Elliptic-curve cryptography I

Diffie-Hellman and clocks

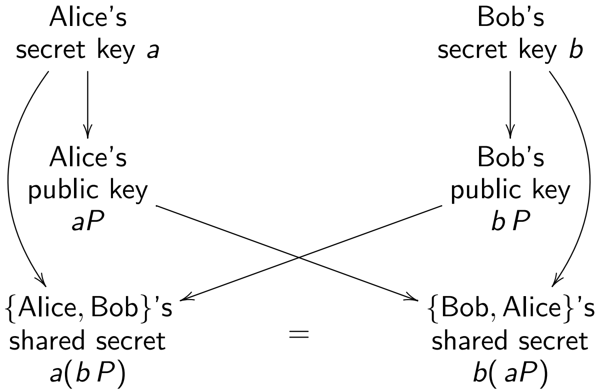
Tanja Lange

Eindhoven University of Technology

2MMC10 – Cryptology

Diffie-Hellman key exchange

Pick some *generator* P, i.e., some group element
(using additive notation here).



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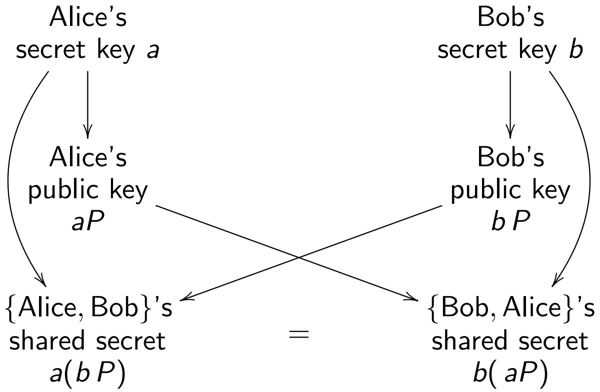
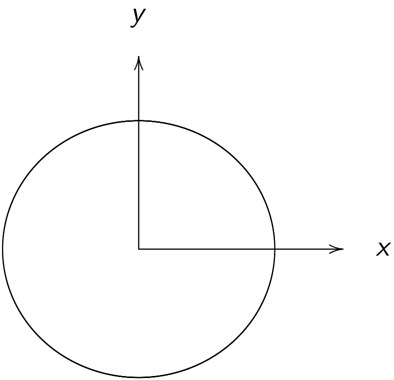


What does P look like? How to compute $P + Q$?

The clock



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This is *not* an elliptic curve.
"Elliptic curve" $\neq$ "ellipse."

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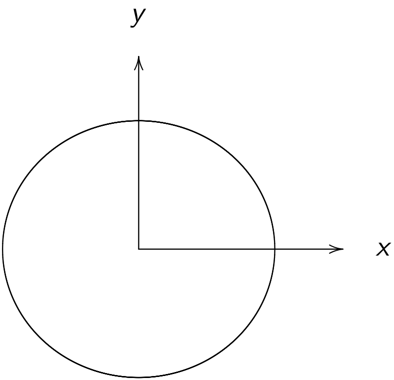

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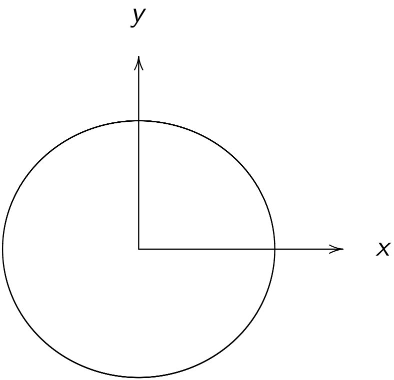

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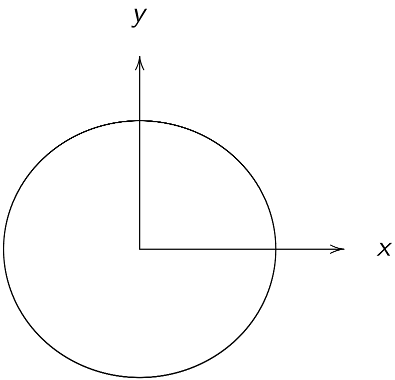

