Discrete logarithm problem III

Random walks and cycle finding

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Eindhoven University of Technology

2MMC10 – Cryptology

Random walks

Target $Q = aP$ in $\langle P \rangle$. Group has n elements.

Make a pseudo-random walk in the group $\langle P \rangle$,
where the next step depends on current point: $W_{i+1} = f(W_i)$.

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Assume that for each point we know $0 \leq a_i, b_i < n$ so that $W_i = a_i P + b_i Q$.

Then $W_i = W_j$ means that

$$a_i P + b_i Q = a_j P + b_j Q$$

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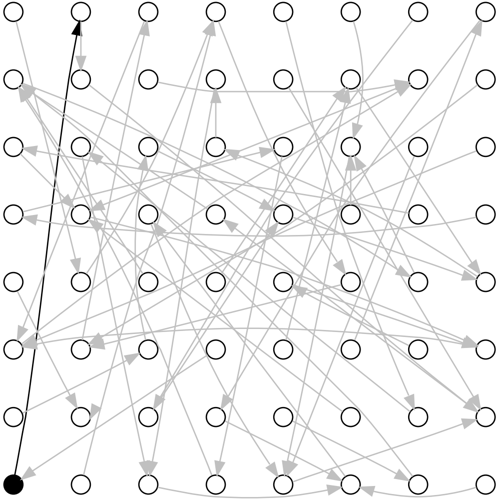
Then $W_i = W_j$ means that

$$a_i P + b_i Q = a_j P + b_j Q \Leftrightarrow (b_i - b_j)Q = (a_j - a_i)P.$$

If $b_i - b_j$ invertible modulo n, the DLP is solved:

$$a \equiv (a_j - a_i)/(b_i - b_j) \bmod n$$

Random walks have collisions



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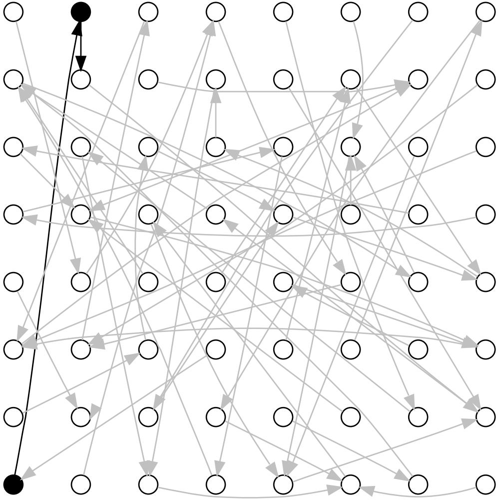

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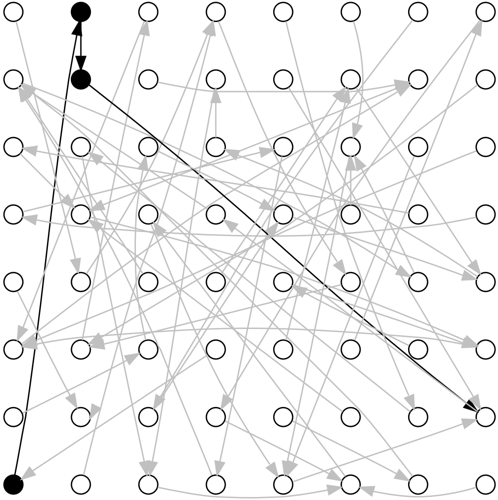

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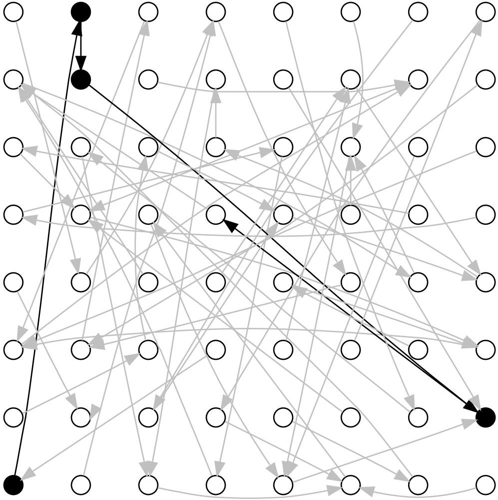

