Discrete logs and other collisions

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Diffie–Hellman key exchange

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Hardness assumptions

- Computational Diffie-Hellman Problem (CDHP):
  Given P, aP, bP compute abP.
- Decisional Diffie-Hellman Problem (DDHP):
  Given P, aP, bP, and cP decide whether $cP = abP$.
- Discrete Logarithm Problem (DLP):
  Given P, aP, compute a.
- If one can solve DLP, then CDHP and DDHP are easy.

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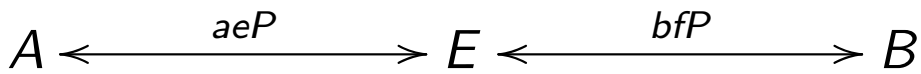
- In many groups, DLP and CDHP are equally hard
  (up to some constants).
- In some groups, DDHP is significantly easier than CDHP.

Practical problems

- Eve can set up a *man-in-the-middle* attack:



  - E chooses e and f, presents eP to Alice as Bob's key, and fP to Bob as Alice's key.
  - E computes both DH keys aeP and bfP.
  - E decrypts everything from A and reencrypts it to B and vice versa.

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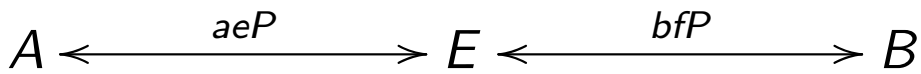

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  - E computes both DH keys aeP and bfP.
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  - This attack requires E to be in charge of the network.
    We typically assume such strong attackers.
- This attack cannot be detected unless A and B compare their keys out of band.

Semi-static DH

- A cryptosystem combining public-key and symmetric-key crypto is called a *hybrid system*[1].
- Alice publishes long-term public key $P_A = aP$,
  keeps long-term private key a.
- Any user can encrypt to Alice using this key:
  - Pick random k and compute $R = kP$.
  - Encrypt message m using symmetric keys derived from $\mathrm{KDF}(kP_A)$, for key-derivation function $\mathrm{KDF} : G \to \{0,1\}^n$,
  - Send ciphertext c along with R.
  - Alice decrypts, by obtaining symmetric key from

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  it means that is not long term.
- Attacker solving DLP or CDHP can *compute* shared secret.
  Attacker solving DDHP can *confirm* guess.

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First approaches to solving DLP

Notation and target

Setting: cyclic group $G = \langle P \rangle$ with n elements;
group operation written additively on these slides.

Given $Q \in G$ as well as P and how to compute in G.

Target: Compute $a = \log_P Q$, i.e. a such that $Q = aP$.
a is called the discrete logarithm of Q to base P.
We typically take $0 \leq a < n$.

For elliptic curves, point counting over $\mathbf{F}_p$ runs in time polynomial in $\log p$.

Number of points in $[p + 1 - 2\sqrt{p}, p + 1 + 2\sqrt{p}]$.

The group is isomorphic to $\mathbf{Z}/m \times \mathbf{Z}/n$, where $m \mid n$ and $m \mid (p - 1)$.

Running example

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Can we find an integer $a \in \{1, 2, 3, \ldots, 500001\}$ such that $aP = (670366, 740819)$?

This point was generated as a multiple of P.
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$1P = (101384, 614510)$, $2P = (102361, 628914)$, $3P = (77571, 87643)$,
$4P = (650289, 31313)$, $\ldots$, $500001P = (101384, 385493) = -P$.
$500002P = \infty$.

At some point we'll find a with $aP = (670366, 740819)$.

Maximum cost of computation: ≤ 500001 additions of P;

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Choosing large, random r moves target around.

Many targets

Computation can be applied to many targets at once.

Given 100 DL targets a_1P, a_2P, ..., $a_{100}P$:
Can find *all* of $a_1, a_2, \ldots, a_{100}$ with $\leq n - 1$ ADDs.

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Solving all 100 DL problems costs about as much as solving one.

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Solving *at least one* out of 100 DL problems is much faster than solving one specific DL problem.

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Can turn one DLP into 100 to benefit: compute $mP = \lfloor n/100 \rfloor P$.
New targets: $aP, (a+m)P, (a+2m)P, \ldots (a+99m)P$.

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Cannot save more than $\sqrt{n}$, else computing targets dominates cost.

Baby-step giant-step algorithm

Systematize and optimize this approach.
Let $0 < m < n$ then there exist $0 \leq a_0 < m$ and $0 \leq a_1 \leq n/m + 1$ with

$$a = a_0 + ma_1.$$

Compute all possibilities for a_0P. This costs m steps and m space.

We first do the small steps to get mP with just one addition.

$a = a_0 + ma_1$ means $aP = (a_0 + ma_1)P$, thus

$$a_0P = aP - a_1mP$$

Each step adds -mP (remember, $-(x, y) = (x, -y)$ on Weierstrass).

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These targets are evenly spaced, so we are guaranteed a match.

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This is a typical divide-and-conquer speedup.
Search for these whenever doing a brute-force search.

Random walks and cycle finding

Random walks

Target $Q = aP$ in $\langle P \rangle$. Group has n elements.

Make a pseudo-random walk in the group $\langle P \rangle$,
where the next step depends on current point: $W_{i+1} = f(W_i)$.

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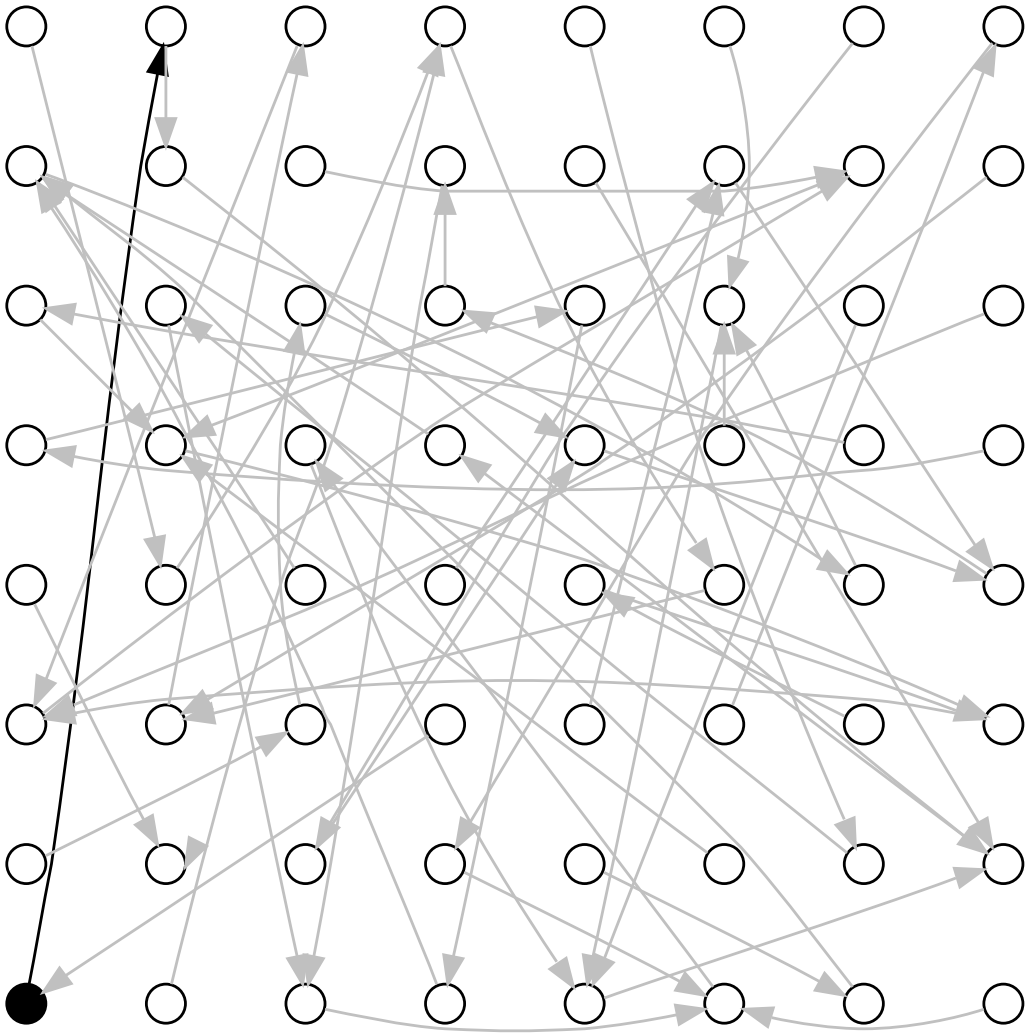
Then $W_i = W_j$ means that

$$a_i P + b_i Q = a_j P + b_j Q \Leftrightarrow (b_i - b_j)Q = (a_j - a_i)P.$$

If $b_i - b_j$ invertible modulo n, the DLP is solved:

$$a \equiv (a_j - a_i)/(b_i - b_j) \bmod n$$

Random walks have collisions



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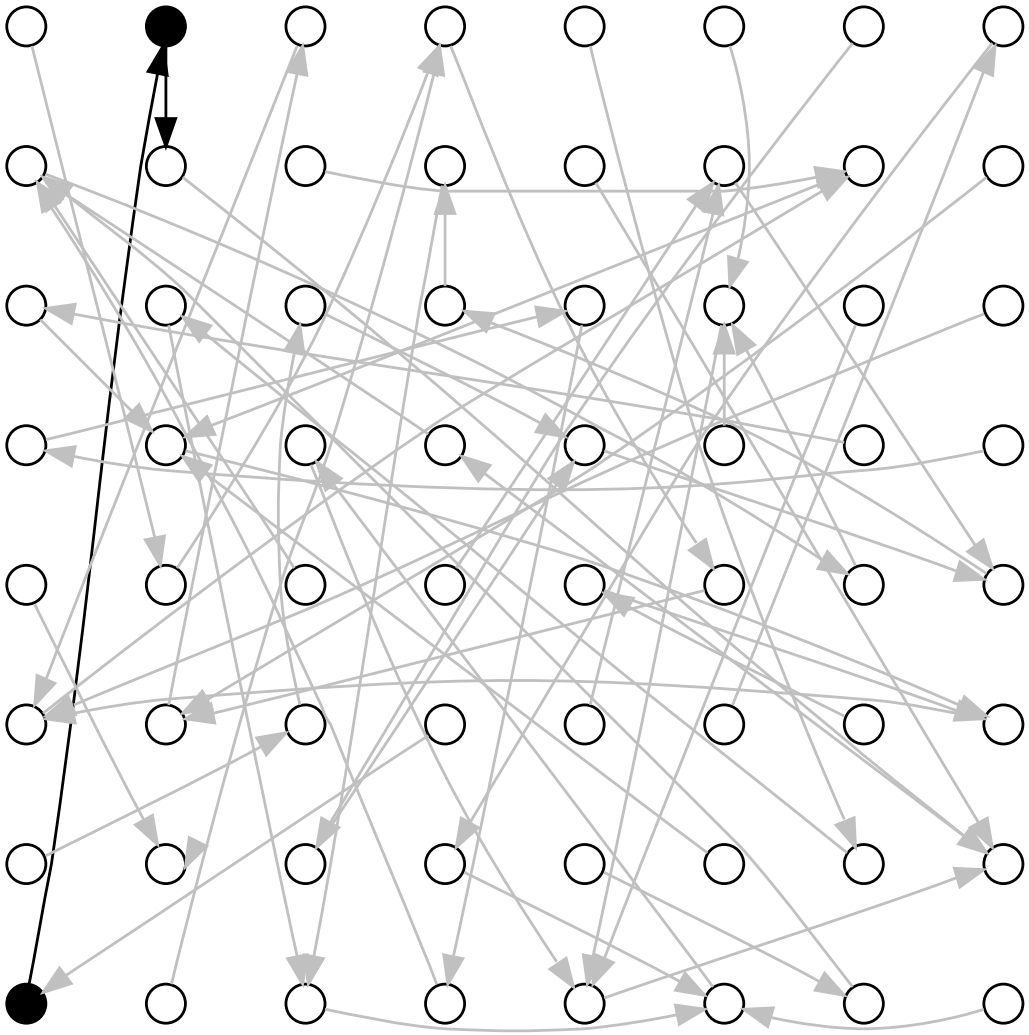

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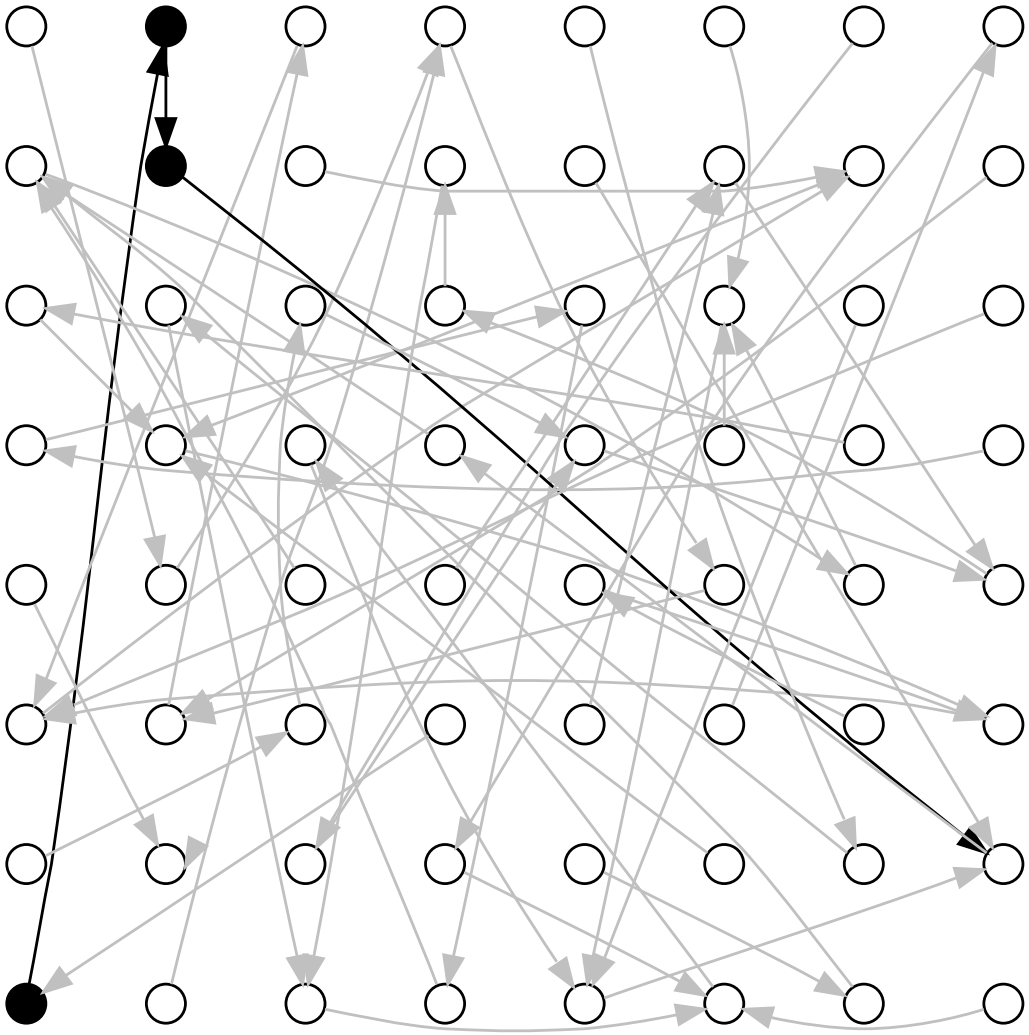

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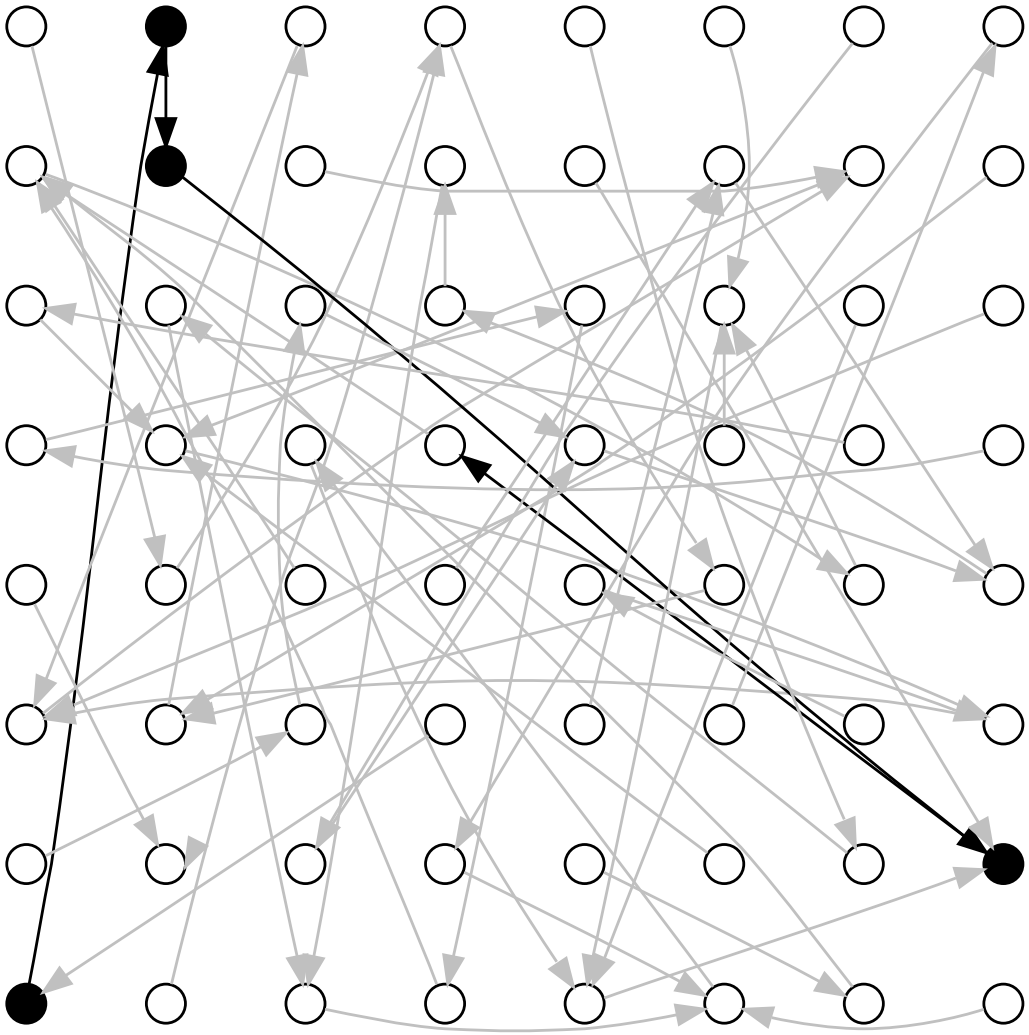