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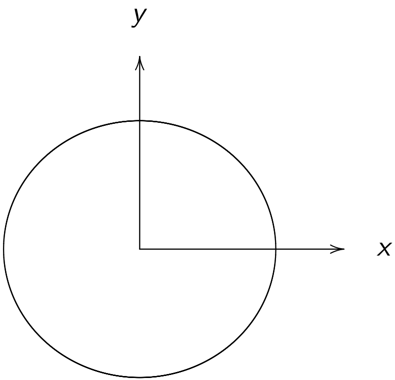

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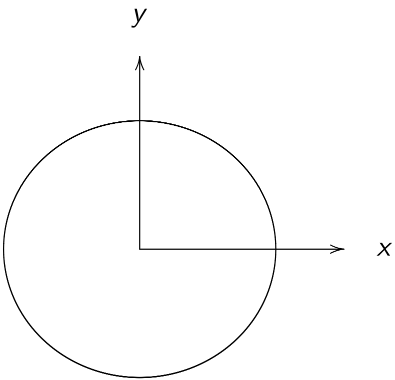

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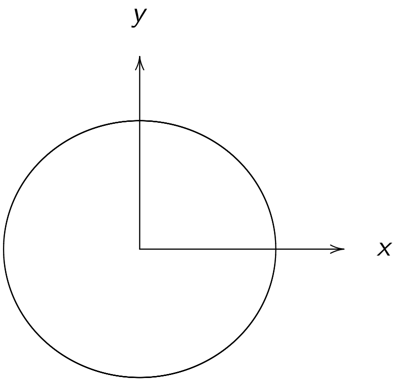

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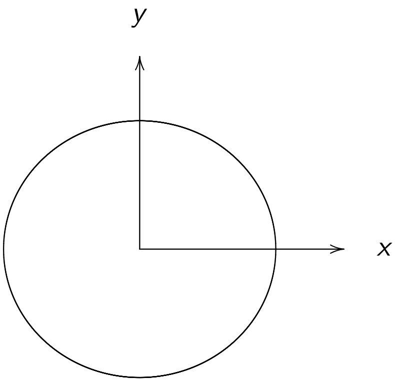

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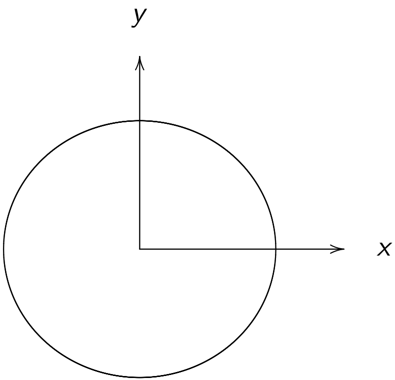

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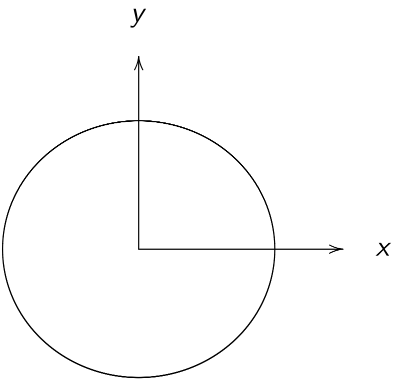

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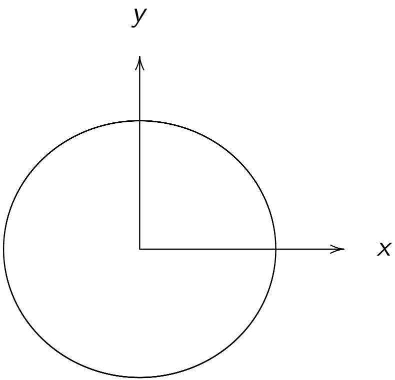