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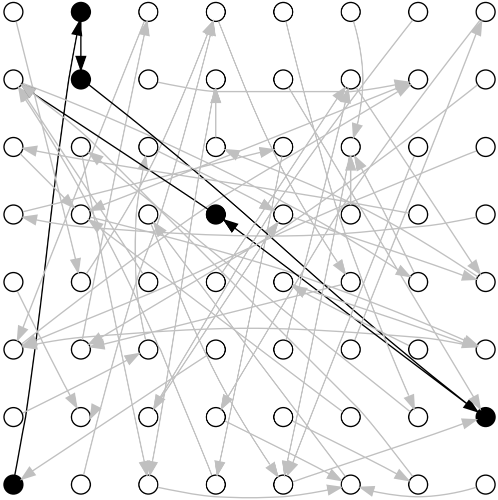

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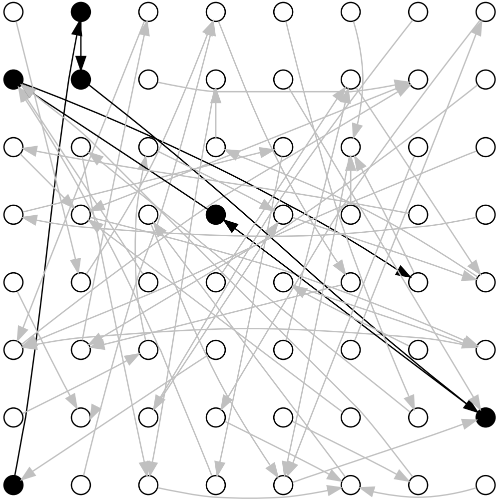

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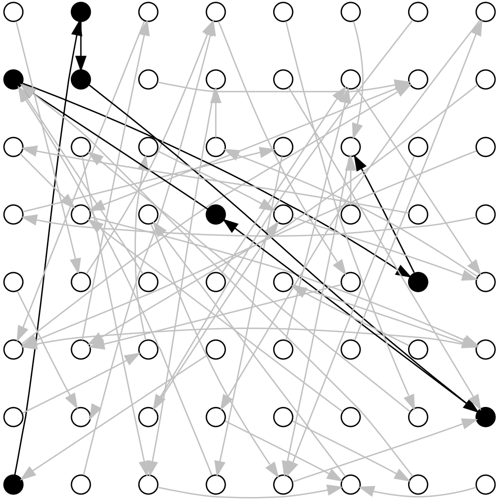

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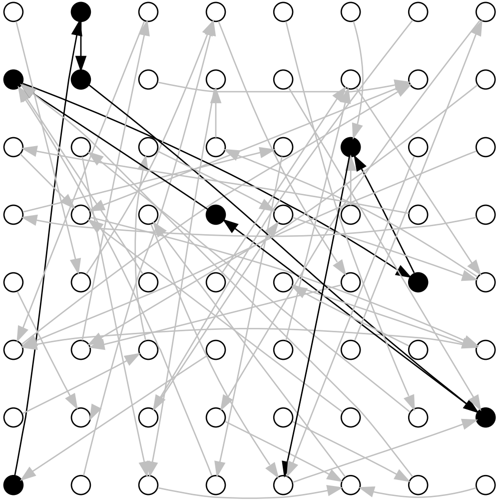

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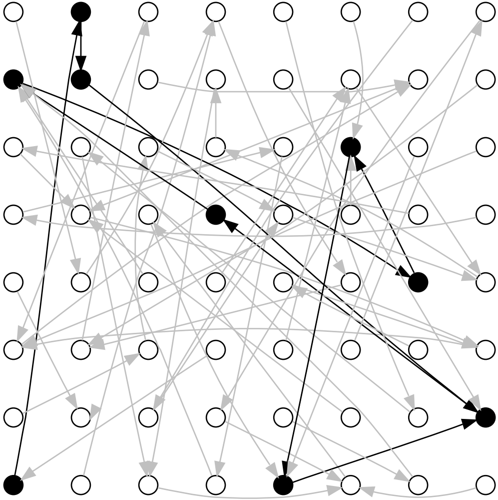

