NTRU and BLISS

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- Security related to lattice problems; pre-version cryptanalyzed with LLL by Coppersmith and Shamir.
- System parameters (n, q), n prime, integer q, $\gcd(3, q) = 1$.
- All computations done in ring $R = \mathbf{Z}[x]/(x^n - 1)$.

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- All computations done in ring $R = \mathbf{Z}[x]/(x^n - 1)$.
- Private key: $f, g \in R$ sparse with coefficients in $\{-1, 0, 1\}$. Additional requirement: f must be invertible in R modulo q.
- Public key $h = 3g/f \bmod q$.
- Can see this as lattice with basis matrix

$$B = \begin{pmatrix} q\,I_n & 0 \\ H & I_n \end{pmatrix},$$

where H corresponds to multiplication by $h/3$ modulo $x^n - 1$.

- (g, f) is a short vector in the lattice as result of

$$(k, f)B = (kq + f \cdot h/3, f) = (g, f)$$

for some polynomial k (from $fh/3 = g - kq$).

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  Additional requirement: f must be invertible in R modulo q and modulo 3.
- Public key $h = 3g/f \bmod q$.
- Encryption of message $m \in R$, coefficients in $\{-1, 0, 1\}$:
  Pick random, sparse $r \in R$, same sample space as f; compute:

$$c = r \cdot h + m \bmod q.$$

- Decryption of $c \in R_q$: Compute

$$a = f \cdot c = f(rh + m) \equiv f(3rg/f + m) \equiv 3rg + fm \bmod q,$$

  move all coefficients to $[-q/2, q/2]$. If everything is small enough then a equals $3rg + fm$ in R and $m = a/f \bmod 3$.

Decryption failures

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move all coefficients to $[-q/2, q/2]$. If everything is small enough then a equals $3rg + fm$ in R and $m = a/f$ mod 3.
Let

$$\begin{aligned} L(d,t) =& \{F \in R | F \text{ has } d \text{ coefficients equal to } 1 \\ & \text{and } t \text{ coefficients equal to } -1, \text{all others } 0\}. \end{aligned}$$

Let $f \in L(d_f, d_f - 1)$, $r \in L(d_r, d_r)$, and $g \in L(d_g, d_g)$ with $d_r < d_g$.
Then $3rg + fm$ has coefficients of size at most

$$3 \cdot 2d_r + 2d_f - 1$$

which is larger than $q/2$ for typical parameters. Such large coefficients are highly unlikely – but annoying for applications and guarantees.
Security decreases with large q; reduction is important.

Evaluation-at-1 attack

Ciphertext equals $c = rh + m$ and $r \in L(d_r, d_r)$, so $r(1) = 0$ and $g \in L(d_g, d_g)$, so $h(1) = g(1)/f(1) = 0$.

This implies

$$c(1) = r(1)h(1) + m(1) = m(1)$$

which gives information about m, in particular if $|m(1)|$ is large.

NTRU rejects extreme messages – this is dealt with by randomizing m via a padding (not mentioned so far).

For other choices of r and h, such as $L(d_r, d_r - 1)$ or such, one knows $r(1)$ and h is public, so evaluation at 1 leaks $m(1)$.

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Could also replace $x^n - 1$ by $\Phi_n = (x^n - 1)/(x - 1)$ to avoid attack.

Mathematical attacks

- Meet-in-the-middle attack;
- Lattice-basis reduction (e.g. LLL, BKZ);
- Hybrid attack, combining both.

Crypto attacks:

- Chosen-ciphertext attacks;
- Decryption-failure attacks;
- Complicated padding systems.

Odlyzko's meet-in-the-middle attack on NTRU

- Idea: split the possibilities for f in two parts

$$h = (f_1 + f_2)^{-1} 3g$$
$$f_1 \cdot h = 3g - f_2 \cdot h.$$

- If there was no g: collision search in $f_1 \cdot h$ and $-f_2 \cdot h$

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- If there was no g: collision search in $f_1 \cdot h$ and $-f_2 \cdot h$
- Solution: look for collisions in $c(f_1 \cdot h)$ and $c(-f_2 \cdot h)$ with

$$c(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}) = (\mathbf{1}(a_0 > 0), \ldots, \mathbf{1}(a_{n-1} > 0))$$

  using that g is small and thus $+g$ often does not change the sign.
- If $c(f_1 \cdot h) = c(-f_2 \cdot h)$ check whether $h(f_1 + f_2)$ is in $L(d_g, d_g)$.
- Basically runs in squareroot of size of search space.

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- If $c(f_1 \cdot h) = c(-f_2 \cdot h)$ check whether $h(f_1 + f_2)$ is in $L(d_g, d_g)$.
- Basically runs in squareroot of size of search space.
- General running time / memory mitm (Christine van Vredendaal)

$$L = \sqrt{|S|}/\sqrt{s}.$$

Attackable rotations

In NTRU, $x^i f$ is simply a rotation of f, so it has the same coefficients, just at different positions. This means, $x^i f$ also gives a solution in the mitm attack: $hx^i f = x^i g$ has same sparsity etc., increasing the number of targets.
Decryption using $x^i f$ works the same as with f for NTRU, so each target is valid.