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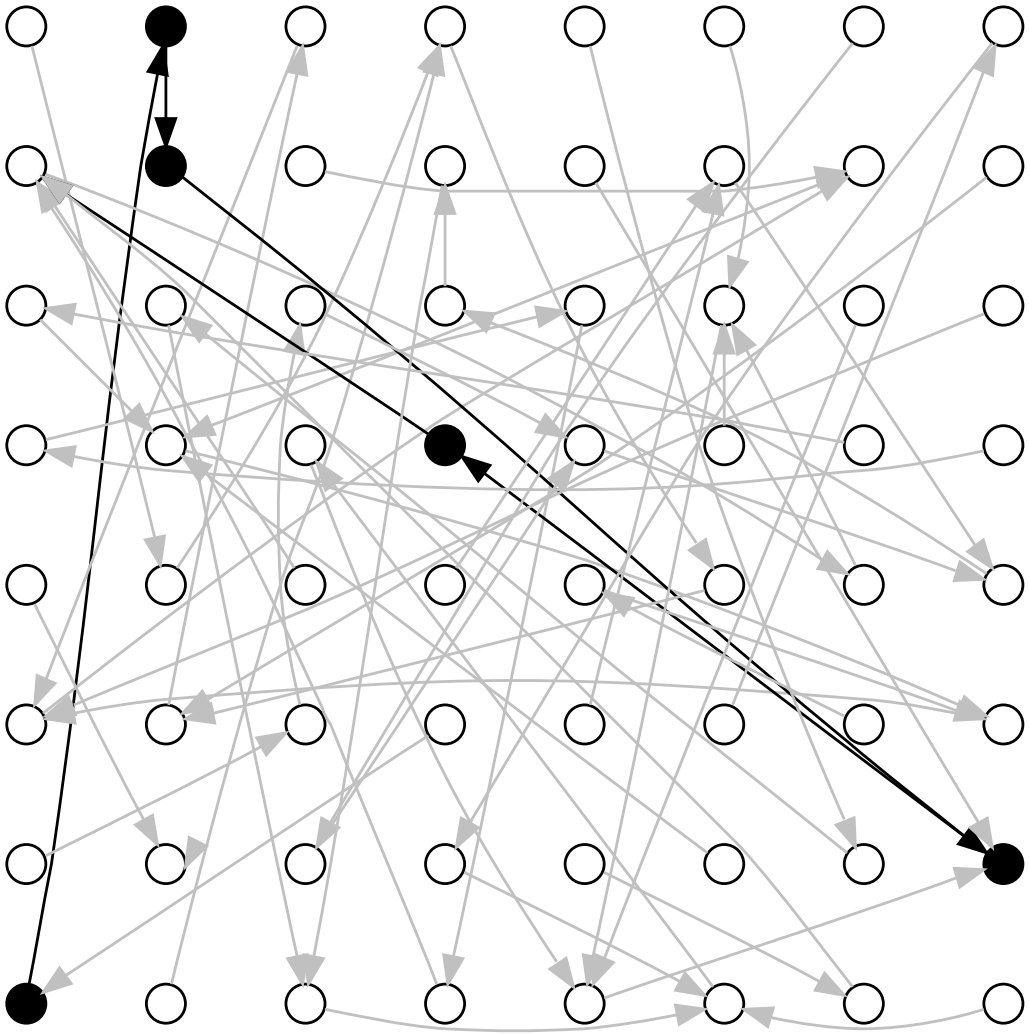

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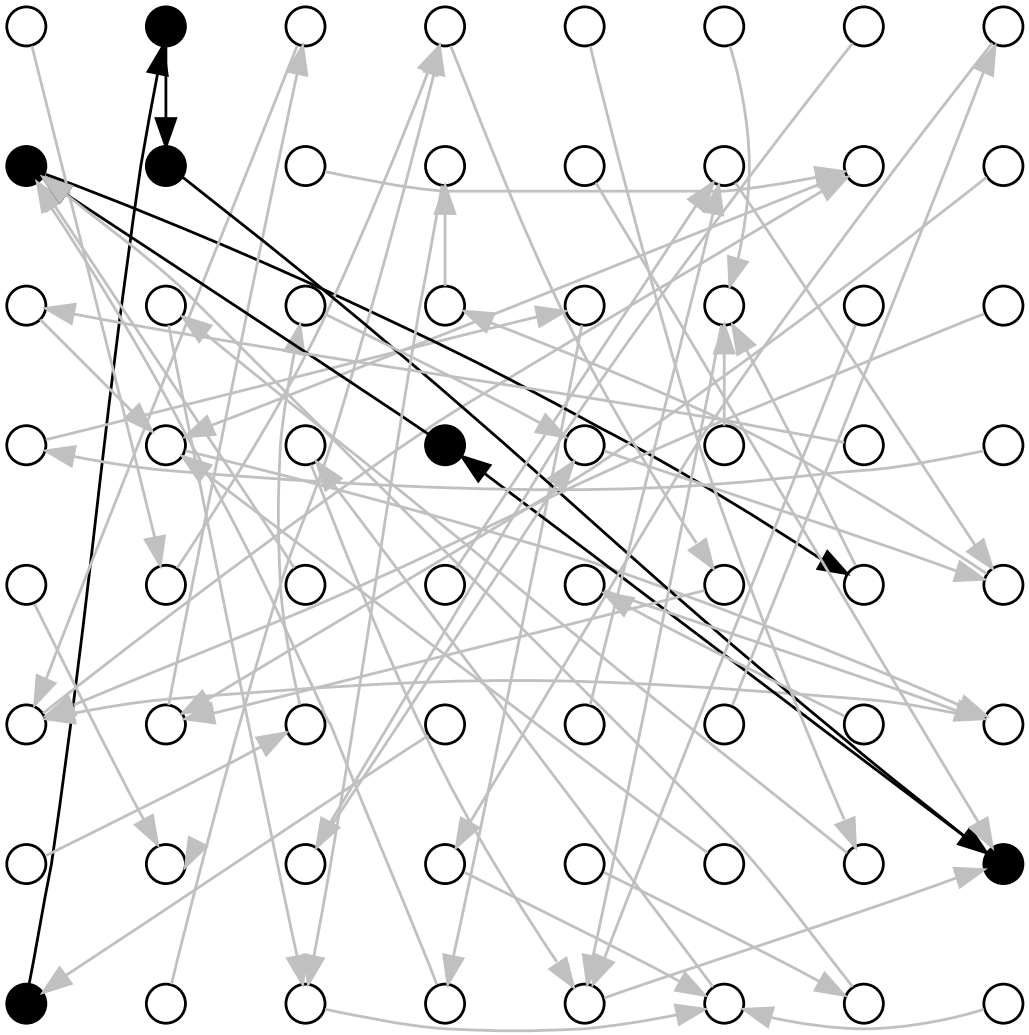

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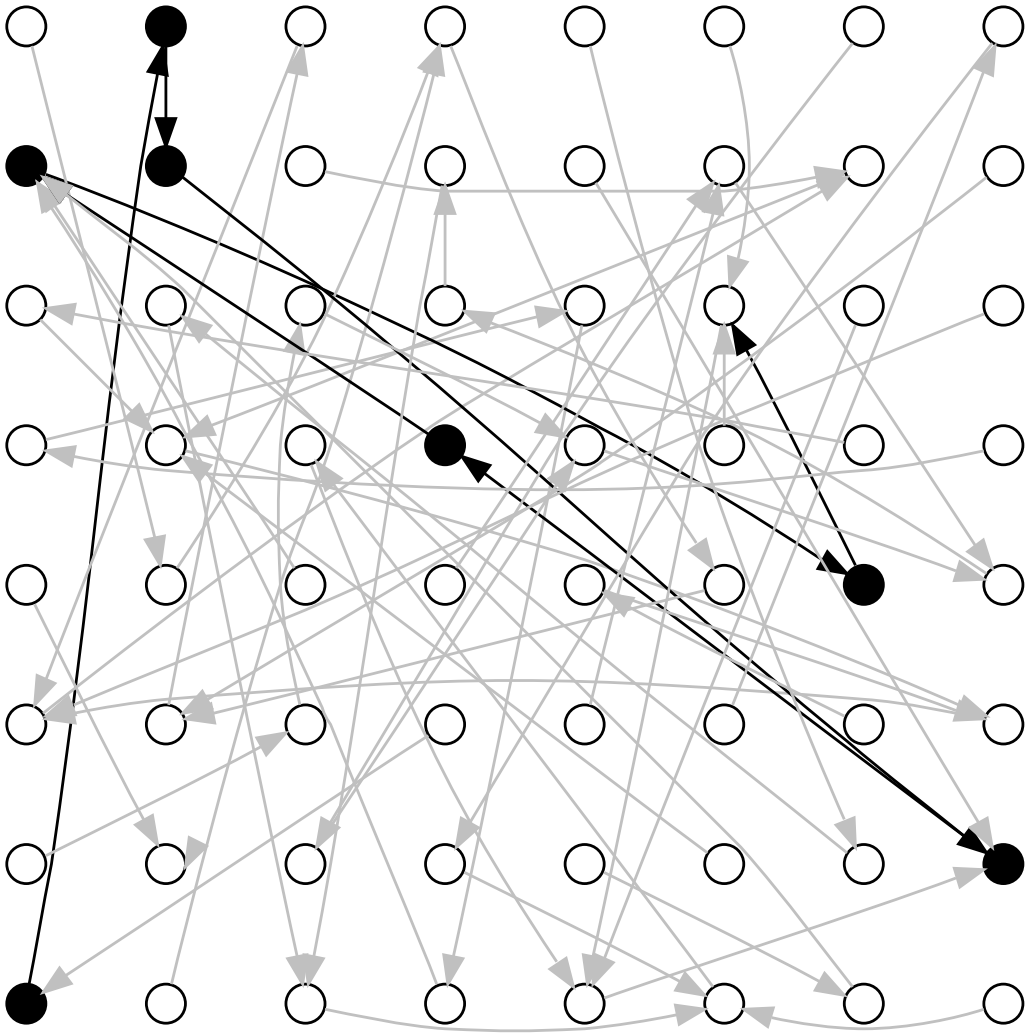

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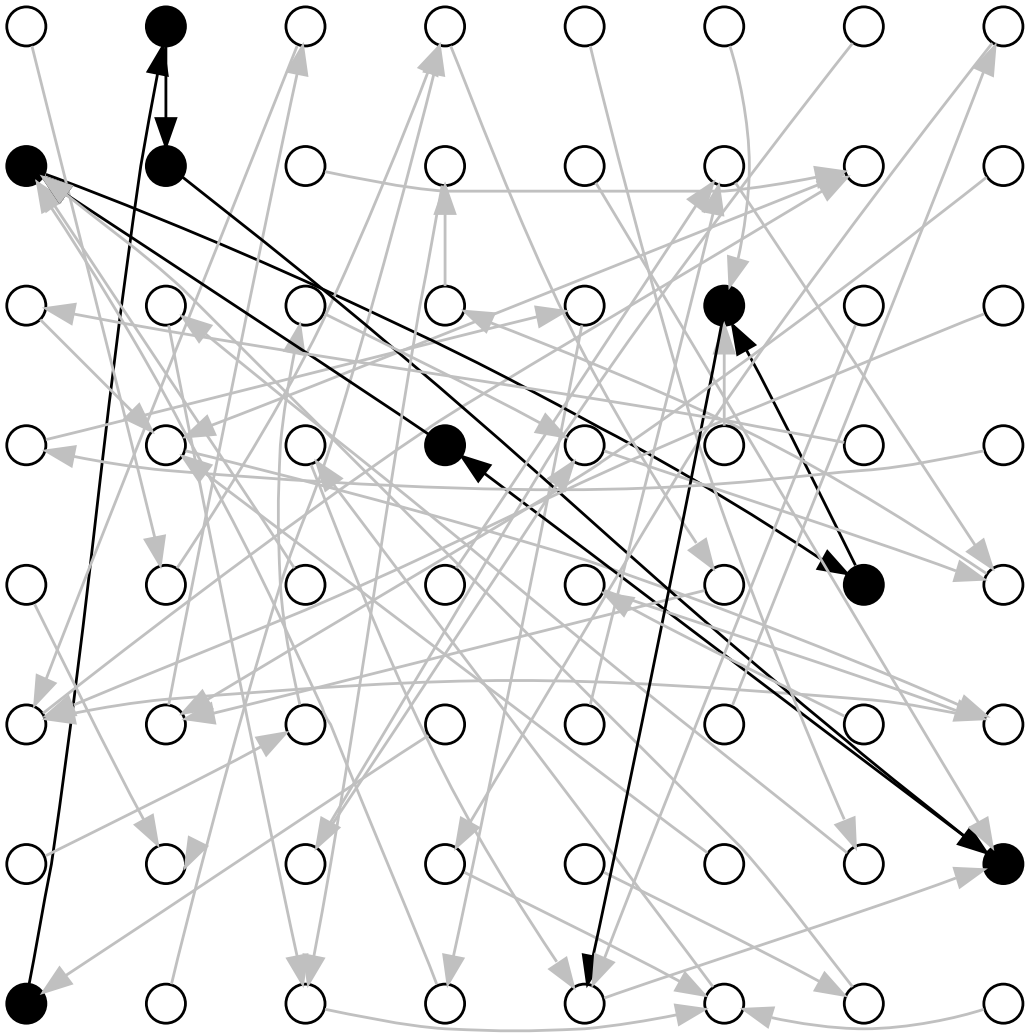

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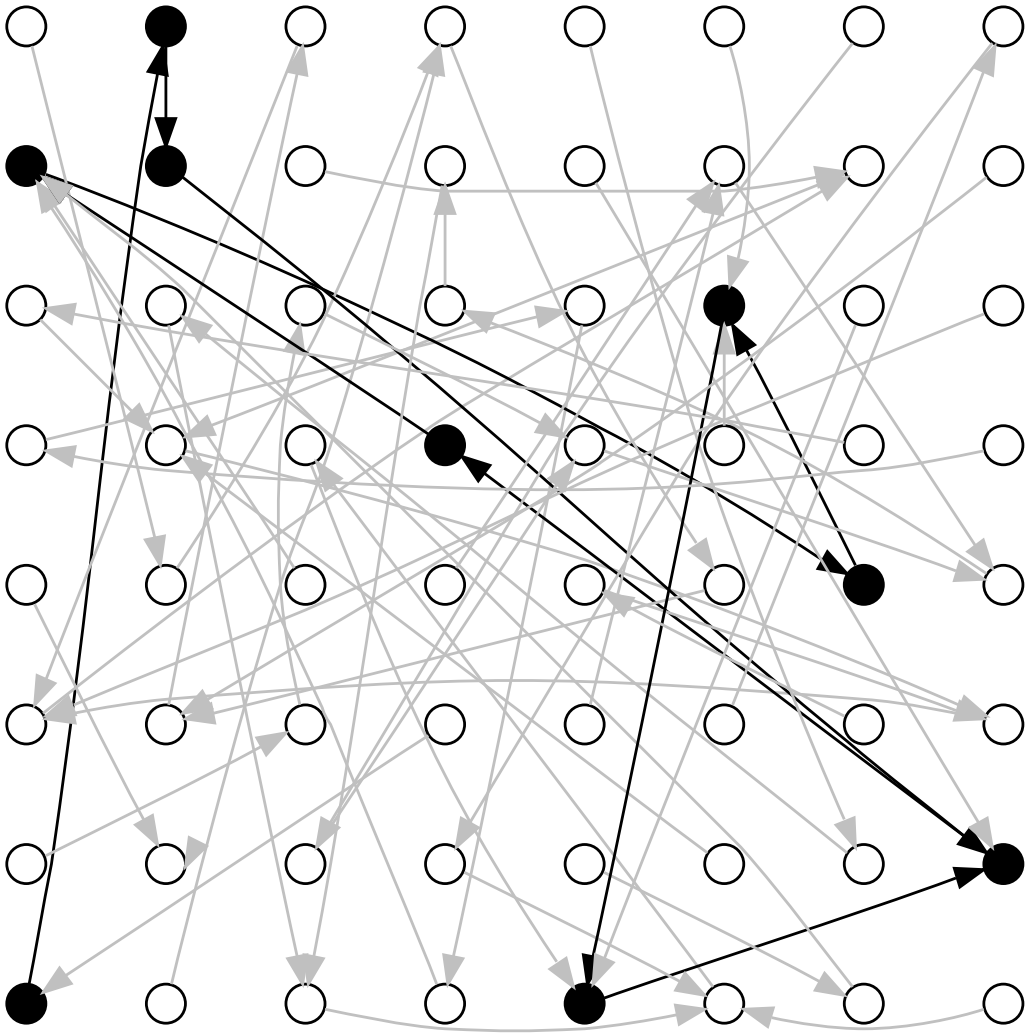

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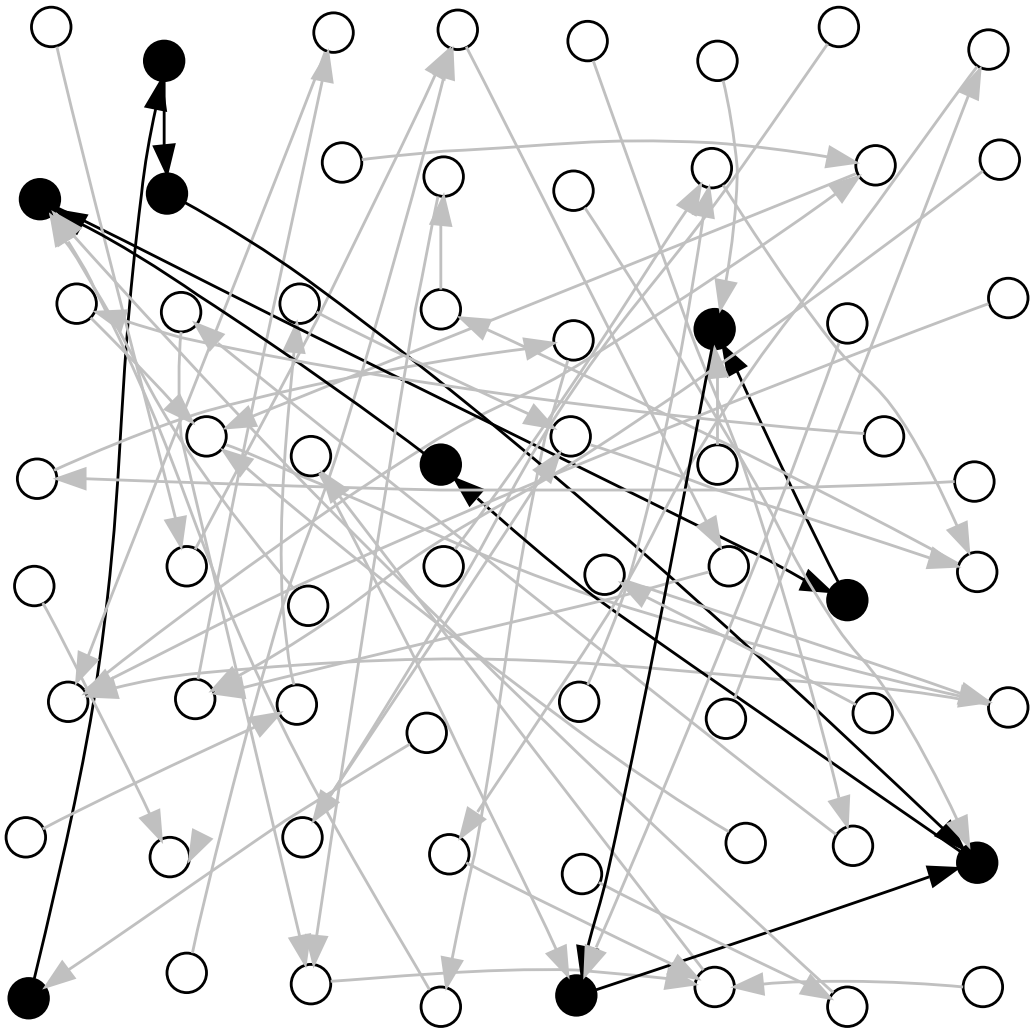

