Algebra and discrete mathematics, homework sheet 1

Due: 25 February 2014, 8:45

You can hand in in groups of two. Please clearly write the names on the sheet.

1. The integers modulo 30 form a monoid, $(\mathbb{Z}/30, \cdot, 1)$, with respect to multiplication.
   (a) Detemine $(\mathbb{Z}/30)^{\times}$, i.e. the group of invertible elements.
   (b) Denote the class $a + 30\mathbb{Z}$ by a. Compute the cyclic submonoids $\langle 3\rangle, \langle 6\rangle$, and $\langle 7\rangle$ and determine for each of them the cycle length and the tail length.

2. Let $(G, \circ, e, x \mapsto \text{inv}(x))$ be a group and let $X \subseteq G$.
   (a) Show that the normalizer of X in G
   $$N(X, G) = \{g \in G | g \circ X = X \circ g\}$$
   is a subgroup of G.
   (b) Show that the centre of G
   $$Z(G) = \{g \in G | h \circ g = g \circ h \text{ for all } h \in G\}$$
   is a subgroup of G.

3. Determine $C(\{(1, 2, 3), (1, 3, 2)\}, S_3)$ and $N(\{(1, 2, 3), (1, 3, 2)\}, S_3)$

4. Let $\circ$ be an operation on the rationals $\mathbb{Q}$ defined as follows:
   $$a \circ b = ab + 2(a + b) + 2,$$
   where addition and multiplication are the regular operations on $\mathbb{Q}$.
   (a) Show that $\mathbb{Q}$ is a monoid with respect to the operation $\circ$. You may use that $(\mathbb{Q}, +, 0, x \mapsto -x)$ and $(\mathbb{Q}, \cdot, 1, x \mapsto x^{-1})$ are commutative groups.
   (b) Is $(\mathbb{Q}, \circ)$ commutative?
   (c) Determine the invertible elements.