RSA VII

Q-sieve

Tanja Lange

Eindhoven University of Technology

2MMC10 – Cryptology

with some slides by

Daniel J. Bernstein

Q sieve

Sieving small integers $i > 0$
using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
5							5	
6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
11								
12	2	2			3			
13								
14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

etc.

Q sieve

Sieving i and $611 + i$ for small i using primes $2, 3, 5, 7$:

1								
2	2							
3					3			
4	2	2						
5							5	
6	2				3			
7								7
8	2	2	2					
9					3	3		
10	2						5	
11								
12	2	2			3			
13								
14	2							7
15					3		5	
16	2	2	2	2				
17								
18	2				3	3		
19								
20	2	2					5	

612	2	2			3	3						
613												
614	2											
615					3			5				
616	2	2	2									7
617												
618	2				3							
619												
620	2	2						5				
621					3	3	3					
622	2											
623												7
624	2	2	2	2	3							
625								5	5	5	5	
626	2											
627					3							
628	2	2										
629												
630	2				3	3		5				7
631												

etc.

Have complete factorization of the "congruences" $i(611 + i)$ for some i's.

$14 \cdot 625 = 2^1 3^0 5^4 7^1$.
$64 \cdot 675 = 2^6 3^3 5^2 7^0$.
$75 \cdot 686 = 2^1 3^1 5^2 7^3$.

$14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$
$= 2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$.

$\gcd(611, 14 \cdot 64 \cdot 75 - 2^4 3^2 5^4 7^2)$
$= 47$.

$611 = 47 \cdot 13$.

Why did this find a factor of 611?
Was it just blind luck:
$\gcd(611, \text{random}) = 47$?

No.

By construction 611 divides $s^2 - t^2$
where $s = 14 \cdot 64 \cdot 75$
and $t = 2^4 3^2 5^4 7^2$.
So each prime > 7 dividing 611
divides either $s - t$ or $s + t$.

Not terribly surprising
(but not guaranteed in advance!)
that one prime divided $s - t$
and the other divided $s + t$.

Why did the first three completely factored congruences have square product?
Was it just blind luck?

Yes. The exponent vectors $(1, 0, 4, 1), (6, 3, 2, 0), (1, 1, 2, 3)$ happened to have sum 0 mod 2.

But we didn't need this luck!
Typically use linear algebra, see Dixon's method.

Collect at least as many relations as length of each vector.

E.g. for $n = 671$:

$$\begin{aligned} 1(n+\ 1) &= 2^5 3^1 5^0 7^1; \\ 4(n+\ 4) &= 2^2 3^3 5^2 7^0; \\ 15(n+15) &= 2^1 3^1 5^1 7^3; \\ 49(n+49) &= 2^4 3^2 5^1 7^2; \\ 64(n+64) &= 2^6 3^1 5^1 7^2. \end{aligned}$$

$\mathbf{F}_2$-kernel of exponent matrix is generated by (0 1 0 1 1) and (1 0 1 1 0); e.g., $1(n+1)15(n+15)49(n+49)$ is a square.

Plausible conjecture:
$\mathbf{Q}$ sieve can separate the odd prime divisors of any n.

Given n and parameter y:

Try to completely factor $i(n+i)$ for $i \in \{1, 2, 3, \ldots, y^2\}$ into products of primes $\leq y$.

Look for nonempty set I of i's with $i(n+i)$ completely factored and with $\prod_{i \in I} i(n+i)$ square.

Compute $\gcd(n, s-t)$ where $s = \prod_{i \in I} i$ and $t = \sqrt{\prod_{i \in I} i(n+i)}$.

Compute t as product of prime powers, no square root needed.

How large does y have to be for this to find a square?

Uniform random integer in $[1, n]$ has $n^{1/u}$-smoothness chance roughly u^{-u}.

Plausible conjecture:
$\mathbf{Q}$ sieve succeeds
with $y = \lfloor n^{1/u} \rfloor$
for all $n \geq u^{(1+o(1))u^2}$;
here $o(1)$ is as $u \to \infty$.

More generally, if $y \in \exp\sqrt{\left(\frac{1}{2c}+o(1)\right)\log n \log\log n}$, conjectured y-smoothness chance is $1/y^{c+o(1)}$.

Find enough smooth congruences by changing the range of i's: replace y^2 with $y^{c+1+o(1)} = \exp\sqrt{\left(\frac{(c+1)^2+o(1)}{2c}\right)\log n \log\log n}$.

Increasing c past 1 increases number of i's but reduces linear-algebra cost. So linear algebra never dominates when y is chosen properly.