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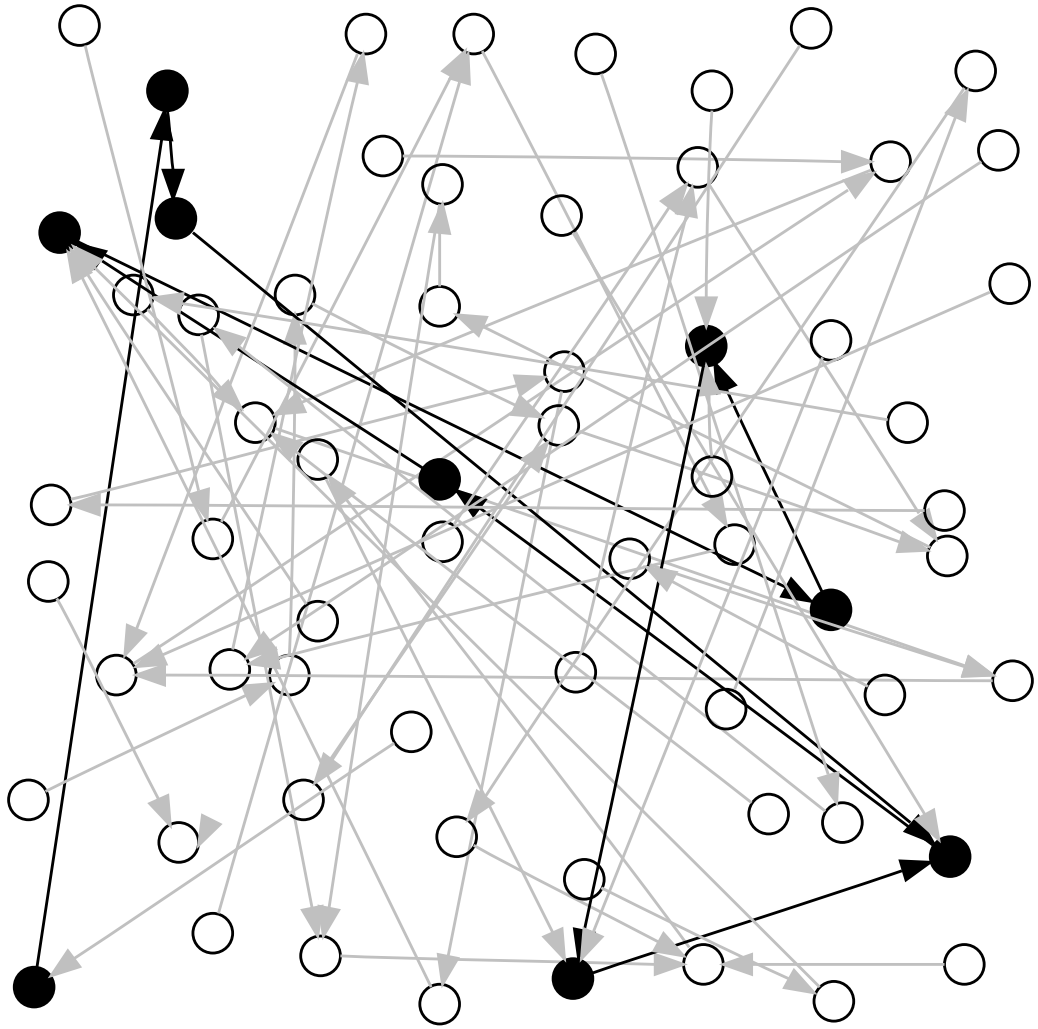

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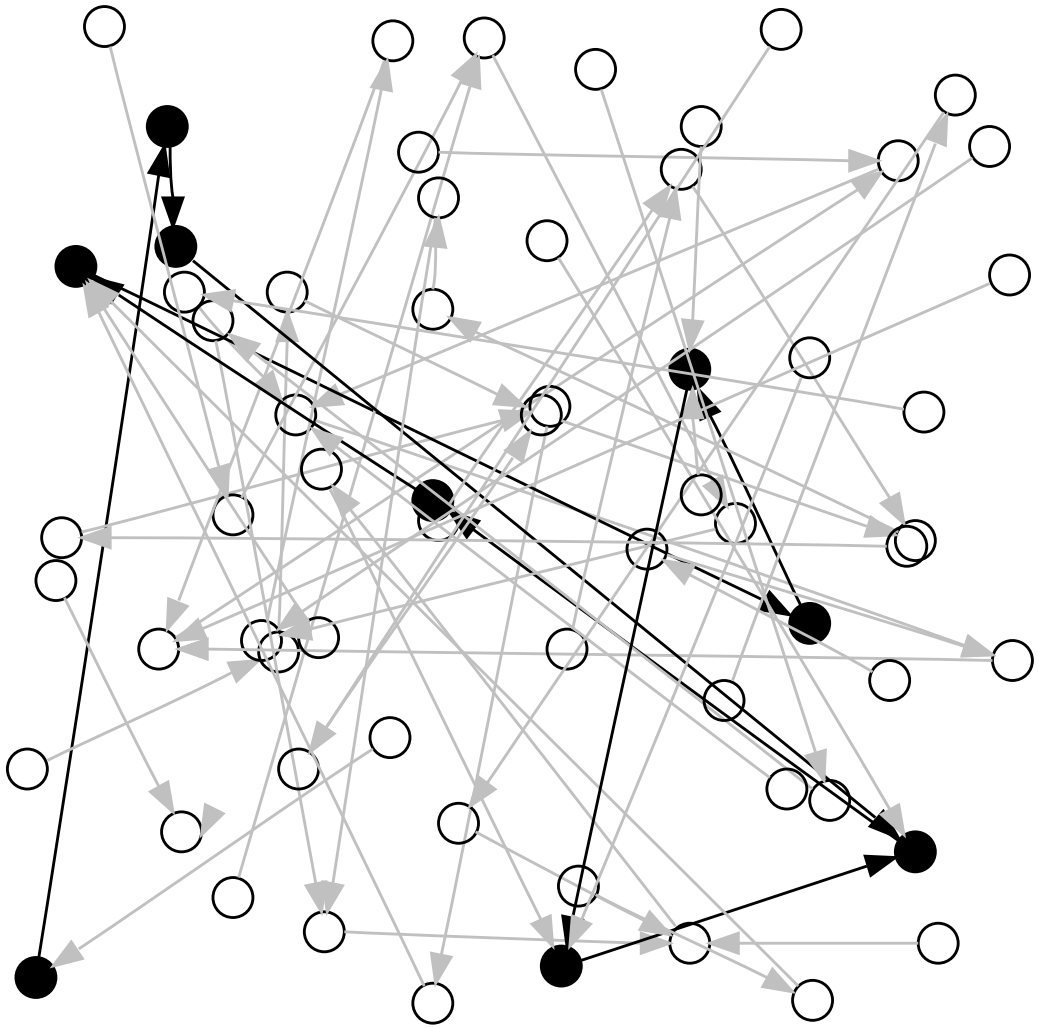

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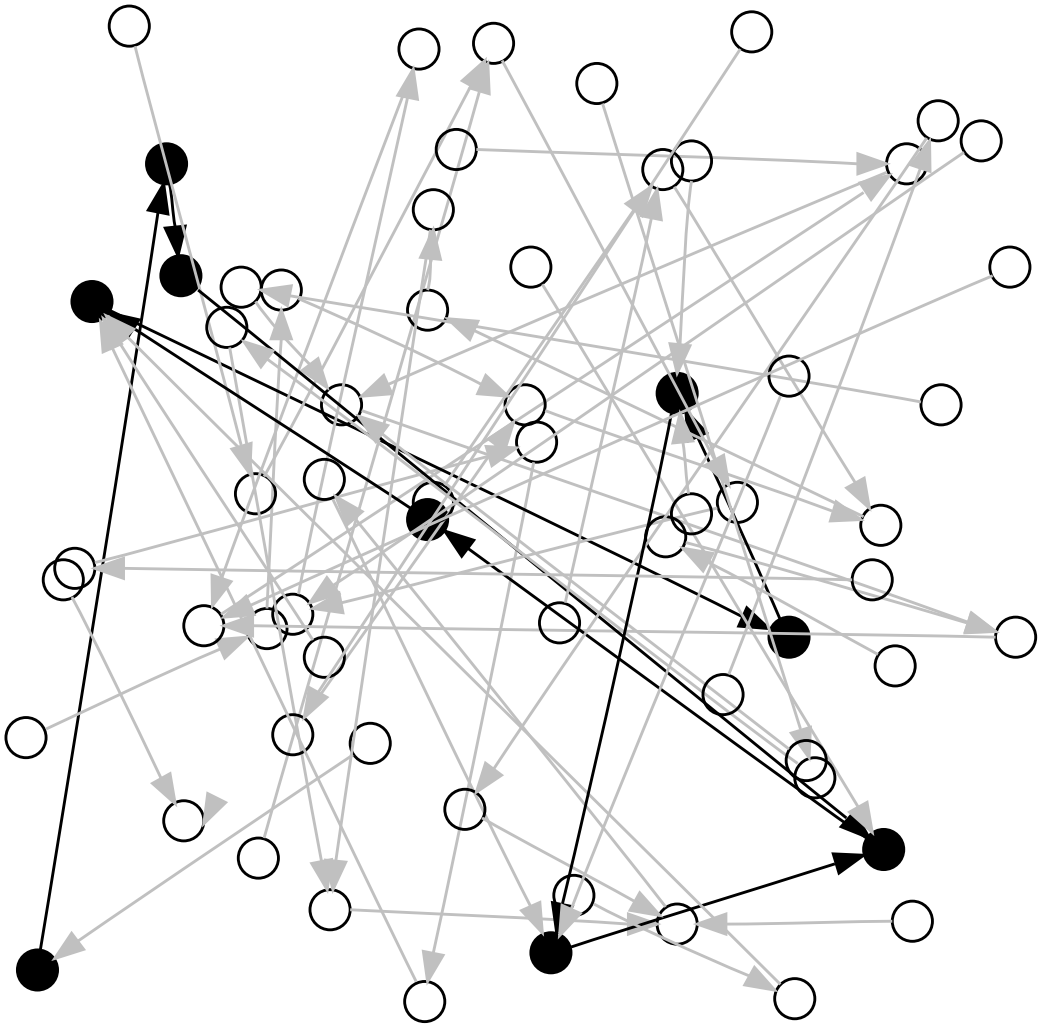

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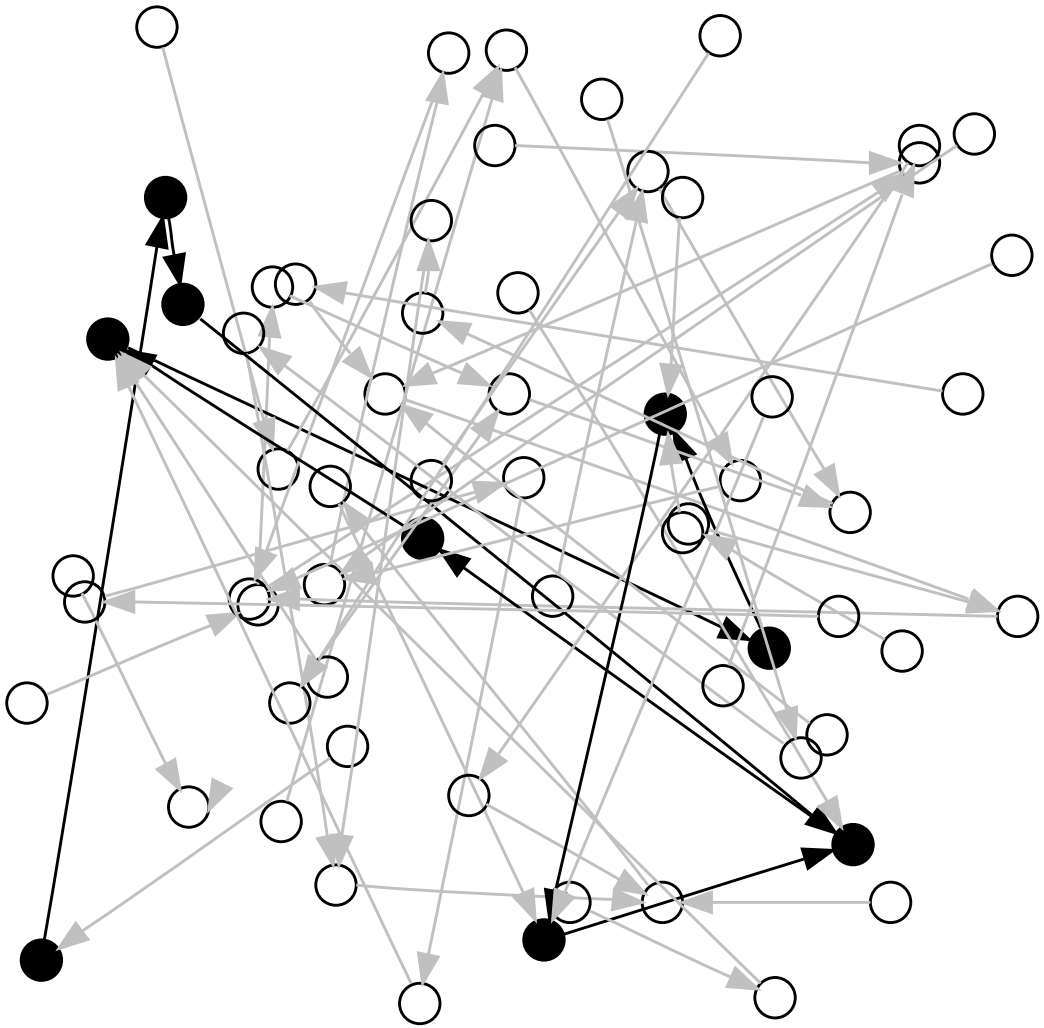

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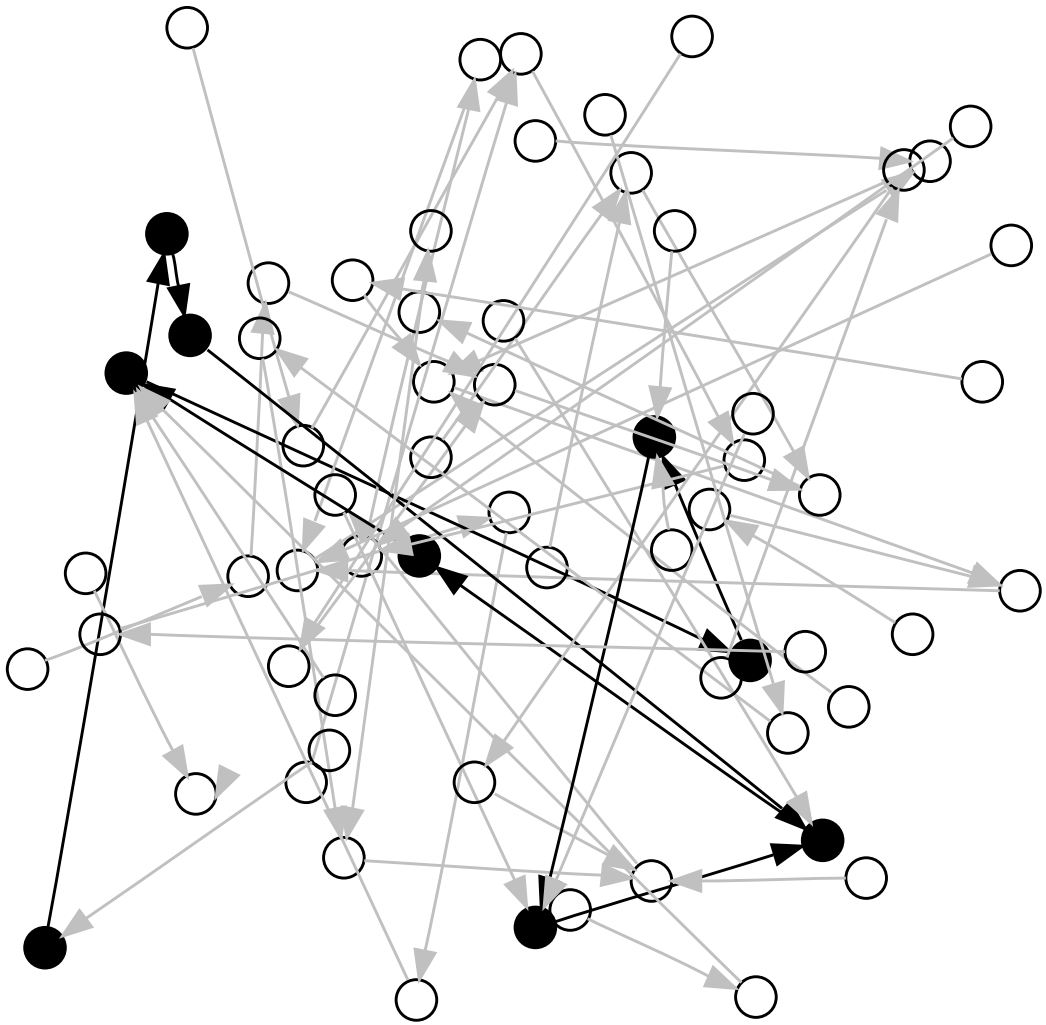

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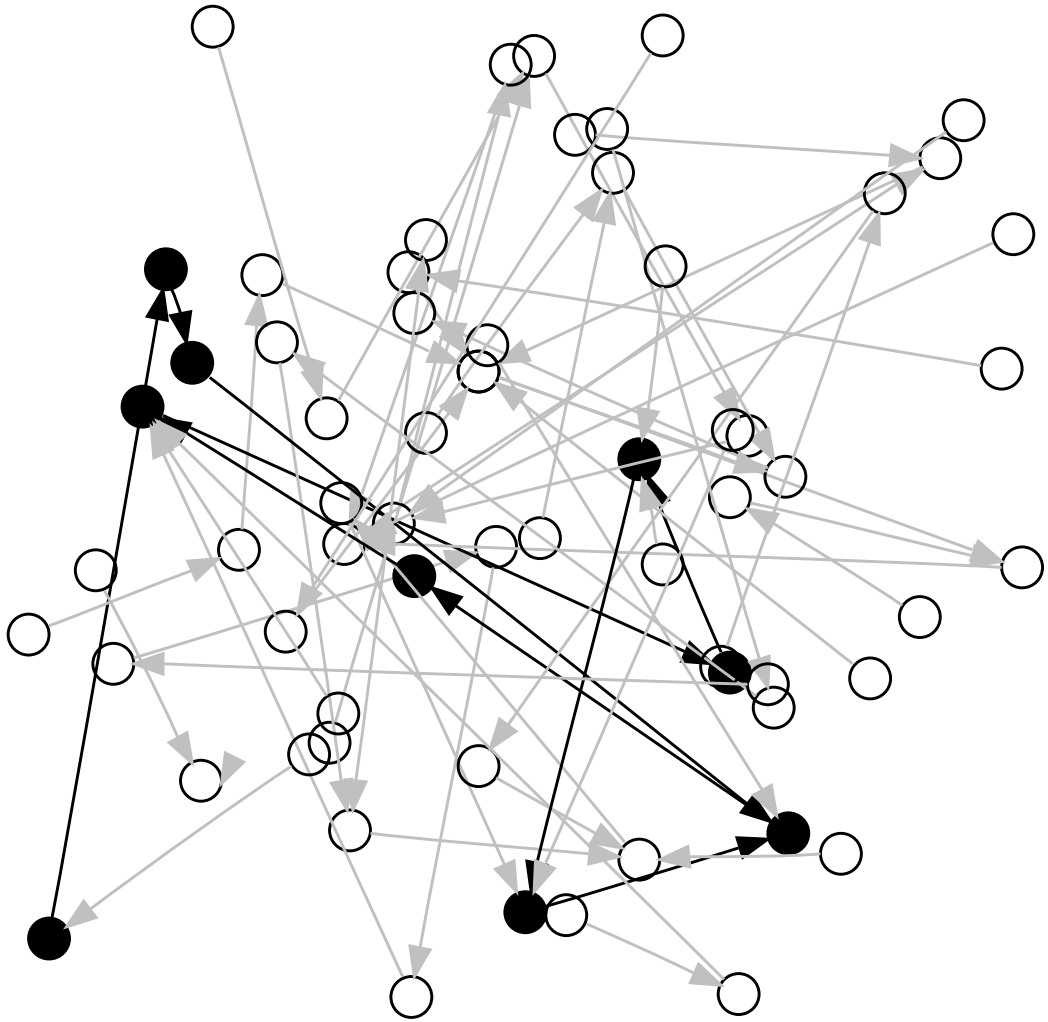