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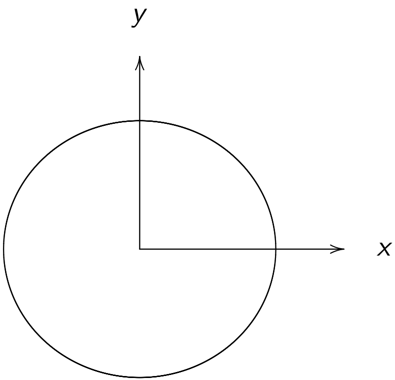

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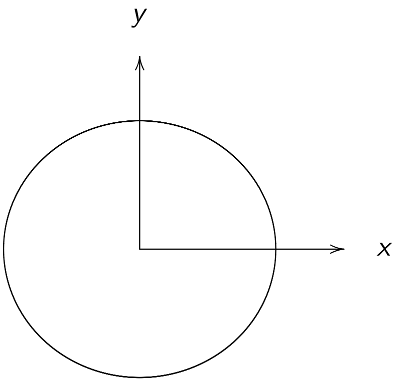

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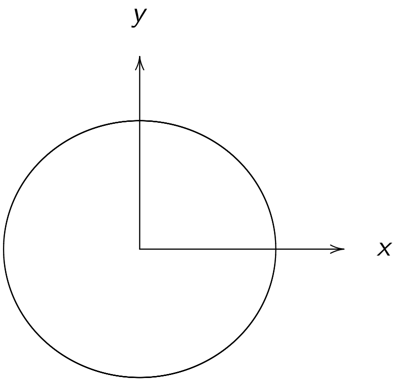

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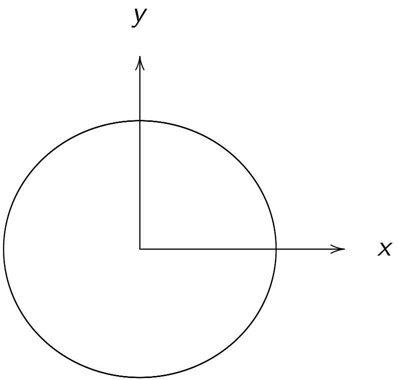

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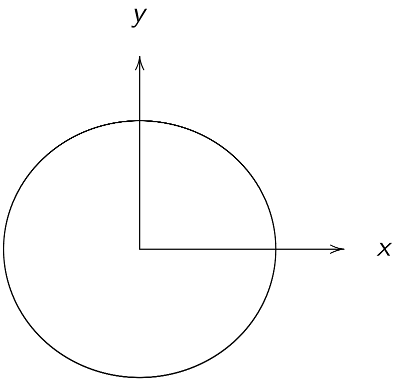

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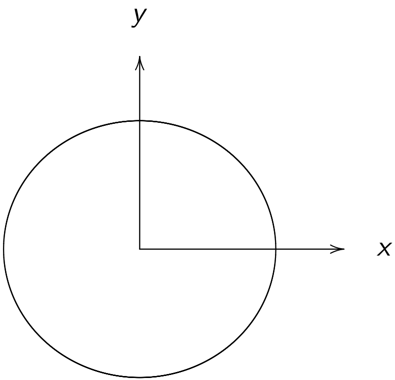

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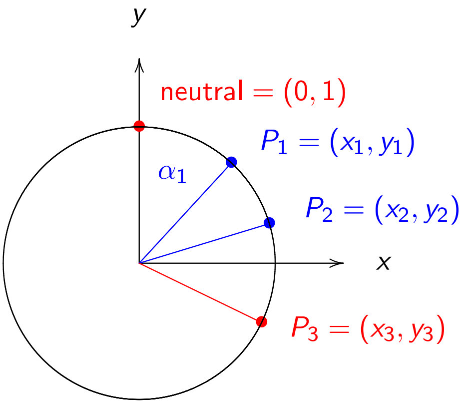
Many more.

Addition on the clock



Neutral element: $(0, 1)$
at angle $\alpha = 0^\circ$.

