RSA VII Q-sieve

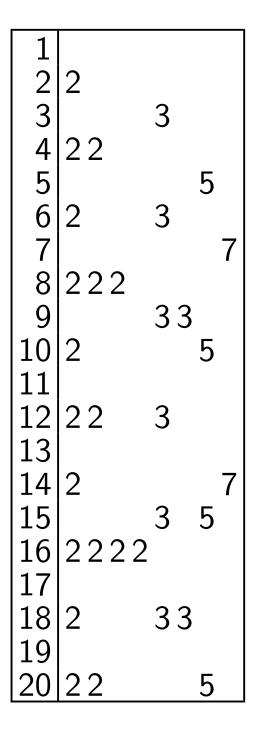
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with some slides by Daniel J. Bernstein

Q sieve

Sieving small integers i > 0using primes 2, 3, 5, 7:



etc.

Q sieve

Sieving *i* and 611 + i for small *i* using primes 2, 3, 5, 7:

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etc.

Have complete factorization of the "congruences" i(611 + i) for some i's.

- $14 \cdot 625 = 2^{1}3^{0}5^{4}7^{1}.$ $64 \cdot 675 = 2^{6}3^{3}5^{2}7^{0}.$ $75 \cdot 686 = 2^{1}3^{1}5^{2}7^{3}.$
- $14 \cdot 64 \cdot 75 \cdot 625 \cdot 675 \cdot 686$ = $2^8 3^4 5^8 7^4 = (2^4 3^2 5^4 7^2)^2$. gcd(611, 14 \cdot 64 \cdot 75 - $2^4 3^2 5^4 7^2$) = 47.

 $611 = 47 \cdot 13.$

Why did this find a factor of 611? Was it just blind luck: gcd(611, random) = 47? No.

By construction 611 divides $s^2 - t^2$ where $s = 14 \cdot 64 \cdot 75$ and $t = 2^4 3^2 5^4 7^2$. So each prime > 7 dividing 611 divides either s - t or s + t.

Not terribly surprising (but not guaranteed in advance!) that one prime divided s - tand the other divided s + t. Why did the first three completely factored congruences have square product? Was it just blind luck?

Yes. The exponent vectors (1, 0, 4, 1), (6, 3, 2, 0), (1, 1, 2, 3) happened to have sum 0 mod 2.

But we didn't need this luck! Typically use linear algebra, see Dixon's method.

Collect at least as many relations as length of each vector.

E.g. for n = 671: $1(n + 1) = 2^5 3^1 5^0 7^1$; $4(n + 4) = 2^2 3^3 5^2 7^0$; $15(n + 15) = 2^1 3^1 5^1 7^3$; $49(n + 49) = 2^4 3^2 5^1 7^2$; $64(n + 64) = 2^6 3^1 5^1 7^2$.

F₂-kernel of exponent matrix is generated by $(0\ 1\ 0\ 1\ 1)$ and $(1\ 0\ 1\ 1\ 0)$; e.g., 1(n+1)15(n+15)49(n+49)is a square.

Plausible conjecture:

Q sieve can separate the odd prime divisors of any *n*.

Given *n* and parameter *y*:

Try to completely factor i(n + i)for $i \in \{1, 2, 3, ..., y^2\}$ into products of primes $\leq y$.

Look for nonempty set I of i's with i(n + i) completely factored and with $\prod_{i \in I} i(n + i)$ square.

Compute gcd(n, s - t) where $s = \prod_{i \in I} i$ and $t = \sqrt{\prod_{i \in I} i(n + i)}$. Compute t as product of prime powers, no square root needed. How large does y have to be for this to find a square?

Uniform random integer in [1, n] has $n^{1/u}$ -smoothness chance roughly u^{-u} .

Plausible conjecture: **Q** sieve succeeds with $y = \lfloor n^{1/u} \rfloor$ for all $n \ge u^{(1+o(1))u^2}$; here o(1) is as $u \to \infty$. More generally, if $y \in$

 $\exp \sqrt{\left(\frac{1}{2c} + o(1)\right)}\log n \log \log n$, conjectured *y*-smoothness chance is $1/y^{c+o(1)}$.

Find enough smooth congruences by changing the range of *i*'s: replace y^2 with $y^{c+1+o(1)} =$ $\exp \sqrt{\left(\frac{(c+1)^2+o(1)}{2c}\right) \log n \log \log n}$.

Increasing *c* past 1 increases number of *i*'s but reduces linear-algebra cost. So linear algebra never dominates when *y* is chosen properly.