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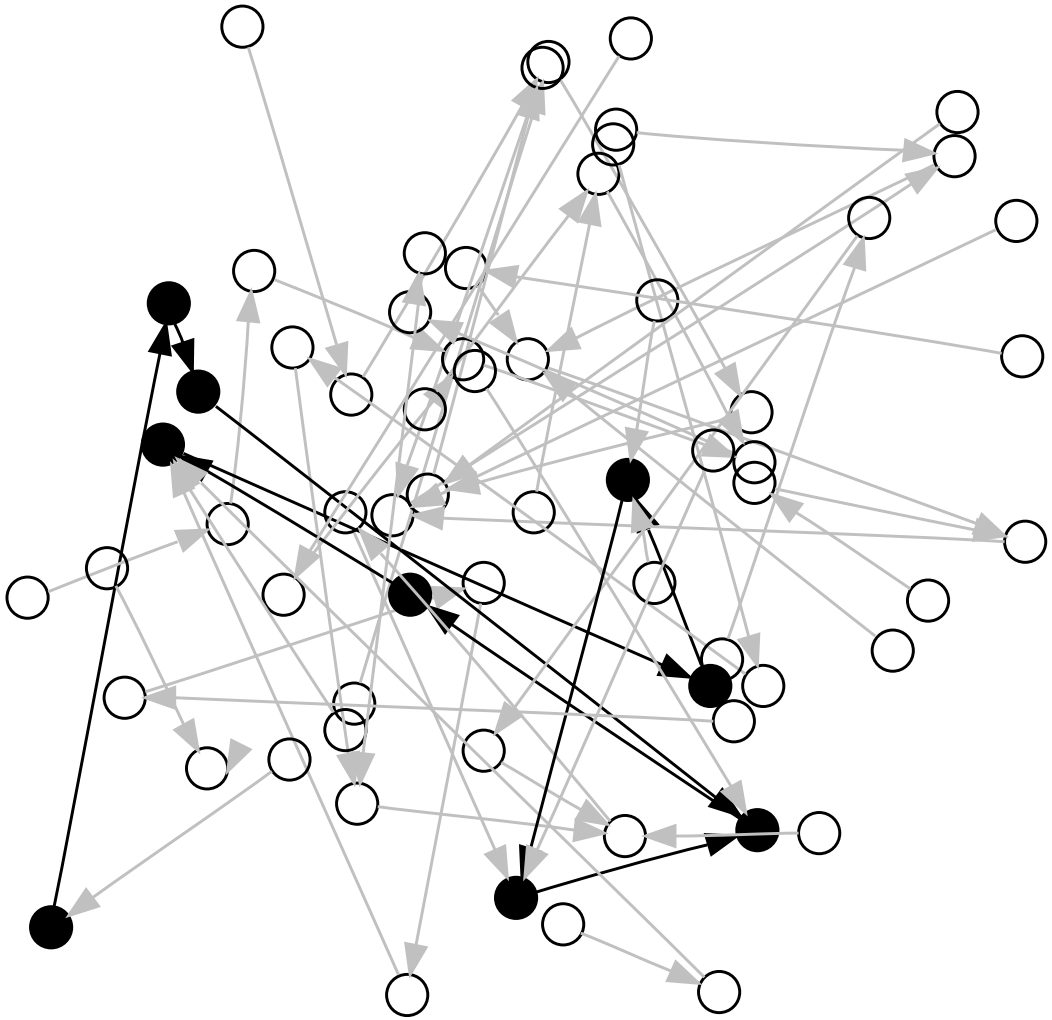

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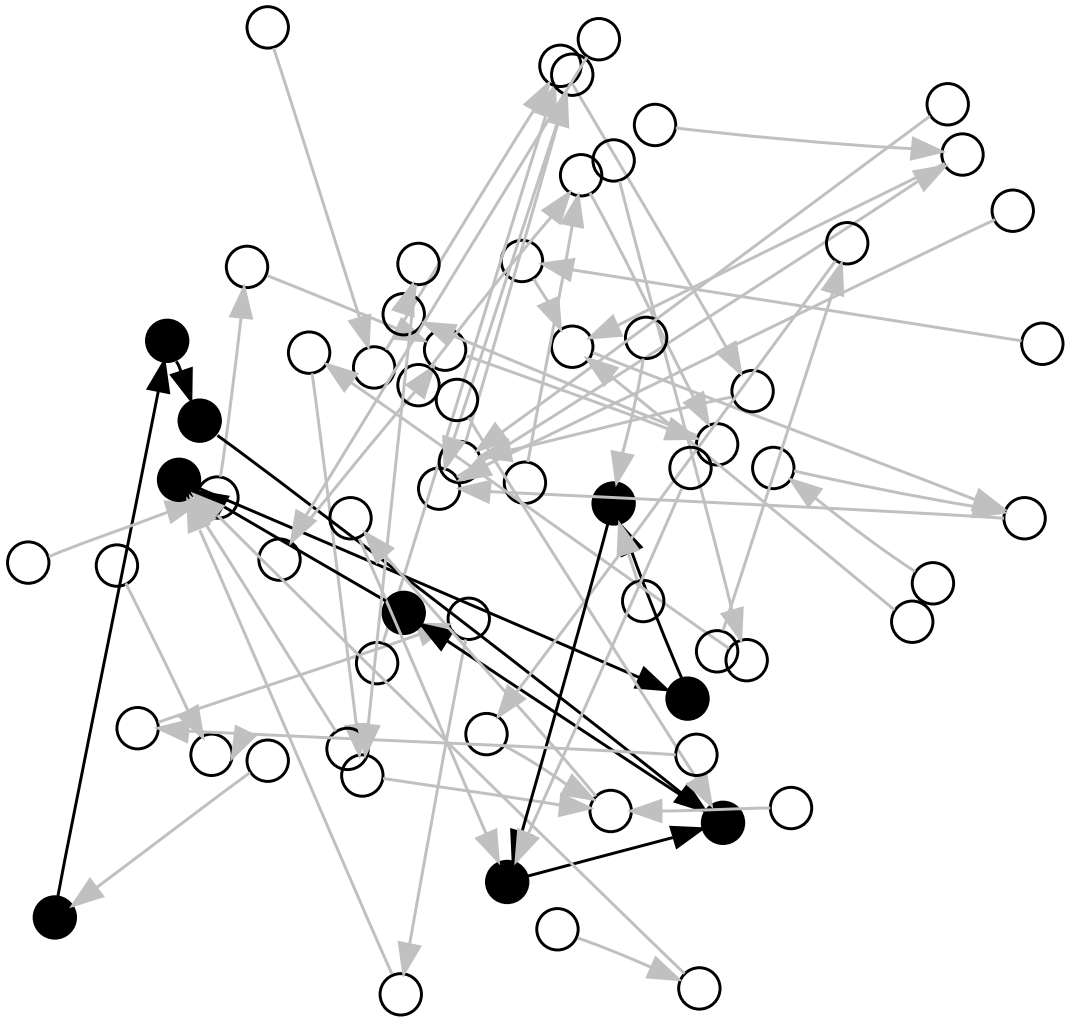

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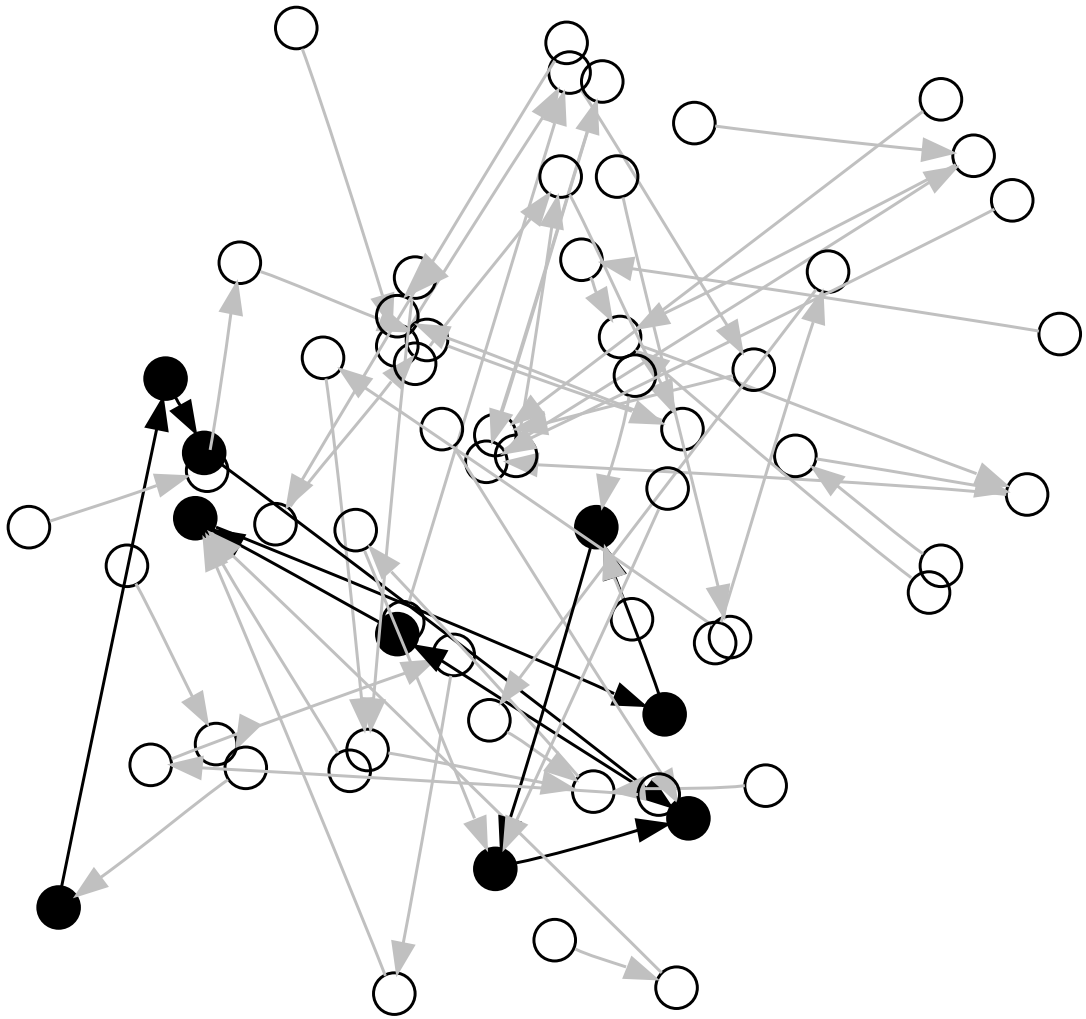

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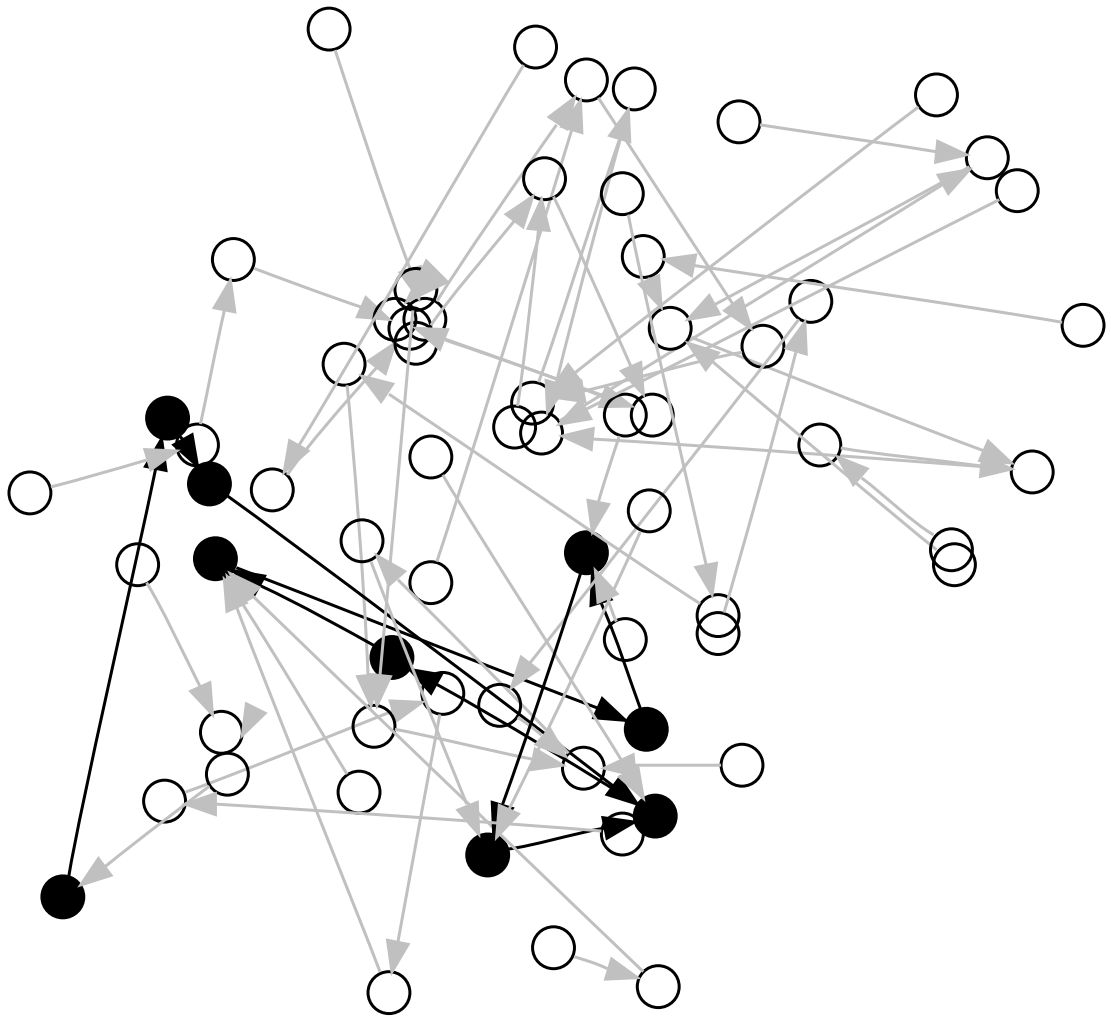

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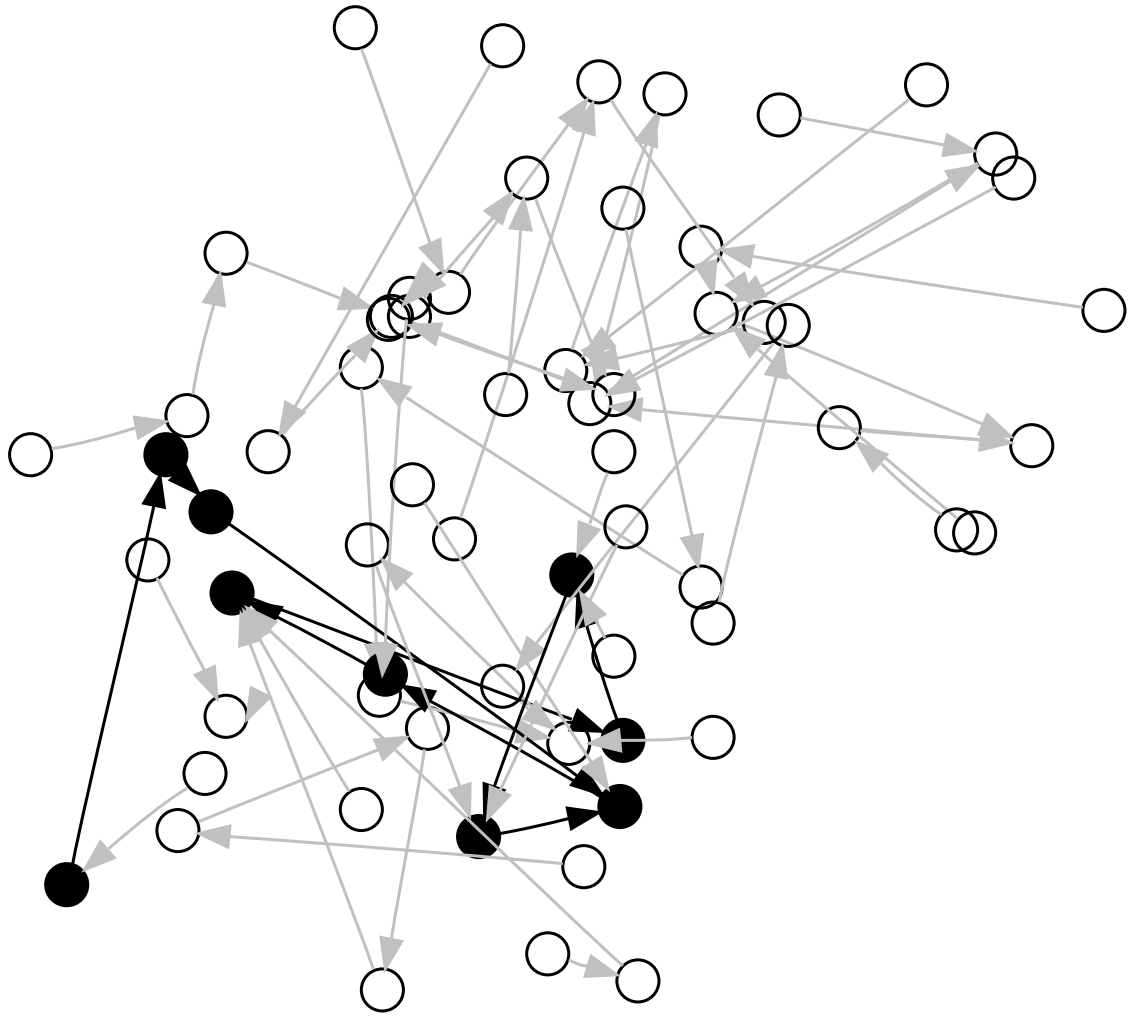

