

# Algebra and discrete mathematics, homework sheet 4

Due: 18 March 2014, 8:45

You can hand in in groups of two. Please clearly write the names on the sheet.

1. Let  $(\mathbb{C}, +, \cdot)$  denote the field of complex numbers with regular addition and multiplication. Let the sets  $M_1$  and  $M_2$  be defined as follows:

$$M_1 = \{a + b\sqrt[3]{6} + c\sqrt[3]{6^2} \mid a, b, c \in \mathbb{Z}\} \subseteq \mathbb{C},$$

$$M_2 = \{a + b\sqrt{2} + c\sqrt{3} \mid a, b, c \in \mathbb{Z}\} \subseteq \mathbb{C}.$$

- (a) Study whether  $(M_1, \cdot)$  is a semigroup.
  - (b) Study whether  $(M_2, \cdot)$  is a semigroup.
  - (c) Is  $(M_1, +, \cdot)$  a subring of  $(\mathbb{C}, +, \cdot)$ ? Why?
  - (d) Is  $(M_1, +, \cdot)$  a subfield of  $(\mathbb{C}, +, \cdot)$ ? Why?
  - (e) Is  $(M_2, +, \cdot)$  a subfield of  $(\mathbb{C}, +, \cdot)$ ? Why?
2. Consider the ring  $R = \mathbb{Z}/2 \times \mathbb{Z}/7 \times \mathbb{Z}/9$ .
- (a) How many elements does  $R$  have?
  - (b) Compute the order of  $R^\times$ .
  - (c) Compute the (multiplicative) order of  $(1, 3, 4) \in R^\times$ .
  - (d) Does there exist an integer  $m$  such that  $R \cong \mathbb{Z}/m$ ? If so, compute it, explain how to compute the ring homomorphism and the inverse of the ring homomorphism, and compute the image of  $(1, 3, 2)$ . If not, why not?
  - (e) Find two elements  $a, b \in R$  so that  $a \cdot b = (0, 0, 0)$  but  $a, b \neq (0, 0, 0)$ .
  - (f) Compute the number of elements  $a \in R$  with  $a \neq (0, 0, 0)$  for which there exists a  $b \neq (0, 0, 0)$  such that  $a \cdot b = (0, 0, 0)$ .
  - (g) How many such zero divisors exist in  $R^\times$ ?

3. Let  $R$  be a ring and let  $\phi : R \rightarrow R$  be a ring homomorphism. Show that the set

$$S = \{s \in R \mid \phi(s) = s\}$$

is a subring of  $R$ .