **are false.**

Should P-256 ECDSA users be worried about this P-256 ECDL algorithm A?

No!

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But A *exists*, and the standard conjecture doesn't see the 2^{170}.

Cryptanalysts *do* see the 2^{170}.

Common parlance: We have a 2^{170} "precomputation" (independent of Q) followed by a 2^{85} "main computation".

For cryptanalysts: This costs 2^{170}, much worse than 2^{128}.

For the standard security definitions and conjectures: The main computation costs 2^{85}, much better than 2^{128}.

What the algorithm does

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1999 Escott–Sager–Selkirk–Tsapakidis, also crediting Silverman–Stapleton:

Computing (e.g.) $\log_P Q_1$, $\log_P Q_2$, $\log_P Q_3$, $\log_P Q_4$, and $\log_P Q_5$ costs only $2.49\times$ more than computing $\log_P Q$.

The basic idea:
compute $\log_P Q_1$ with rho;
compute $\log_P Q_2$ with rho, *reusing* distinguished points produced by Q_1; etc.

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cost $\Theta(n^{1/2}\ell^{1/2})$
for n discrete logarithms
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2004 Hitchcock–
Montague–Carter–Dawson:
View computations of
$\log_P Q_1, \ldots, \log_P Q_{n-1}$ as
precomputatation for main
computation of $\log_P Q_n$.
Analyze tradeoffs between
main-computation time and
precomputation time.

2012 Bernstein–Lange:

(1) Adapt to interval of length ℓ inside much larger group.

(2) Analyze tradeoffs between main-computation time and precomputed table size.

(3) Choose table entries more carefully to reduce main-computation time.

(4) Also choose iteration function more carefully.

(5) Reduce space required for each table entry.

(6) Break $\ell^{1/4}$ barrier.

Applications:

(7) Disprove the standard 2^{128} P-256 security conjectures.

(8) Accelerate trapdoor DL etc.

(9) Accelerate BGN etc.; this needs (1).

Bonus:

(10) Disprove the standard 2^{128} AES, DSA-3072, RSA-3072 security conjectures.

Credit to earlier Lee–Cheon–Hong paper for (2), (6), (8).

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(If this fails, rerandomize Q.)

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The following sketch
is not the state of the art —
but good enough to break
the 2^{128} assumption.

Let $g \in \mathbf{F}_p^*$ have order q, $h = g^k$.
Goal: Find k.

Precomputation:
Take $y = 2^{110}$,
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with $h_2 \in \{1, 2, 3, \ldots, 2^{1535}\}$,
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into primes $\le y$.

If this fails, try again
with hg, hg^2, etc.

Analysis

About $y/\log y \approx 2^{103.75}$ primes $\le y$ for a total of $2^{109.33}$ bytes to store all small DLs.

Can write h as h_1/h_2 with probability $\approx (6/\pi^2) 2^{3071}/p$.

h_i is y-smooth with probability very close to $u^{-u} \approx 2^{-53.06}$ where $u = 1535/110$.

Overall the attack requires between $2^{107.85}$ and $2^{108.85}$ iterations; batch smoothness detection is fast.

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(4) Add effectivity. Include cost for finding the algorithm.

(5) Add uniformity. Clearly stops attacks but breaks most theorems.

Abandons goal of defining concrete security of AES etc.