Security against Odlyzko's meet-in-the-middle attack

- Number of choices for f is

$$\binom{n}{t}\binom{n-t}{t-1}$$

because f has $2t-1$ non-zero coefficients.

- Number of rotations is n.
- Running time / memory against NTRU

$$L = \frac{\sqrt{\binom{n}{t}\binom{n-t}{t-1}}}{\sqrt{n}}.$$

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- Memory requirement can be reduced.

Security against lattice sieving

- Recall $h = 3g/f$ in $\mathcal{R}/q$.
- This implies that for $k \in \mathcal{R}$: $f \cdot h/3 + k \cdot q = g$.
- NTRU lattice

$$\begin{pmatrix} k & f \end{pmatrix} \begin{pmatrix} qI_n & 0 \\ H & I_n \end{pmatrix} = \begin{pmatrix} g & f \end{pmatrix}.$$

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- Keypair (g, f) is a short vector in this lattice.
- Asymptotically sieving works in $2^{0.292 \cdot 2p + o(p)}$ using $2^{0.208 \cdot 2p + o(p)}$ memory.
- Crossover point between sieving and BKZ is still unclear.
- Memory is more an issue than time.

Hybrid attack

Howgrave-Graham combines lattice basis reduction and meet-in-the-middle attack.

- Idea: reduce submatrix of the NTRU lattice, then perform mitm on the rest.

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- Idea: reduce submatrix of the NTRU lattice, then perform mitm on the rest.
- Use BKZ on submatrix B to get B':

$$C \cdot \begin{pmatrix} qI_n & 0 \\ H & I_n \end{pmatrix} = \left(\begin{array}{ccc} qI_w & 0 & 0 \\ \hline * & B' & 0 \\ \hline * & * & I_{w'} \end{array}\right).$$

- Guess options for last w' coordinates of f, using collision search (as before).
- If the Hermite factor of B' is small enough, then a rounding algorithm can detect collision of halfguesses.

Security against the hybrid attack

- Balance the costs of the BKZ and mitm phase.

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- Hoffstein, Pipher, Schanck, Silverman, Whyte, and Zhang [HPSWZ15] published simplfied analyzis tool.
- Compute BKZ costs with Chen-Nguyen simulator.
- Estimate the mitm costs by estimating the size of the projected space [HPSWZ15].

How about other interesting candidates?

- Bimodal Lattice Signature Scheme (BLISS) (CRYPTO '13 by Léo Ducas and Alain Durmus and Tancrède Lepoint and Vadim Lyubashevsky)
- Pretty short and efficient; already included in strongSwan (library for IPsec-based VPN).
- Needs noise from discrete Gaussian distribution.
- Security is related to lattice-based problem; direct reduction to SIS_q = Short Integer Solution mod q.

Background

- Work in $R = \mathbf{Z}[x]/(x^n + 1)$, $n = 2^r$, and $R_q = (\mathbf{Z}/q)[x]/(x^n + 1)$ for q prime.
- Switch representation between polynomial and vector notation.

$$f(x) = \sum_{i=0}^{n-1} f_i x^i \Leftrightarrow f = (f_{n-1}, f_{n-2}, \ldots, f_1, f_0).$$

- Polynomial multiplication then corresponds to vector-matrix multiplication. Let $f, g, \in R_q$, then

$$f \cdot g = fG = gF,$$

where $F, G \in (\mathbf{Z}/q)^{n \times n}$ match vectors of $x^i f$ and $x^j g$.

$$\begin{pmatrix} f_0 & -f_{n-1} & -f_{n-2} & \ldots & -f_1 \\ f_1 & f_0 & -f_{n-1} & \ldots & -f_2 \\ \vdots & \vdots & \ddots & \vdots & \\ f_{n-1} & f_{n-2} & f_{n-3} & \ldots & f_0 \end{pmatrix}$$

Simplified BLISS

- Secret key $S = (s_1, s_2) = (f, 2g+1) \in R_q^2$, f, g sparse in $\{0, \pm 1\}^n$.
- Public key $A = (a_1, a_2) \in R_{2q}^2$, with key equation $a_1 s_1 + a_2 s_2 \equiv q \bmod 2q$.
- Computed as $a_q = (2g+1)/f \bmod q$ (restart if f is not invertible); then $A = (2a_q, q-2) \bmod 2q$.

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- Computed as $a_q = (2g+1)/f \bmod q$ (restart if f is not invertible);
  then $A = (2a_q, q-2) \bmod 2q$.
- $2a_q s_1 + (q-2)s_2 \equiv 2(2g+1)/f \cdot f + (q-2)(2g+1) \equiv q \bmod 2q$.
- Attacker can verify key guess for f with key equation; g computable; $-S$ just as good as S.