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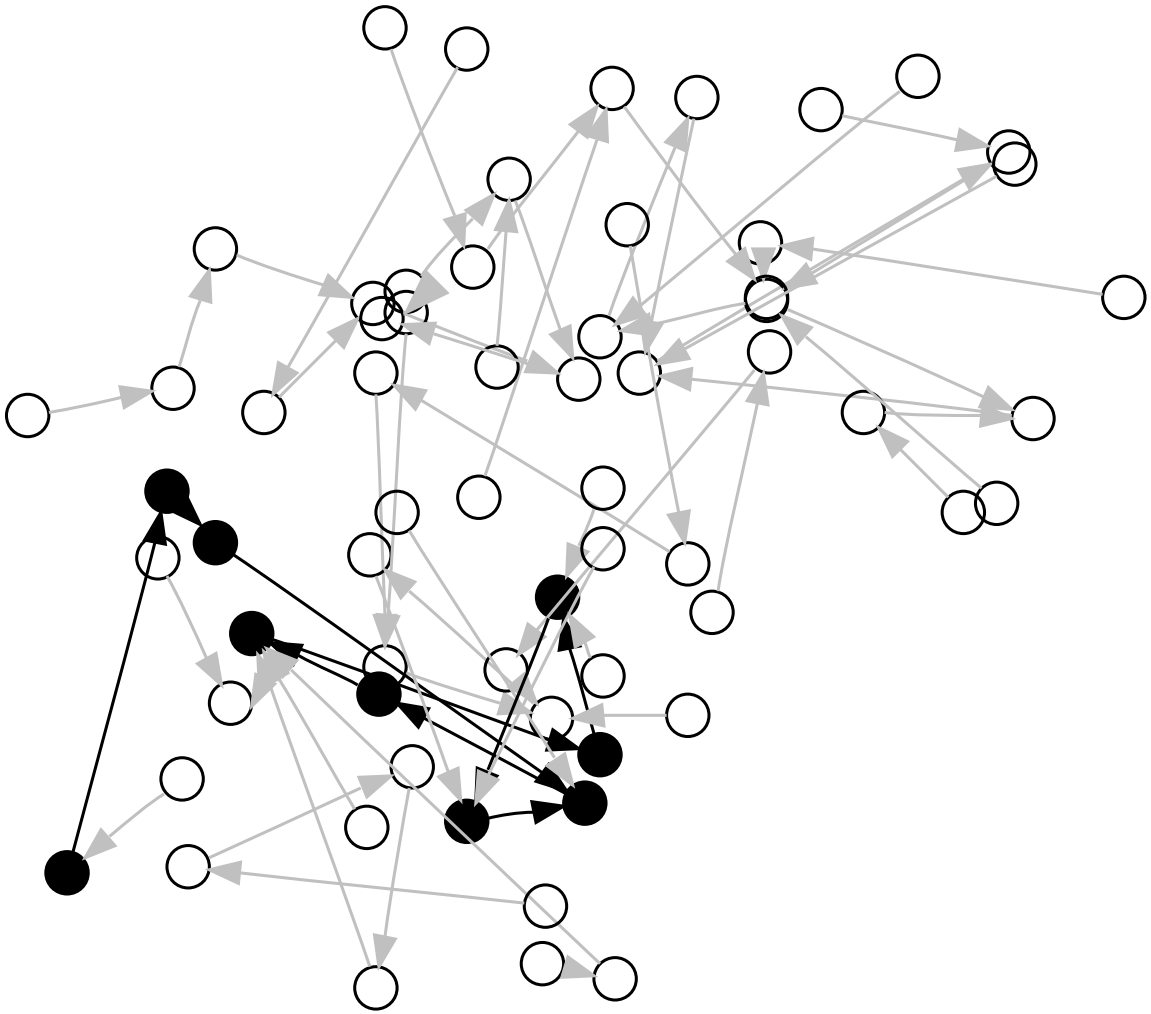

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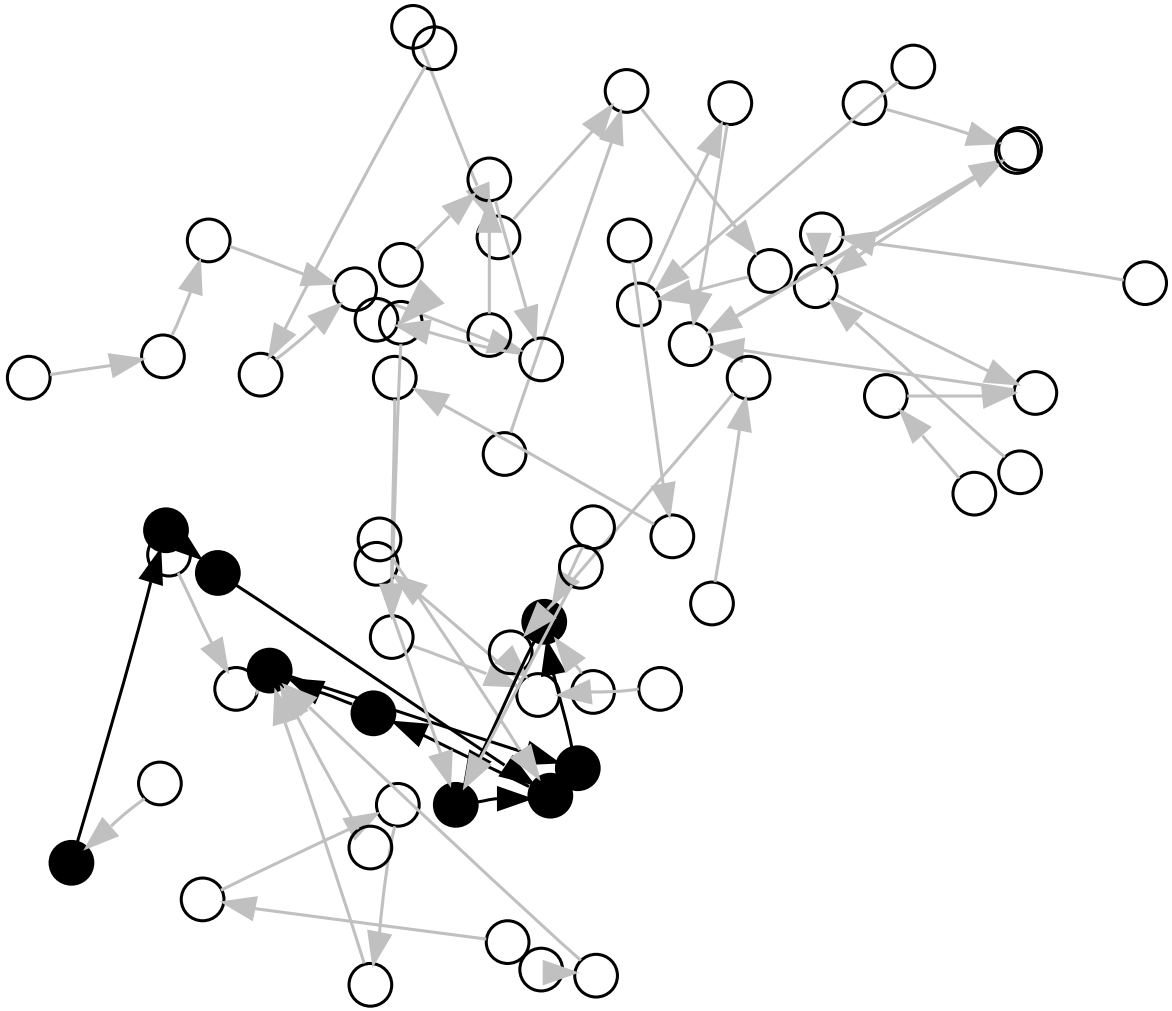

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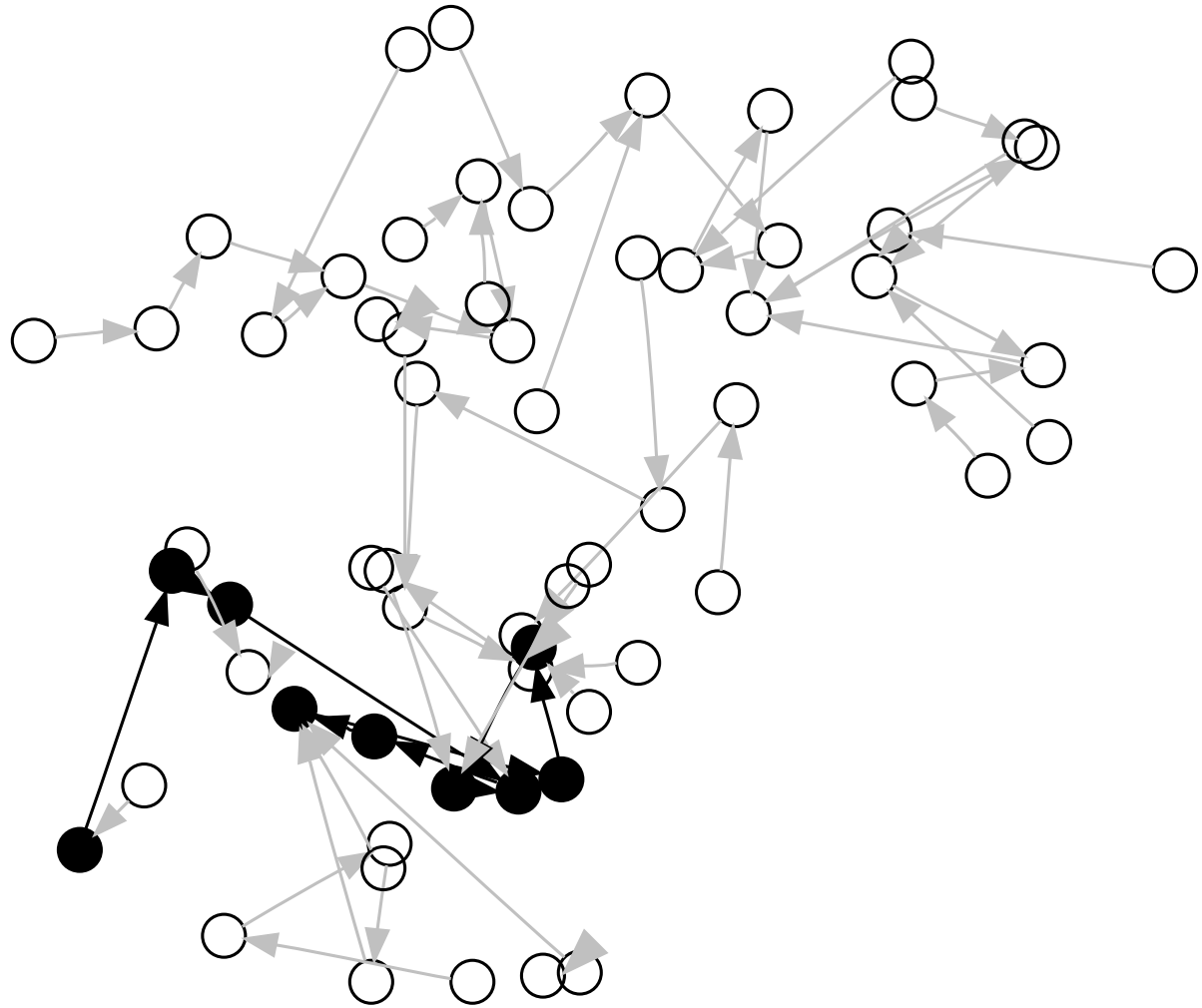

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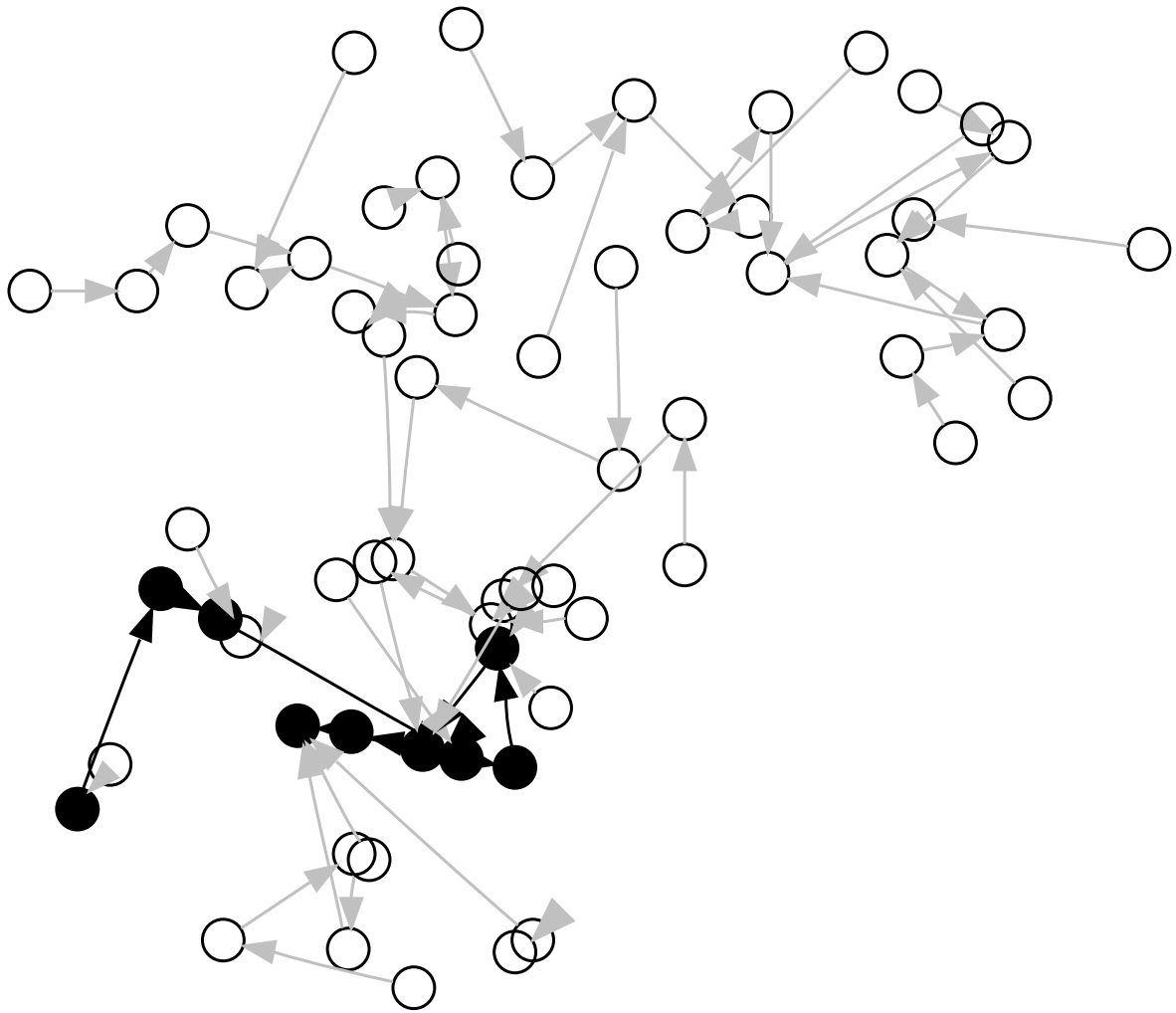

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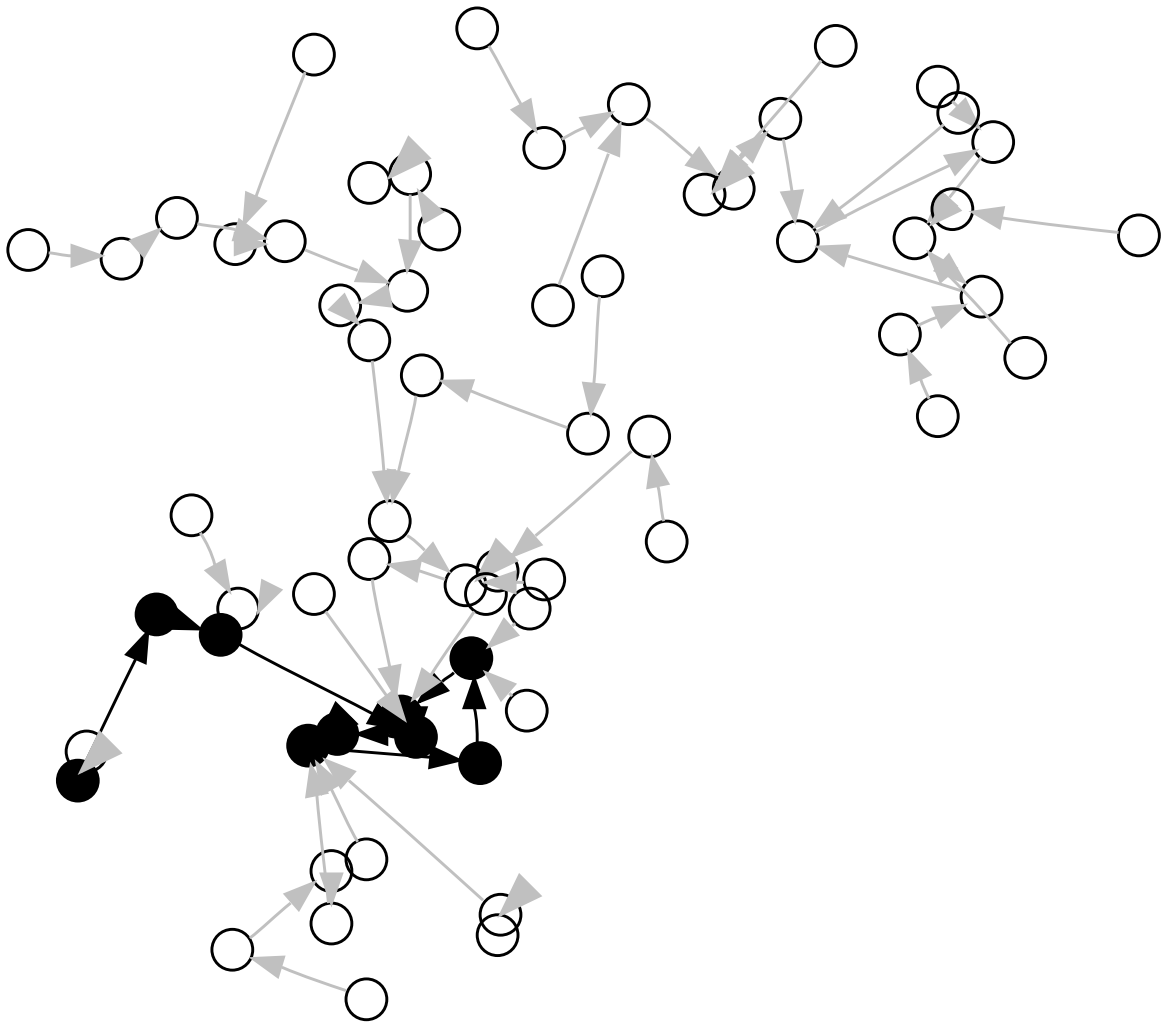