Adding two points $\cong$
adding angles α_1 and α_2
(taken modulo 360°).

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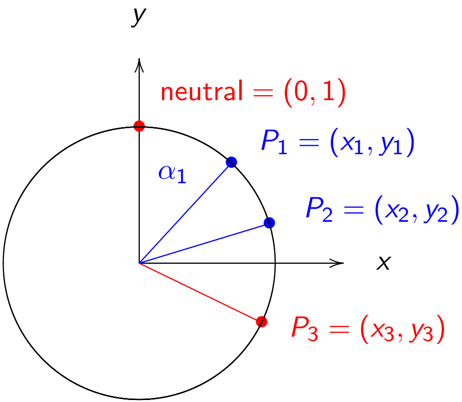

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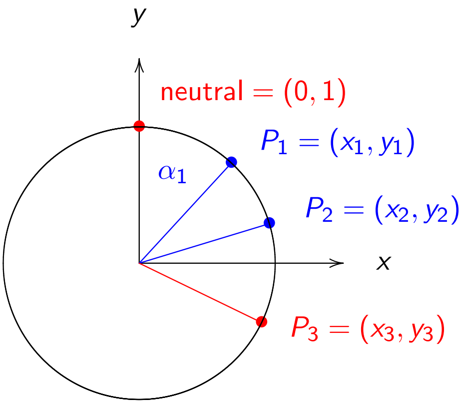

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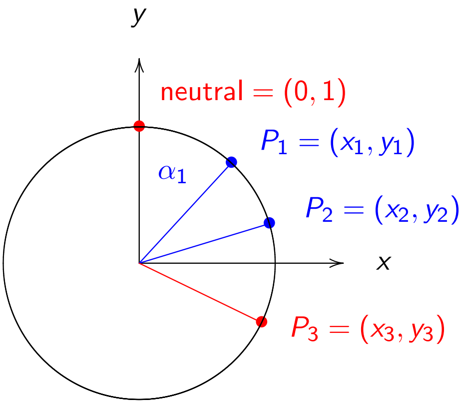

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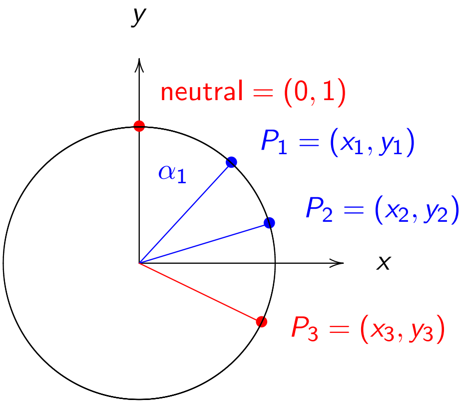

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Points form group under addition of angles.

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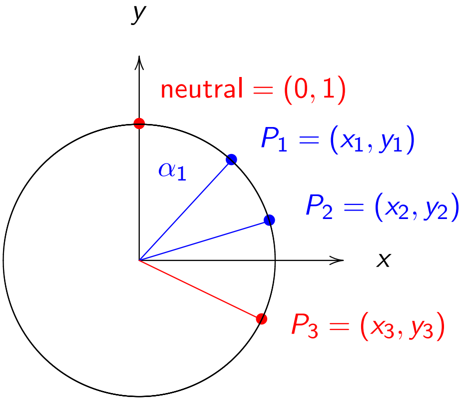

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How about $(3/5, 4/5) + (3/5, 4/5)$?