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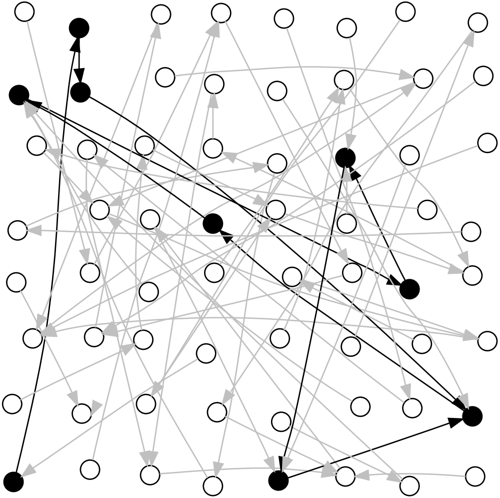

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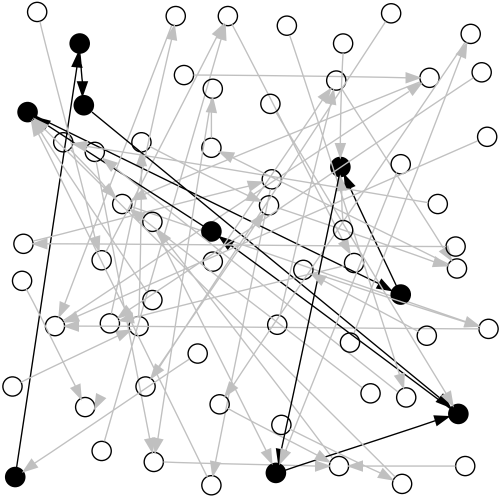

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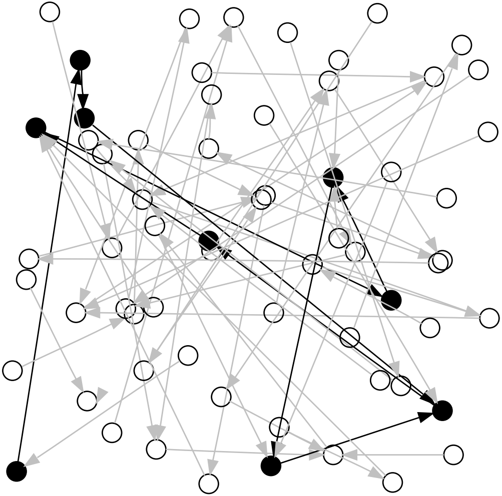

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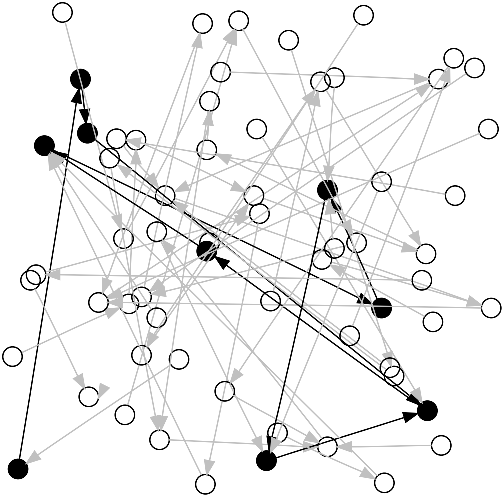

