Exercise sheet 6, 21 December 2017

For this exercise sheet you should not use your computer for more functions than a pocket calculator offers you (though with more digits) – unless explicitly stated.
The first six exercises are to recap finite fields and groups; skip them if you feel comfortable in that area – or just do them quickly.

1. Write all elements of $\mathbb{Z}/13$. For each element determine the order in $(\mathbb{Z}/13, +)$. What orders do you observe; what orders could be possible?
2. Write all elements of $\mathbb{Z}/6$. For each element determine the order in $(\mathbb{Z}/6, +)$. What orders do you observe; what orders could be possible?
3. Write all elements of $(\mathbb{Z}/13)^*$. For each element determine the order in $((\mathbb{Z}/13)^*, \cdot)$. What orders do you observe; what orders could be possible?
4. Write all elements of $(\mathbb{Z}/6)^*$. For each element determine the order in $((\mathbb{Z}/6)^*, \cdot)$. What orders do you observe; what orders could be possible?
5. Show that $\mathbb{F}_{61}^* = \langle 2 \rangle$, i.e. show that the order of 2 in $\mathbb{F}_{61}$ is 60.
6. Determine the smallest generator $g \in (\mathbb{Z}/4969)^*$ that is larger than 1000. Do this by testing whether $1000+i$ is a generator, starting from $i = 1$ and incrementing i if it is not. Try to make each test as cheap as possible. For this exercise I suggest you use modular exponentiation on your computer but don't just ask it for the order.
7. For this exercise you can use your computer. Use the $p - 1$ method with $k = \mathrm{lcm}(1, 2, 3, 4, 5, \ldots, 50)$ and base 2 to factor $n = 400428248257$. If you get stuck on the precision of your computer, remember that the exponentiation is modulo n and that you learned the square-and-multiply method to deal with large exponents. Alternatively, for the last step you can compute the exponentiation in pieces, using the factors of k.
8. For this exercise you should use a pocket calculator (or your computer with just basic functions). Use the $p-1$ method with $k = \mathrm{lcm}\{1, 2, 3, \ldots, 6\}$ and base 2 to factor $n = 101617$.
9. The integer $p = 103$ is prime. You are the eavesdropper and know that Alice and Bob use the Diffie-Hellman key-exchange in a cyclic subgroup of $(\mathbb{Z}/p, +)$ with generator $g = 2$. You observe $h_a = 23$ and $h_b = 42$. What is the shared key of Alice and Bob?
10. The integer $p = 103$ is prime. You are the eavesdropper and know that Charlie and Dave use the Diffie-Hellman key-exchange in a cyclic subgroup of $(\mathbb{Z}/p, +)$ with generator $g = 2$. You observe $h_a = 21$ and $h_b = 39$. What is the shared key of Alice and Bob?
11. The integer $p = 10007$ is prime. You are the eavesdropper and know that Alice and Bob use the Diffie-Hellman key-exchange in a cyclic subgroup of $(\mathbb{Z}/p, +)$ with generator $g = 1234$. You observe $h_a = 2345$ and $h_b = 4567$. What is the shared key of Alice and Bob?
12. This problem is about the DH key exchange. The public parameters are that the group is $(\mathbb{F}_{1009}^*, \cdot)$ and that it is generated by $g = 11$.
    (a) Compute Diffie-Hellman the public key belonging to the secret key $b = 548$.
    (b) Alice's Diffie-Hellman public key is $h_a = 830$. Compute the shared DH key with Alice using b from the previous part.
    (c) Alice and Bob keep the prime but change the generator to $g = 1008$. (This changes the subgroup generated). Simulate one round of DH key exchange. Why would you avoid this generator in practice?

13. The integer $p = 17$ is prime. You are the eavesdropper and know that Alice and Bob use the Diffie-Hellman key-exchange in $\mathbb{F}_{17}^*$ with generator $g = 3$. You observe $h_a = 12$ and $h_b = 14$. Use the Baby-Step Giant-Step algorithm to compute the secret key of Alice and Bob. Compute the shared key using both h_a^b and h_b^a.