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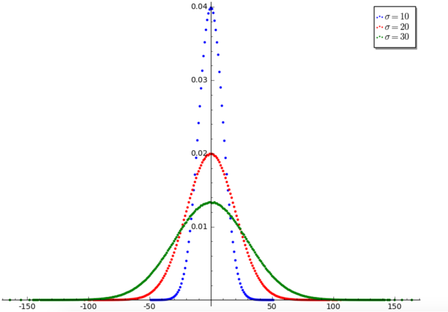
- To sign m, sample y from discrete n-dim Gaussian D_σ^n.
- $c = H(Ay \bmod 2q, m)$ // H special sparse hash function.
- Signature: (z, c) with $z = y + (-1)^b s_1 \cdot c \bmod 2q$. // b random
  Algorithm uses rejection sampling so that it does not leak s_1.
- Accept signature if z is short and $c = H(Az + qc \bmod 2q, m)$.
  Works: $Az + qc \equiv A(y + (-1)^b Sc) + qc \equiv A(y + (-1)^b s_1 c) + qc \equiv Ay + ((-1)^b AS + q) \equiv Ay \bmod 2q$

Discrete Gaussian



- Step 1 in signature algorithm: $y \leftarrow D_\sigma^m$
- This is required to achieve (provable) security and small signature size.
- Not straightforward to do in practice: high precision required.
- Side-channel attack on sampling gives (part of) y.
- Can get $\pm s_1 = (z - y)/c \in R_q$ if we know y, the error vector/polynomial; (c needs to be invertible).
- Full details in `https://eprint.iacr.org/2016/300` (with Groot Bruinderink, Hülsing, and Yarom).

LLL conditions

Lenstra Lenstra Lovasz (1982)

- On input a basis $\{v_1, v_2, \ldots, v_n\}$ output a short vector v_1'.

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- On input a basis $\{v_1, v_2, \ldots, v_n\}$ output a short vector v'_1.
- Actually, LLL outputs a new, shorter and more orthogonal basis.
- LLL uses many elements from Gram-Schmidt orthogonalization:
  - for $j = 1$ to n
  - for $i = 1$ to $j - 1$
  - $\mu_{ij} = \frac{\langle v_i^*, v_j \rangle}{\langle v_i^*, v_i^* \rangle}$
  - $v_j^* = v_j - \sum_{i=1}^{j-1} \mu_{ij} v_i^*$
- Note that the μ_{ij} are not integers, so the v_j^* are not in the lattice.
- A lattice basis is LLL reduced for parameter $0.25 < \delta < 1$ if
  - $|\mu_{ij}| \leq 0.5$ for all $1 \leq j < i \leq n$,
  - $(\delta - \mu_{i-1i}^2)||v_{i-1}^*||^2 \leq ||v_i^*||^2$.
- This guarantees $||v_1|| \leq 2^{(n-1)/4} \det(L)$, where $\det(L)$ is the determinant of the lattice.

LLL algorithm (from Cohen, GTM 138, transposed)

Input: Basis $\{v_1, v_2, \ldots, v_n\}$ of lattice L, $0.25 < \delta < 1$
Output: LLL reduced basis for L with parameter δ

1. $k \leftarrow 2,\ k_{\max} \leftarrow 1,\ v_1^* \leftarrow v_1,\ V_1 = \langle v_1, v_1 \rangle$
2. If $k \leq k_{\max}$ go to step 3; else $k_{\max} \leftarrow k,\ v_k^* \leftarrow v_k$. For $j = 1$ to $k-1$
   - put $\mu_{jk} \leftarrow \langle v_j^*, v_k \rangle / V_j$ and $v_k^* \leftarrow v_k^* - \mu_{jk} v_j^*$

   $V_k = \langle v_k, v_k \rangle$
3. Execute RED$(k, k-1)$. If $(\delta - \mu_{i-1i}^2)V_{k-1} > V_k$ execute SWAP(k) and $k \leftarrow \max\{2, k-1\}$; else for $= k-2$ down to 1 execute RED(k, j) and $k \leftarrow k+1$.
4. If $k \leq n$ to to step 2; else output basis $\{v_1, v_2, \ldots, v_n\}$.

- RED(k, j): If $|\mu_{jk}| \leq 0.5$ return; else $q \leftarrow \lfloor \mu_{jk} \rceil$, $v_k \leftarrow v_k - q v_j$, $\mu_{jk} \leftarrow \mu_{jk} - q$, for $i = 1$ to $j-1$ put $\mu_{ik} \leftarrow \mu_{ik} - q\mu_{ij}$ and return.
- SWAP(k): Swap v_k and v_{k-1}. If $k > 2$ for $j = 1$ to $k-2$ swap μ_{jk} and μ_{jk-1} and update all variables to match (see p.88 in Cohen)