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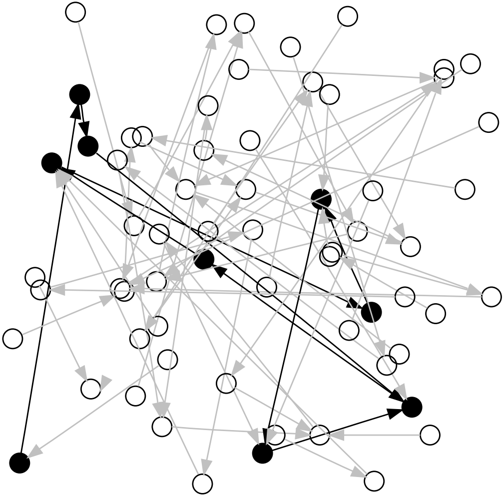

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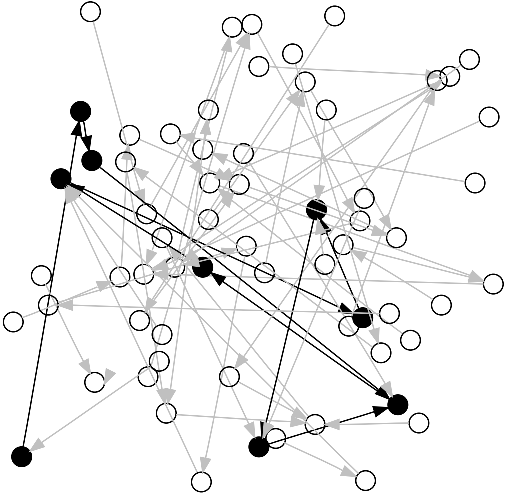

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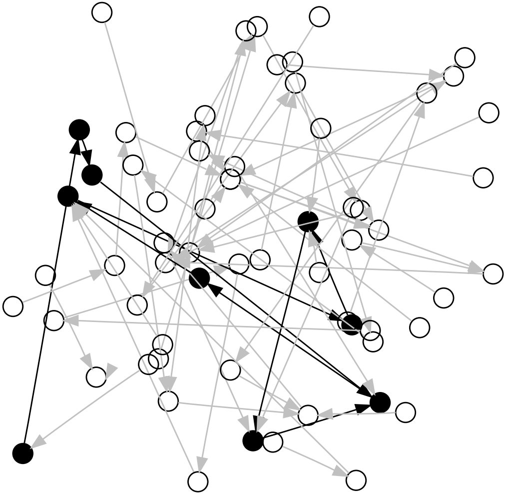

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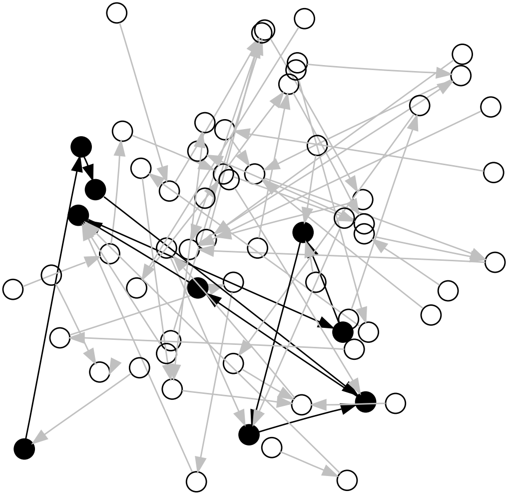

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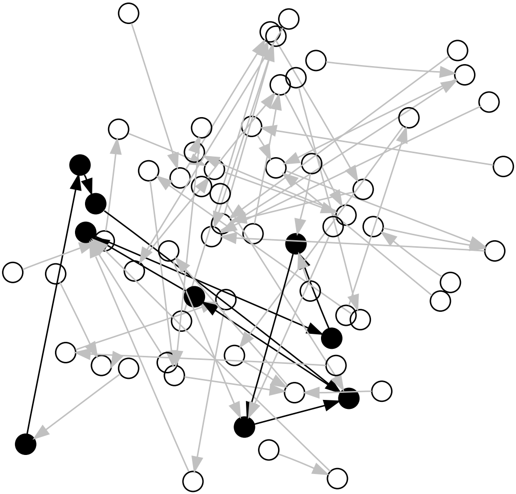