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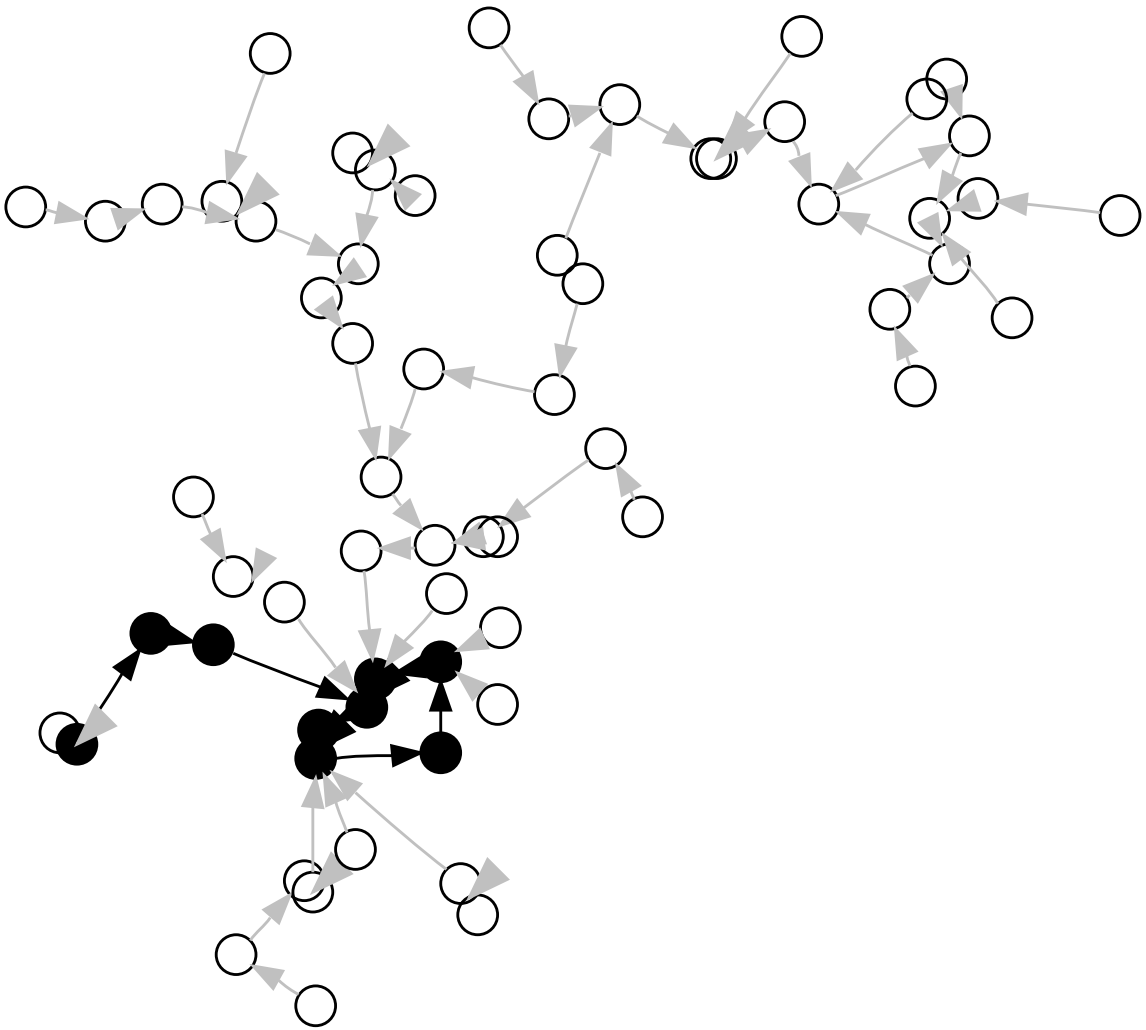

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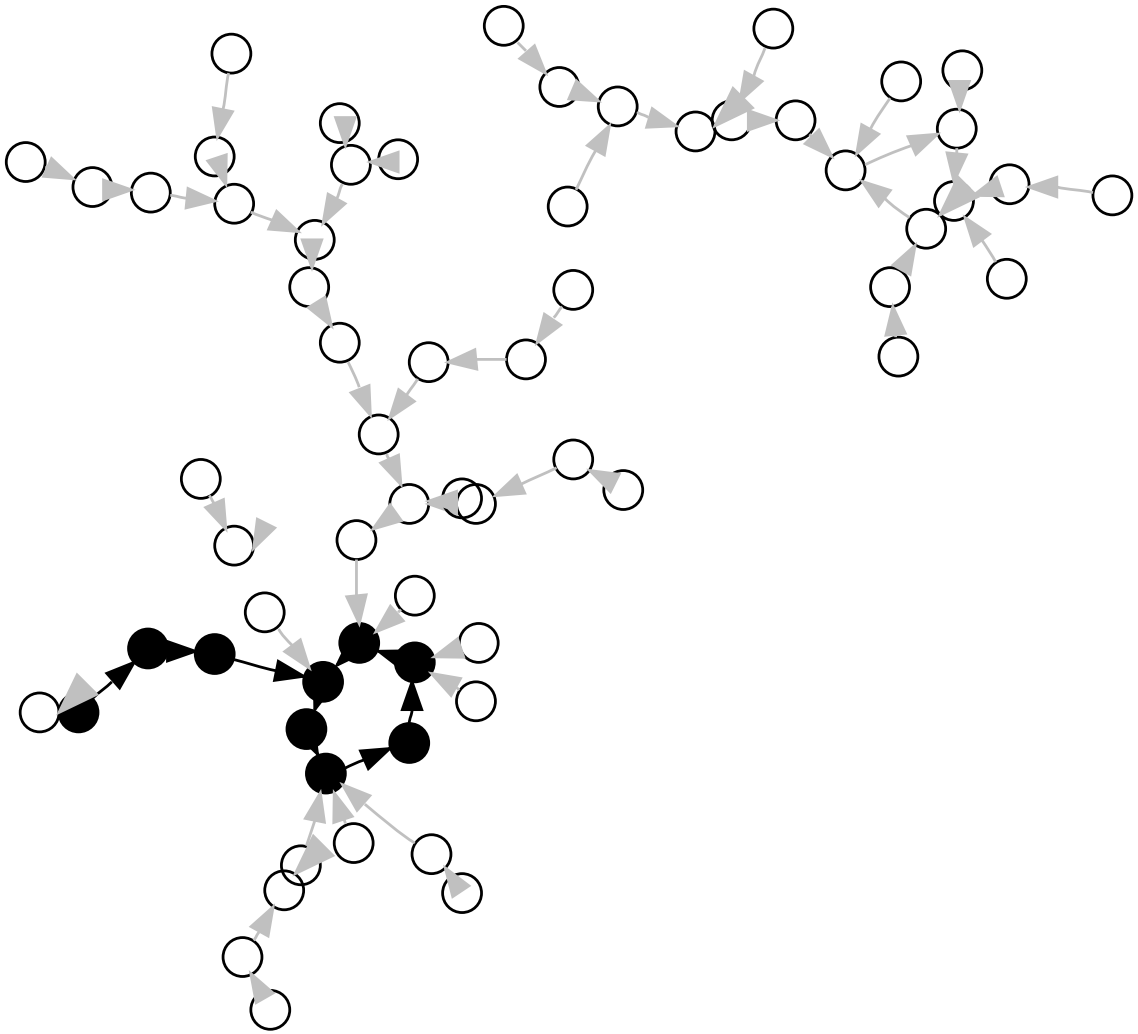

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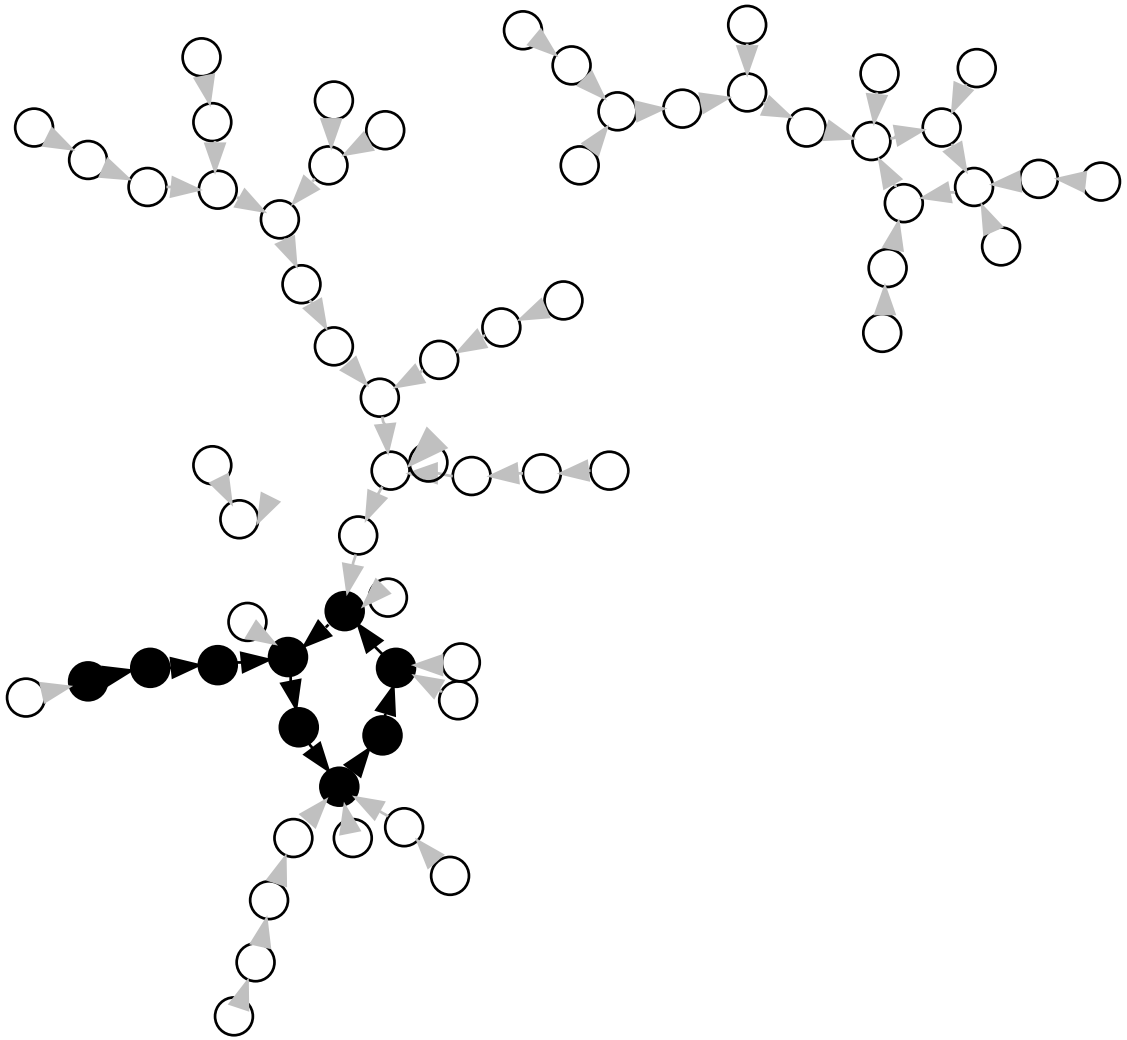


For random mappings:

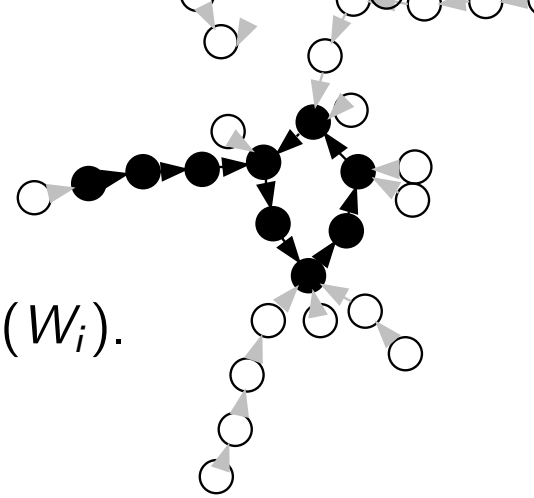
Tail length: $\sqrt{\pi n/8}$

Cycle length: $\sqrt{\pi n/8}$

See Flajolet & Odlyzko URL.

Collision after: $2\sqrt{\pi n/8} = \sqrt{\pi n/2}$ steps.

Eliminate the storage



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where the next step depends on current point: $W_{i+1} = f(W_i)$.

After about $\sqrt{\pi n/2}$ steps we have a collision:
$W_i = W_j$.

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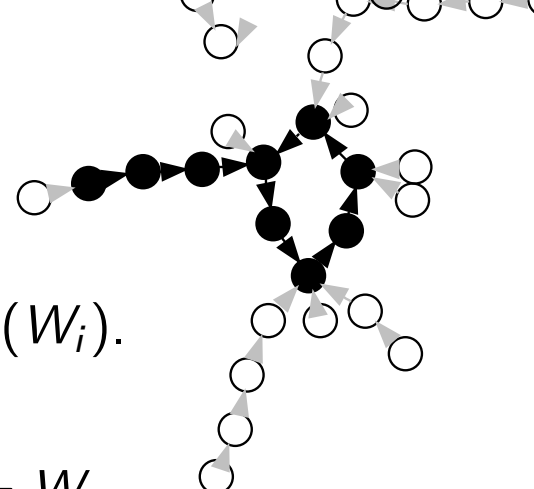

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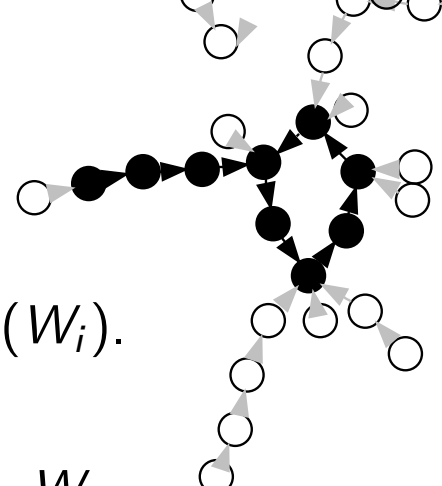

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Cycle-finding algorithms (e.g., Floyd) quickly detects this without requiring storage.

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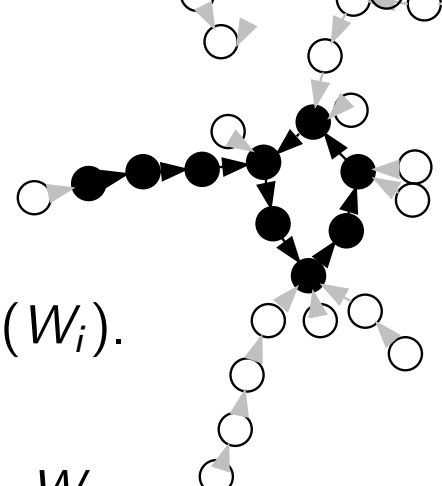

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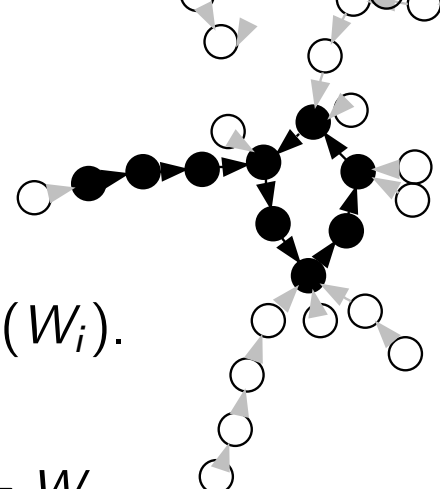

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Start at $S_0 = F_0 = W_0$.
Each step does $S_{i+1} = f(S_i)$ and $F_{i+1} = f(f(F_i))$.
Once both walks enter the cycle, they will collide.

Cycle length: roughly $\sqrt{\pi n/8}$.
Fast walk loops at most once after slow walk enters cycle to catch it.

Pollard's rho method

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The motivation is to remove the storage costs by turning solving the DLP into a random walk on elements of G.

This is a common strategy which we will also encounter for factorization and collision attacks for hash functions (cryptographic functions mapping to short outputs for which it should be hard to find collisions).

By the birthday paradox, after $\sqrt{\pi n/2}$ random draws from a set of n elements, one element will be drawn twice with 50% probability.

Two problems need to be solved to use the rho method successfully:

1. Design step function so that it "randomly" samples elements (so that the birthday paradox applies) while being deterministic (so we can use Floyd's cycle finding method to remove storage).
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This means that collisions are meaningful if $\gcd(n, b_i - b_j) = 1$.

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A double-scalar multiplication is cheaper than two separate scalar multiplications. (Share doublings, add P, Q, or $P + Q$.)

Pollard rho for DLP: Attempt 2 – additive walks

[Note: We have a solution, we're just haggling about the price.]

Idea: we do not need n^2 directions to look random.
Having some fixed set of step directions suffices.

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Compute and store k steps $R_j = c_j P + d_j Q, 0 \leq j < k$.

Fixed (small) number of double-scalar multiplications,
fixed (small) amount of storage.

Define the step function, costing 1 ADD per step, as

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Typical choice: take $s(W) \equiv x(W) \bmod k$, where $x(W)$ is the x-coordinate of W and $\mathbf{F}_p$ is represented as integers in $[0, p-1]$.
This is why we use affine not projective coordinates. Want unique representation.

Rho for DLP with additive walks

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Additive walks induce anti-collisions (see next page), this delays collisions.
If there are k steps then the runtime increases by a factor of

$$1/\sqrt{1 - 1/k} \approx 1 + 1/(2k).$$

Anti-collisions in additive walks

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Additive walks need more steps by a factor of

$$1/\sqrt{1 - 1/k} \approx 1 + 1/(2k).$$

Schoolbook method for Pollard rho

The schoolbook method uses a step function with only 3 types of steps.

This would be very bad for randomness with the $1 + 1/(2k)$ formula!

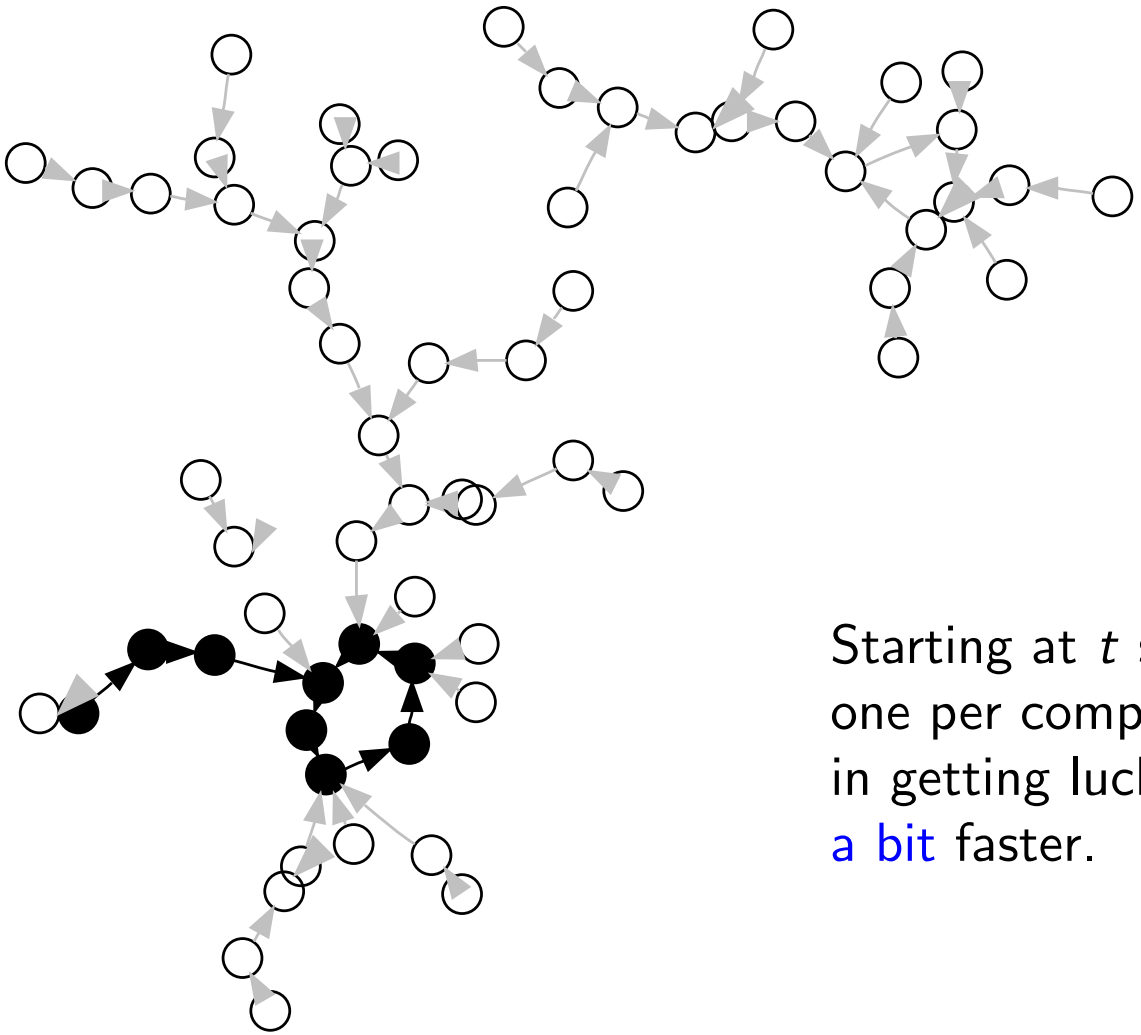
The method escapes that by using doubling as one of the steps.

$$W \leftarrow \begin{cases} W+P \\ W+Q \\ 2W \end{cases}, a \leftarrow \begin{cases} a+1 \\ a \\ 2a \end{cases}, b \leftarrow \begin{cases} b \\ b+1 \\ 2b \end{cases}, \text{ for } s(W) = \begin{cases} 0 \\ 1 \\ 2 \end{cases}.$$

Typically $s(W)$ takes an integer representation of $x(W)$ and outputs the remainder of division by 3.