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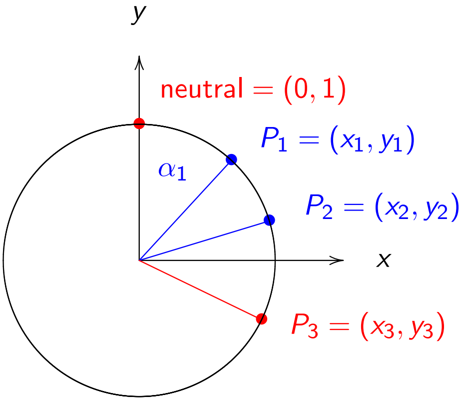

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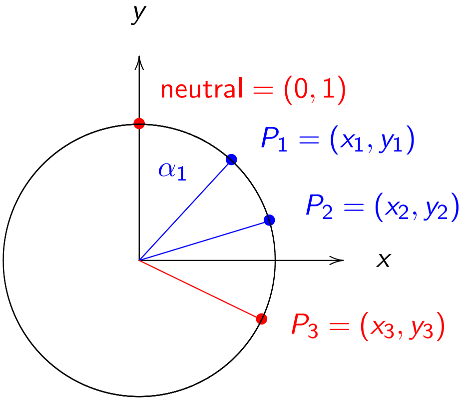

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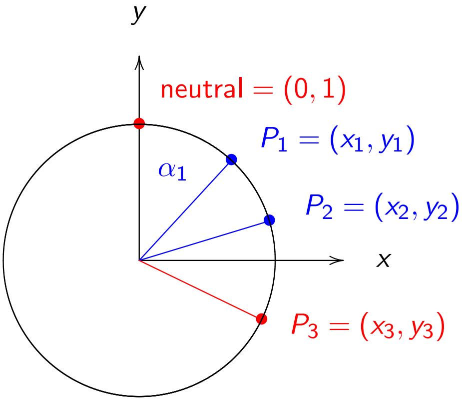

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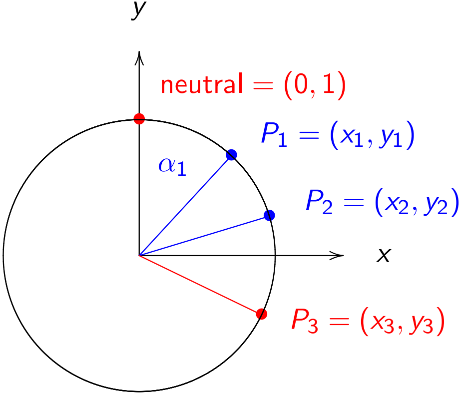

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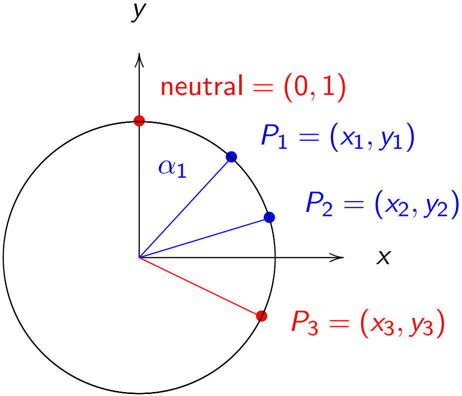

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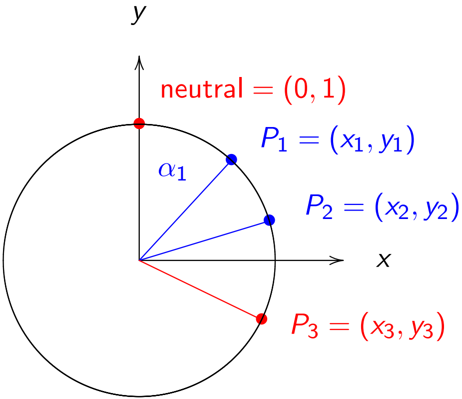

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$$\left(\frac{3}{5}, \frac{4}{5}\right) + \left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{3}{5} \cdot \frac{4}{5} + \frac{4}{5} \cdot \frac{3}{5}, \frac{4}{5} \cdot \frac{4}{5} - \frac{3}{5} \cdot \frac{3}{5}\right) = \left(\frac{24}{25}, \frac{7}{25}\right)$$

We write $kP = \underbrace{P + P + \cdots + P}_{k \text{ copies}}$ for $k \geq 0$.