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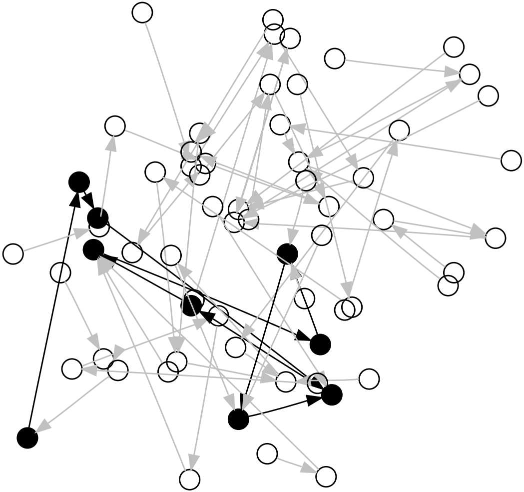

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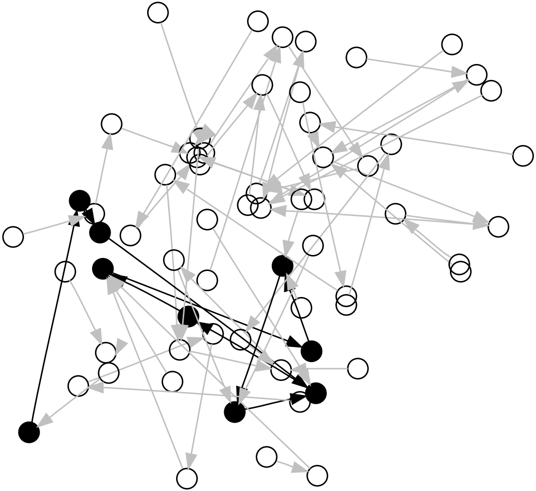

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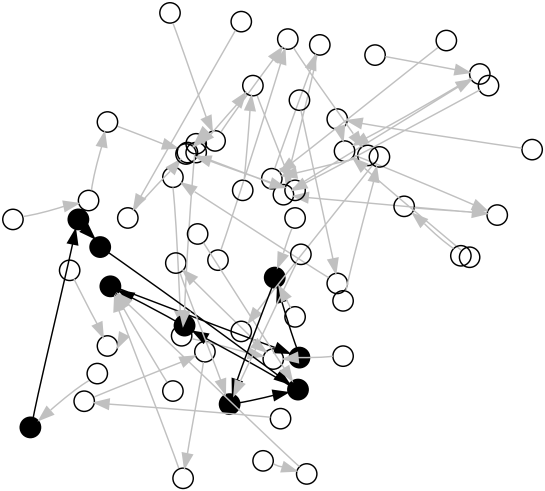

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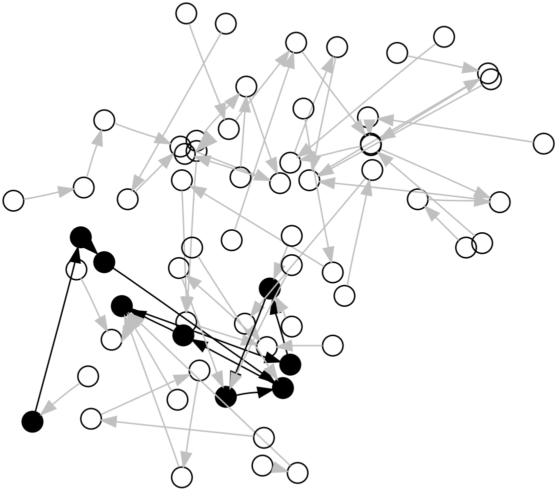