Paralel collision search

How to use more than one computer efficiently?



Starting at t starting points,
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in getting lucky
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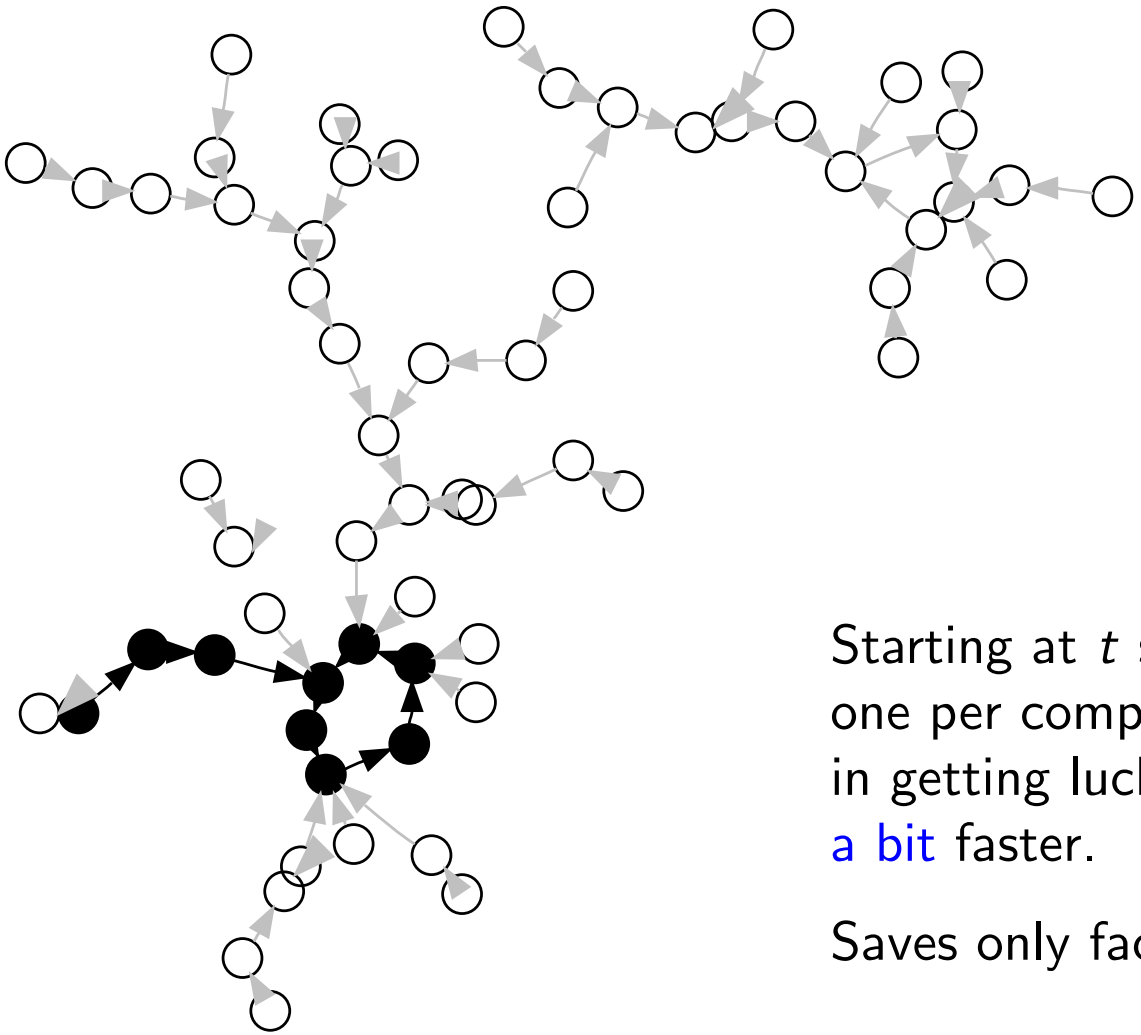


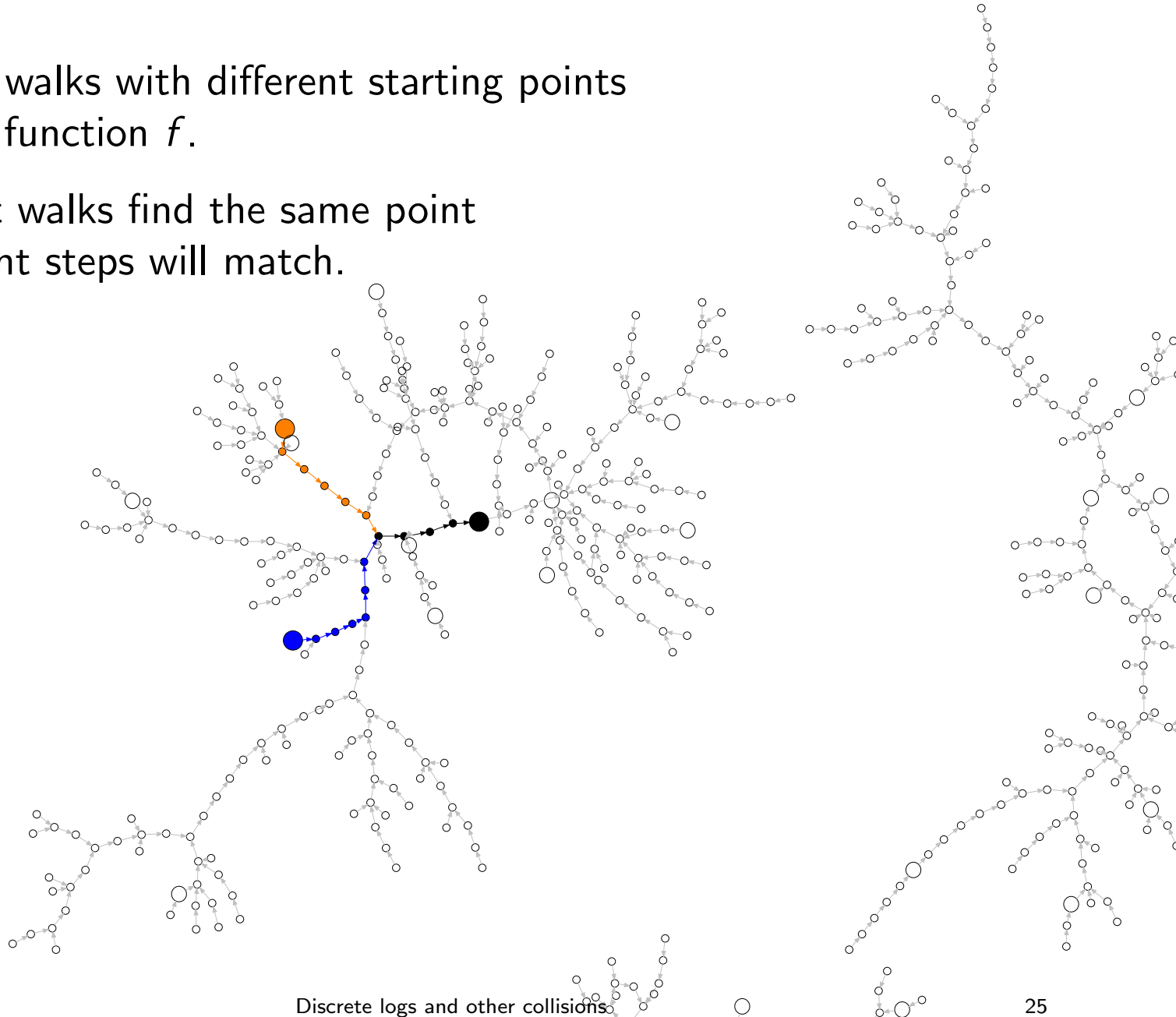
Starting at t starting points, one per computer, helps in getting lucky a bit faster.

Saves only factor of $\sqrt{t}$.

Van Oorschot and Wiener parallel collision search

Perform many walks with different starting points but same step function f.

If two different walks find the same point their subsequent steps will match.



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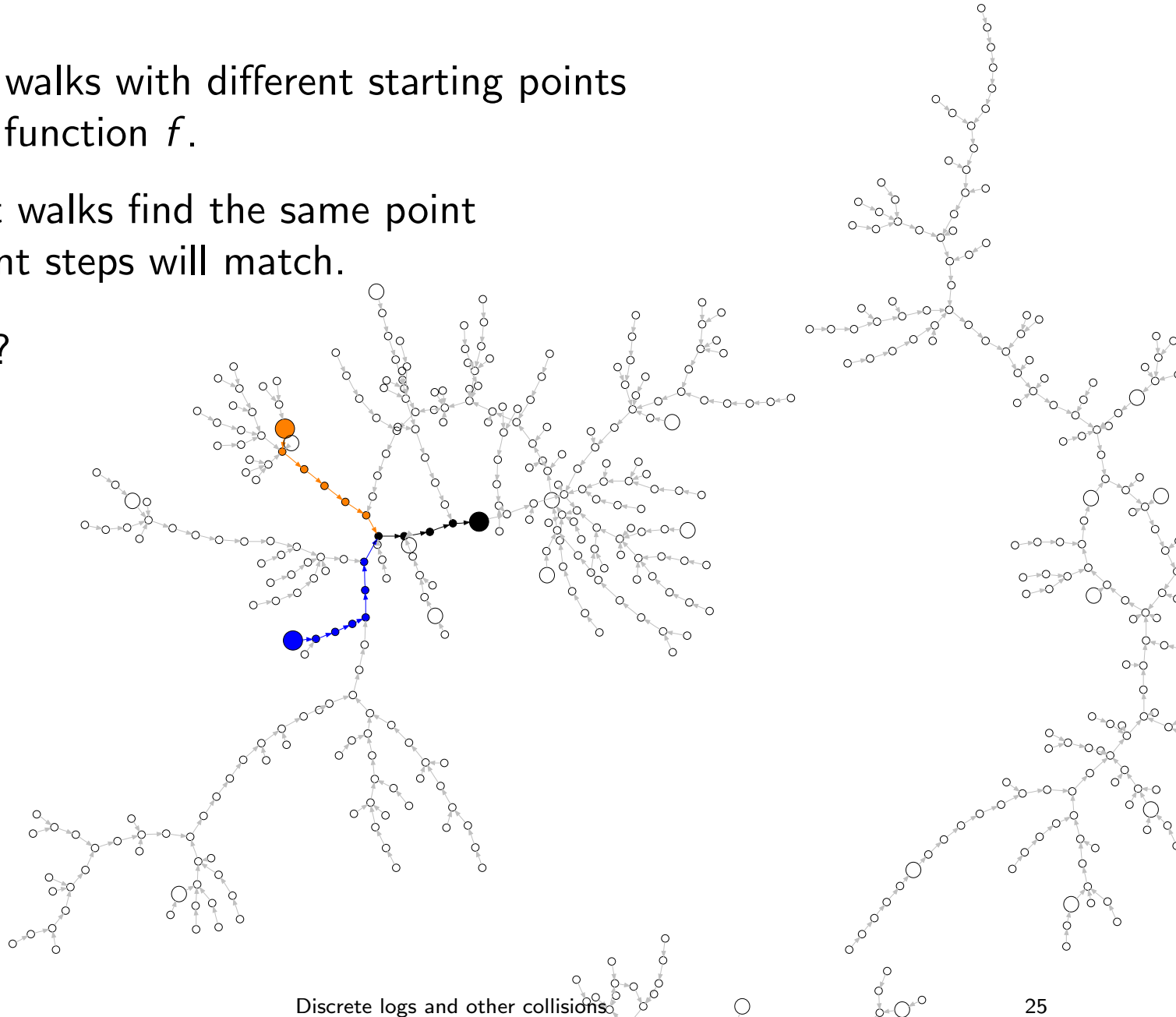

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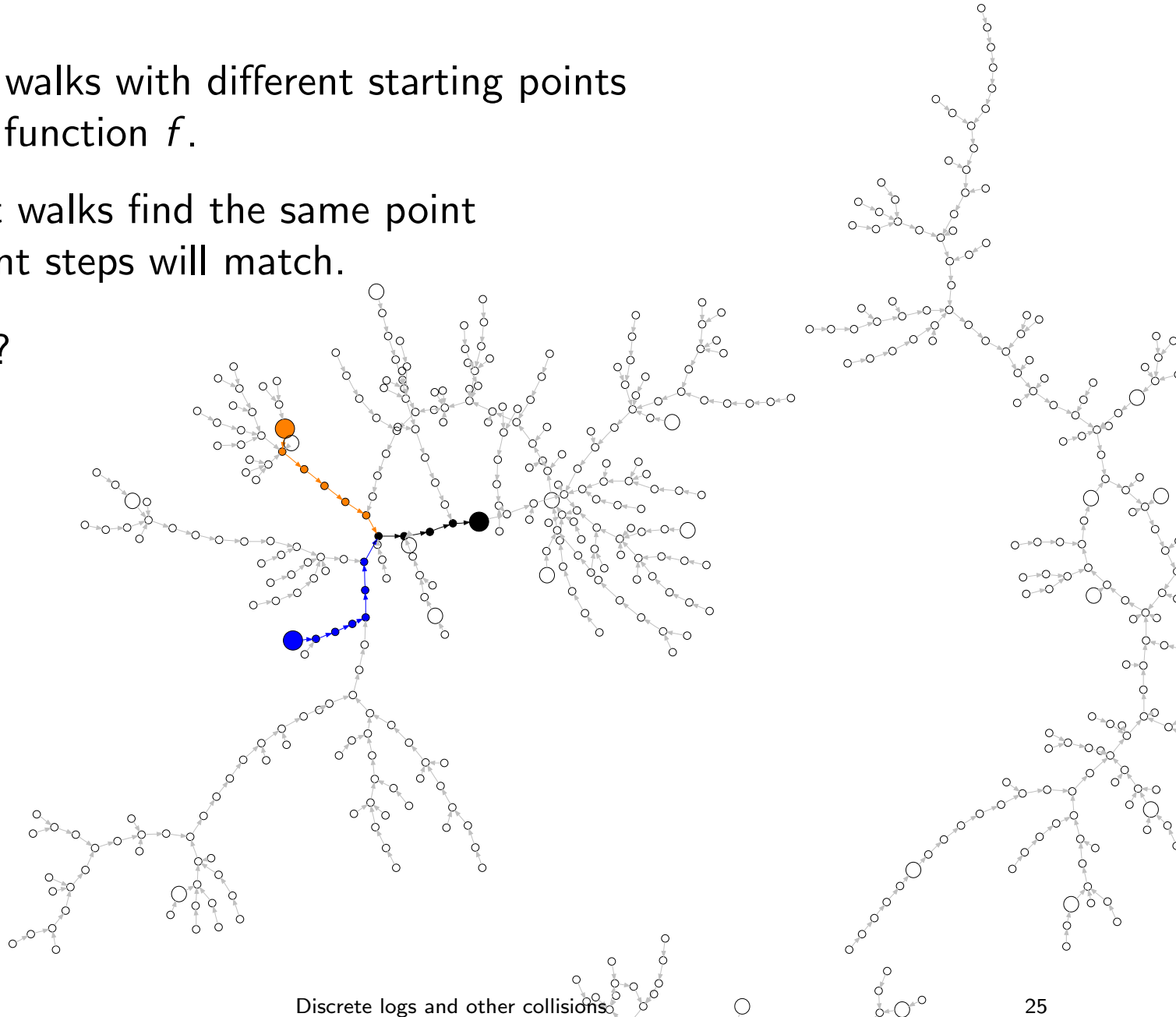

