Algebra and discrete mathematics, homework sheet 6

Due: 31 March 2015, 8:45

You can hand in groups of two or three; specify names and student numbers. To hand in send email to `l.groot.bruinderink@student.tue.nl` with your program. Please include your program as a .txt or .sage file or save it as a worksheet.

If `K` is a field then `K.extension()` generates an extension field over `K`. Note that this command wants you to specify the variable name of the extension field, i.e. `L.<b>=K.extension()`. Check the documentation to see what arguments go into the `()`.
Note that you require an irreducible polynomial for this. To generate a field which contains a third root of 1, i.e. a number ζ satisfying $\zeta^3 = 1$ you would like to use `K.<a>=QQ.extension(x^3-1)` but that gives an error since x^3-1 is not irreducible. Use `(x^3-1).factor()` to get the irreducible factors and use the correct one.
`matrix([[1,2],[3,4]])` creates the 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

1. Generate an extension field of $\mathbb{Q}$ containing α satisfying $\alpha^3 = 2$, i.e. $\mathbb{Q}(\sqrt[3]{2})$. What is the extension degree, i.e. the dimension of this field over $\mathbb{Q}$ as a vector space.

2. Generate an extension field of $\mathbb{Q}$ containing ζ satisfying $\zeta^3 = 1$. What is the extension degree, i.e. the dimension of this field over $\mathbb{Q}$ as a vector space.

3. Prove that $\beta = \frac{1}{\sqrt{2}} + \sqrt{3}$ and $\gamma = \sqrt[3]{3} - \sqrt{3}$ are algebraic numbers using the algorithm of example 7.4.43 for this. For this part, please expand the powers of β and γ by hand and use sage to check that that's correct. Then use sage to solve the linear algebra part. Finally use `.minpoly()` to check your results.