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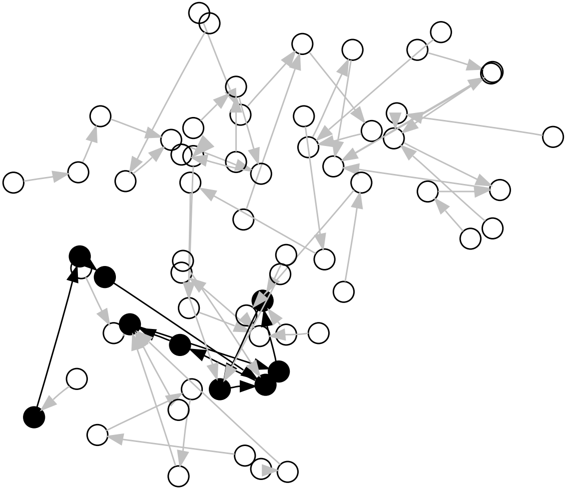

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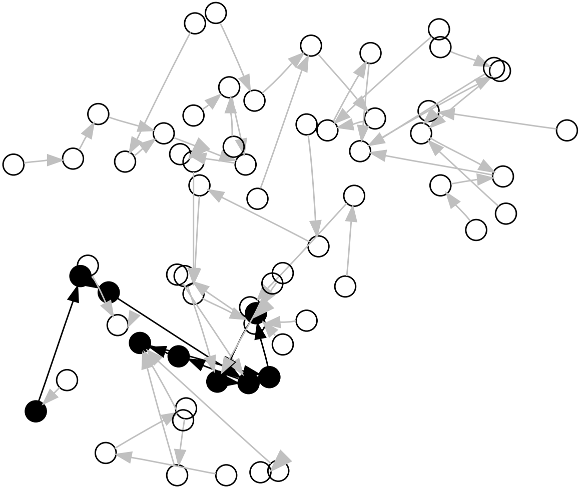

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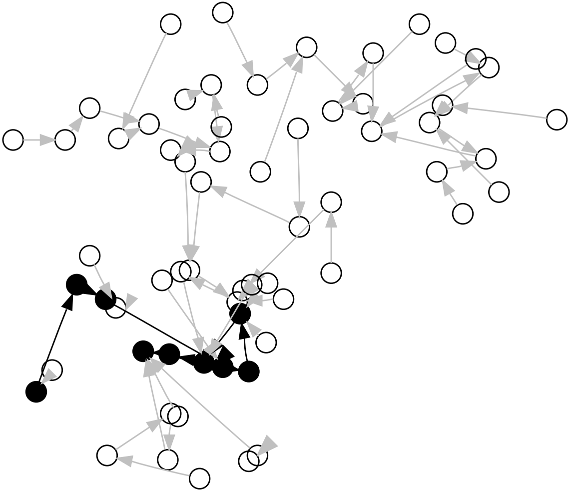

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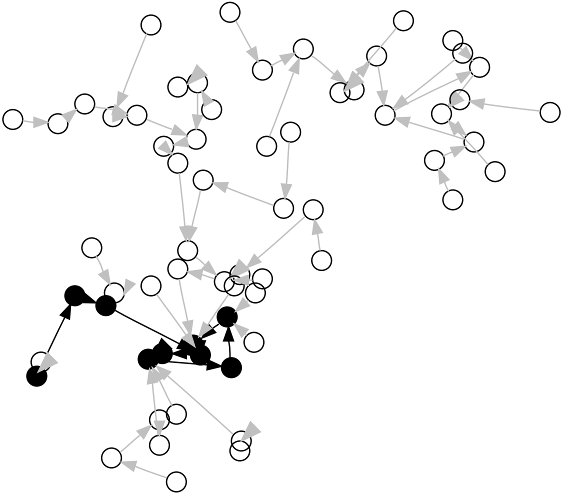

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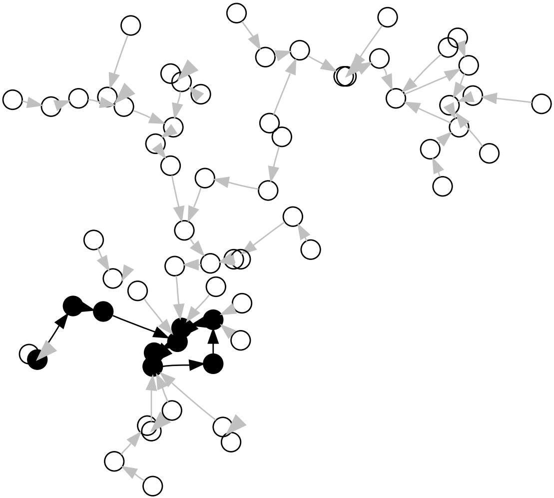

