Algebra and discrete mathematics, homework sheet 6

Due: 01 April 2014, 8:45

You can hand in alone or in groups of two; specify names and student numbers. To hand in send email to tanja@hyperelliptic.org with your program. Please include your program as a .txt or .sage file or save it as a worksheet.

If K is a field then K.extension() generates an extension field over K. Note that this command wants you to specify the variable name of the extension field, i.e. L.<b>=K.extension(). Check the documentation to see what arguments go into the ().

Note that you require an irreducible polynomial for this. To generate a field which contains a third root of 1, i.e. a number ζ satisfying $\zeta^3 = 1$ you would like to use K.<a>=QQ.extension(x^3-1) but that gives an error since $x^3 - 1$ is not irreducible. Use (x^3-1).factor() to get the irreducible factors and use the correct one.

matrix([[1,2],[3,4]]) creates the 2×2 matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

1. Generate an extension field of $\mathbb{Q}$ containing α satisfying $\alpha^3 = 2$, i.e. $\mathbb{Q}(\sqrt[3]{2})$. What is the extension degree, i.e. the dimension of this field over $\mathbb{Q}$ as a vector space.

2. Generate an extension field of $\mathbb{Q}$ containing ζ satisfying $\zeta^3 = 1$. What is the extension degree, i.e. the dimension of this field over $\mathbb{Q}$ as a vector space.

3. Prove that $\beta = \frac{1}{\sqrt{2}} + \sqrt{3}$ and $\gamma = \sqrt[3]{3} - \sqrt{3}$ are algebraic numbers using the algorithm of example 7.4.43 for this. For this part, please expand the powers of β and γ by hand and use sage to check that that's correct. Then use sage to solve the linear algebra part. Finally use .minpoly() to check your results.

4. Show that the quotient ring $\mathbb{Q}[X]/(X^2 + X + 2)$ is isomorphic to $\mathbb{Q}(\sqrt{-7})$. I.e. create both structures in sage and ask sage whether they are isomorphic.