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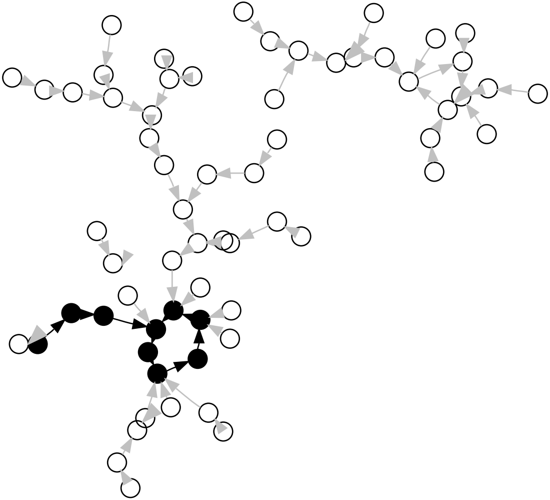

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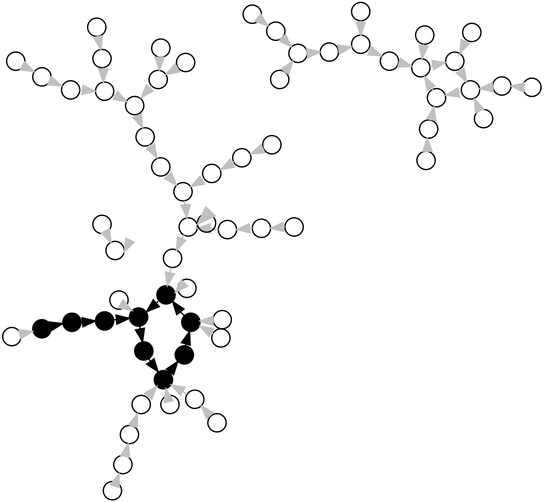


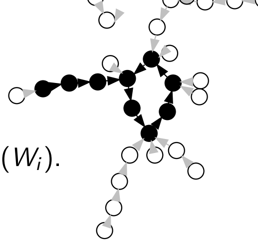
For random mappings:

Tail length: $\sqrt{\pi n/8}$

Cycle length: $\sqrt{\pi n/8}$

See Flajolet & Odlyzko URL.

Eliminate the storage



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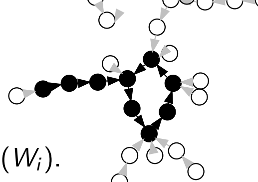

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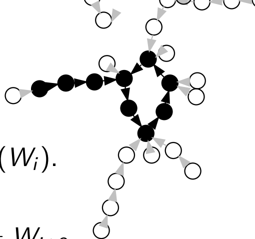

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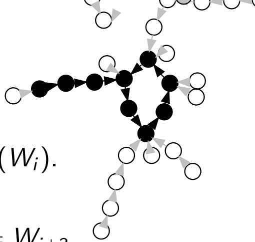

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Only compares $F_i \stackrel{?}{=} S_i$, so does not need any older values.

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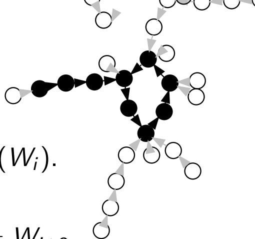

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Start at $S_0 = F_0 = W_0$.
Each step does $S_{i+1} = f(S_i)$ and $F_{i+1} = f(f(F_i))$.
Once both walks enter the cycle, they will collide.

Cycle length: roughly $\sqrt{\pi n/8}$.
Fast walk might need to loop a few times.