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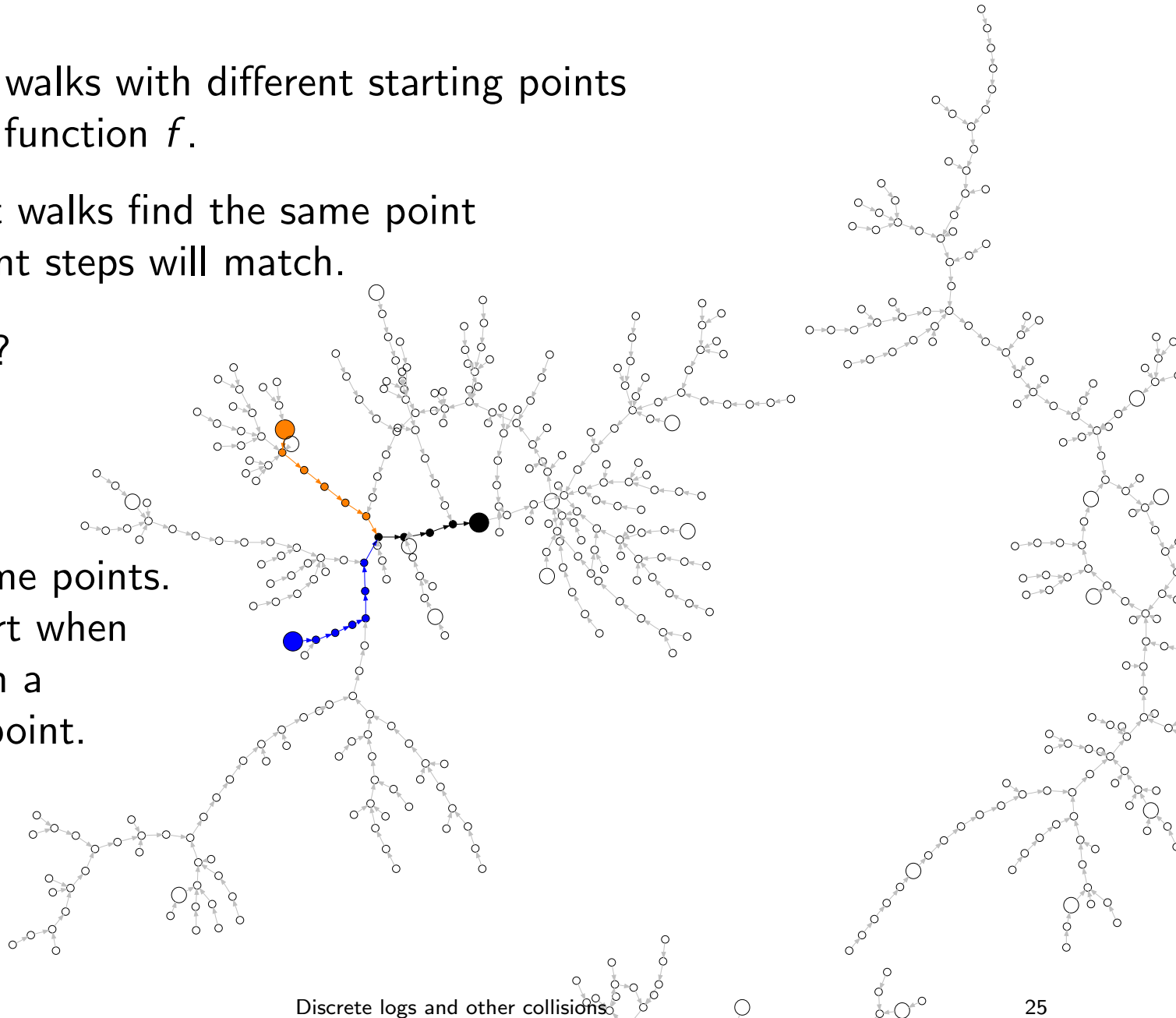
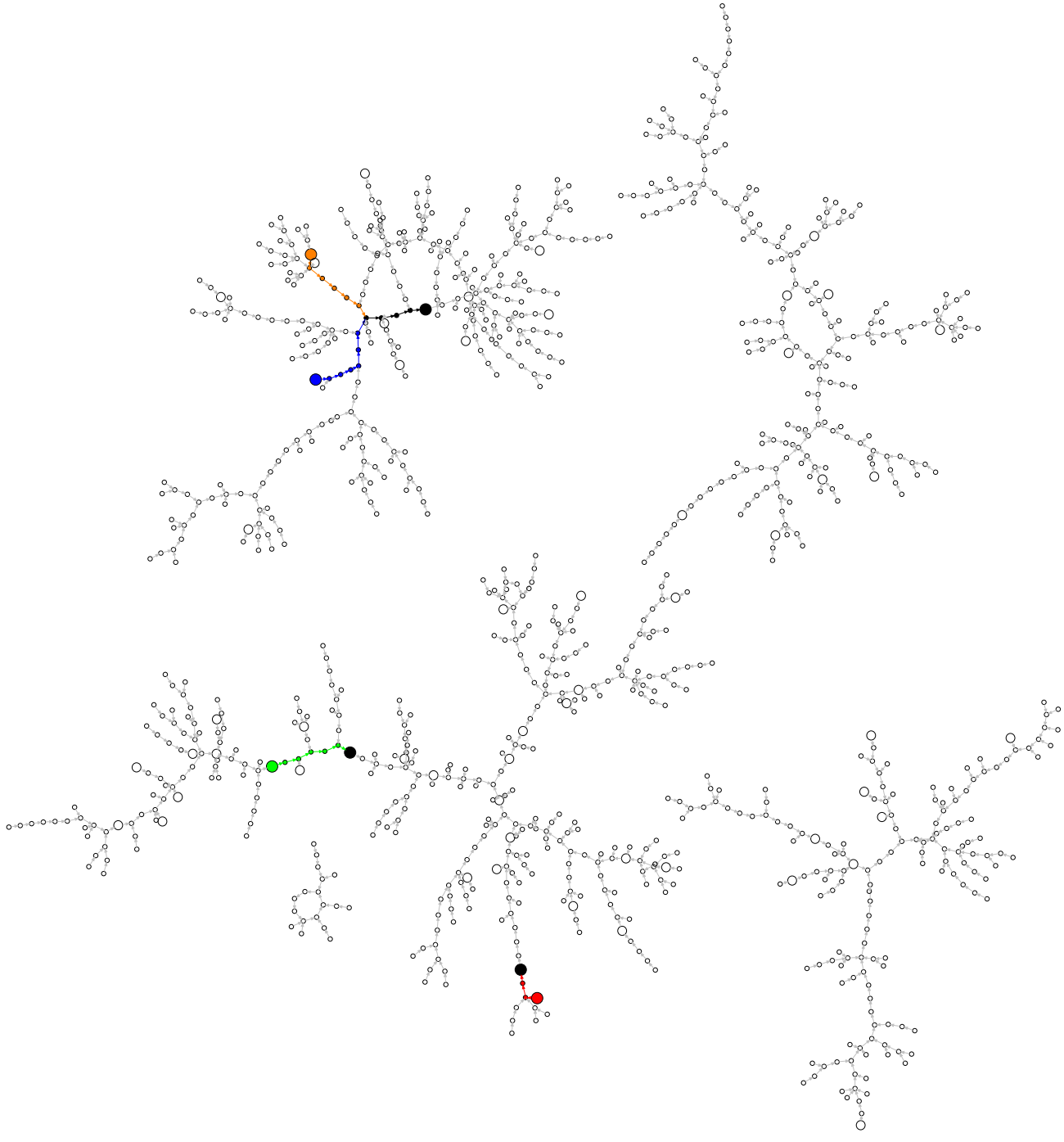
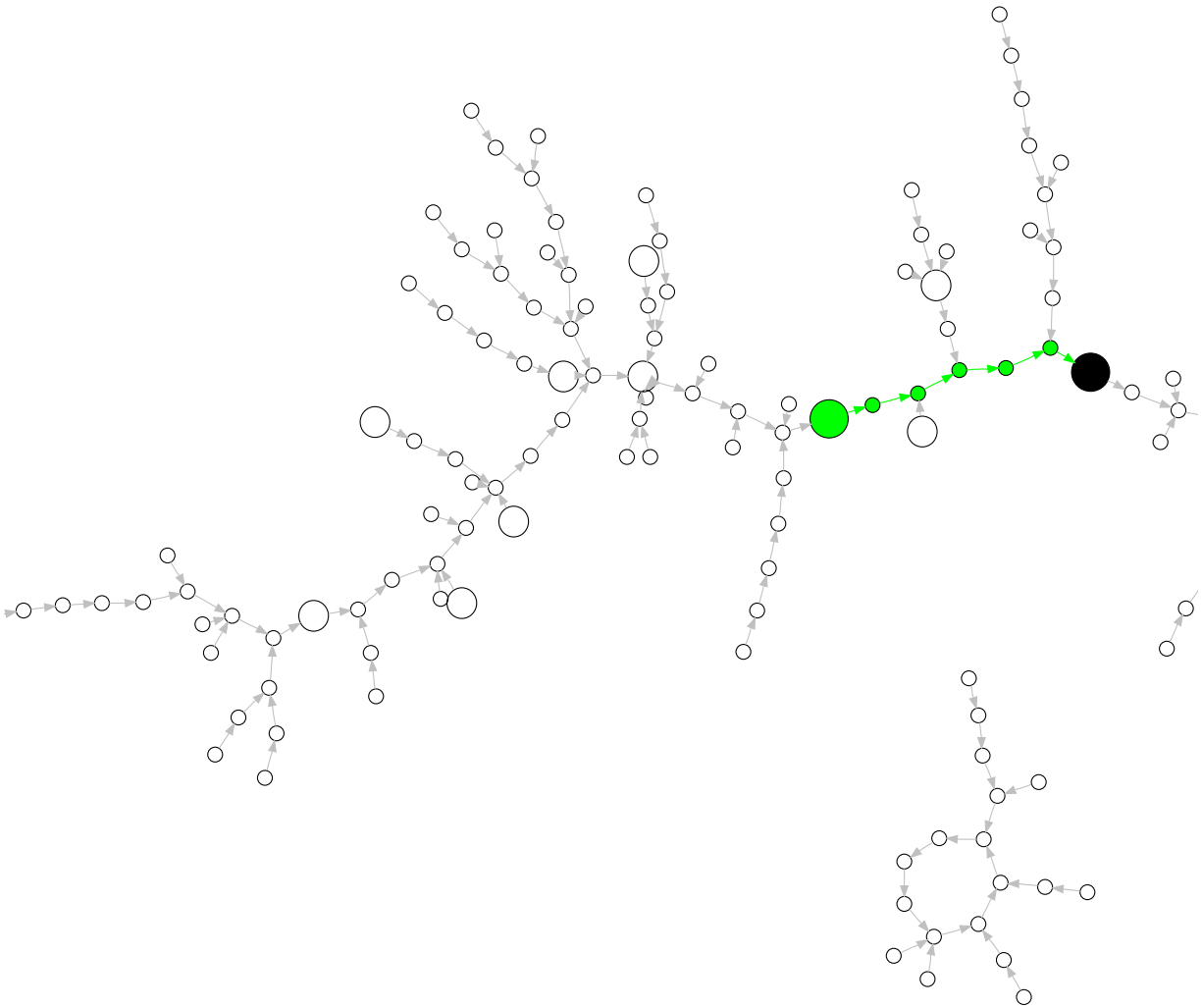
Idea: mark some points. Stop and report when getting to such a distinguished point.

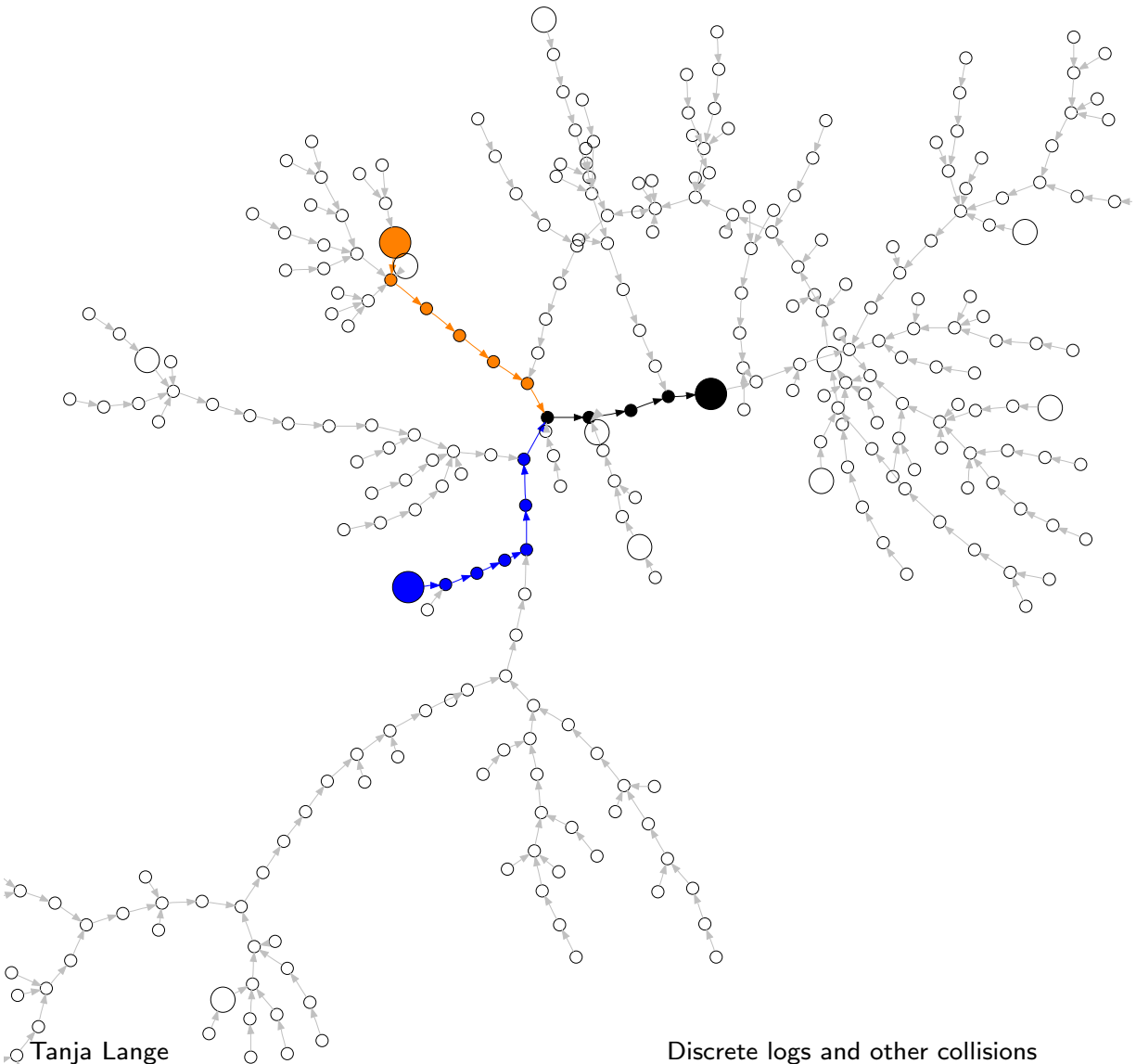




Walk to distinguished point, report to some server



Lucky case: two walks end in same distinguished point



Parallel Pollard rho

- Perform many walks with different starting points.
- Terminate each walk once it hits a distinguished point.
- Report the distinguished point and its a_i and b_i to server.
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Infrequent: small number of very long walks, little storage on server, long delay before a collision is recognized. Must not hit cycle!

More frequent: shorter walks, more storage, more communication.
Too frequent: very short, degrades to random search.

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Several speedups, e.g. identifying W and $-W$, some pitfalls.

Summary of DL systems

All attacks in this unit are *generic* attacks, i.e., they work in any group.

We did not cover the Pohlig-Hellman attack, another generic attack. See lectures 6 & 7 of playlist Discrete Logarithms on my YouTube Channel Cryptology for this.

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In summary, Pohlig-Hellman reduces security of DLP to security of largest prime order subgroup. Many groups are much weaker than their size n predicts!

Let $n = \prod p_i^{e_i}$, $\ell = \max\{p_i\}$.

Breaking DLP costs $O(\sqrt{\ell})(\log n)^{O(1)}$ bit operations.

O ignores all constants and lower order terms. $(\log n)^{O(1)}$ covers scalar multiplications and cost of group operations as well as operations in Pohlig-Hellman.

Hash functions overview and Pollard rho for collision finding

Goals of cryptographic hash functions

What do we want from a hash function $H : \{0,1\}^* \to \{0,1\}^n$?

For any string x, think of $H(x)$ as an n-bit fingerprint of x.

Goals:

- $H(x)$ looks totally random;
- nobody can find two different strings x, x' with $H(x) = H(x')$;
- any tiny change from x to x' makes a totally new $H(x')$;
- nobody can compute $H(x)$ without knowing all of x;
- nobody can compute a secret x given only $H(x)$;
- ...

Warning: Some hash goals are difficult to mathematically define.

Generic hardness of preimage resistance

Goal: Given $y \in H(\{0,1\}^*)$,
finding $x \in \{0,1\}^*$ with $H(x) = y$ is hard.

Here y is given, and is known to be the image of some $x \in \{0,1\}^*$.
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Generic attack: Try $\approx 2^n$ random choices of x.
If the output of H is distributed uniformly then
each x has a $1/2^n$ chance of $H(x) = y$.

e.g. $\approx 2^{128}$ tries if $n = 128$: very expensive.

Generic hardness of second-preimage resistance

Goal: Given $x \in \{0,1\}^*$, finding $x' \in \{0,1\}^*$
with $x \neq x'$ and $H(x') = H(x)$ is hard.

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Typically there are many other $x' \neq x$ with the same image,
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Generic attack: Try $\approx 2^n$ random choices of $x' \neq x$.
Same speed as for first preimages.

Generic hardness of collision resistance

Goal: Finding $x, x' \in \{0,1\}^*$
with $x \neq x'$ and $H(x') = H(x)$ is hard.

Attacker has full flexibility to choose any output y.
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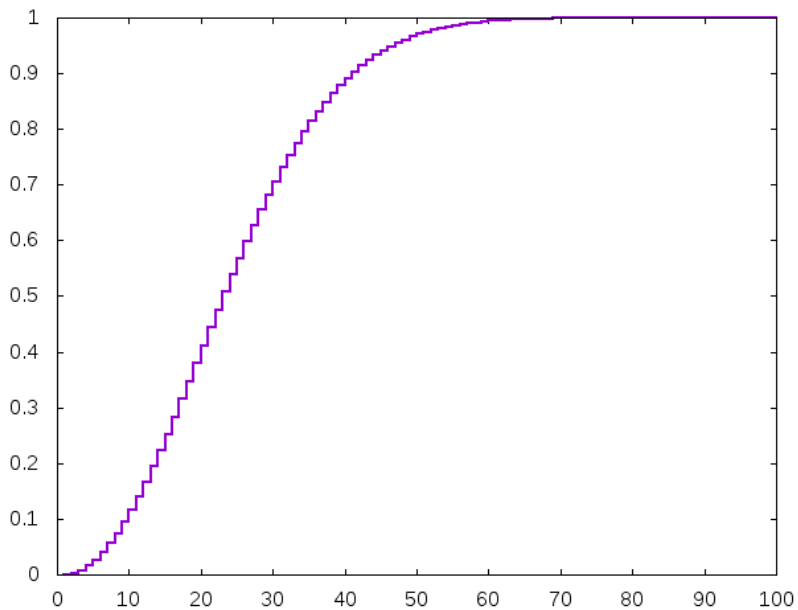
Generic attack: Try $\approx 2^{n/2}$ random choices of x. This number is much lower than 2^n because there is no restriction on the target.

The "birthday paradox": if one draws
$\approx 1.17\sqrt{m}$ elements at random
from a set of m elements,
then with $\approx 50\%$ probability
one has picked one element twice.

Note: this needs lots of storage,
if implemented as sketched here.



Pollard's rho method for collision finding

Avoid storage by performing random walks and checking for collisions.
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Just pick random starting point $W_0 \in \{0,1\}^\ell$ and iterate

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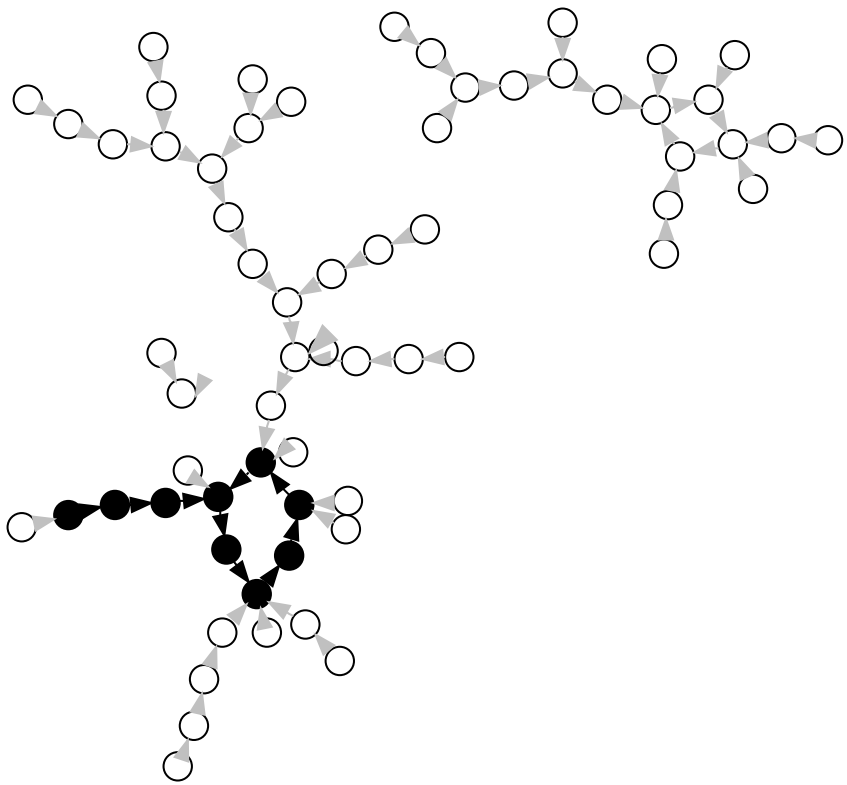
Find collision using Floyd.
If $W_{2i} = W_i$ we have met on the cycle, but we need the initial collision and h is not invertible.

Requires to remember the number i of small steps taken.
Cycle length is divisor of i.

Compare $W_j \stackrel{?}{=} W_{i+j}$ for $j=0,1,2,\ldots$ to find collision at end of tail.

Several improvements designed to avoid full recomputation.



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When we notice a collision in a distinguished point (DP), we might have walked along the same path for many steps.

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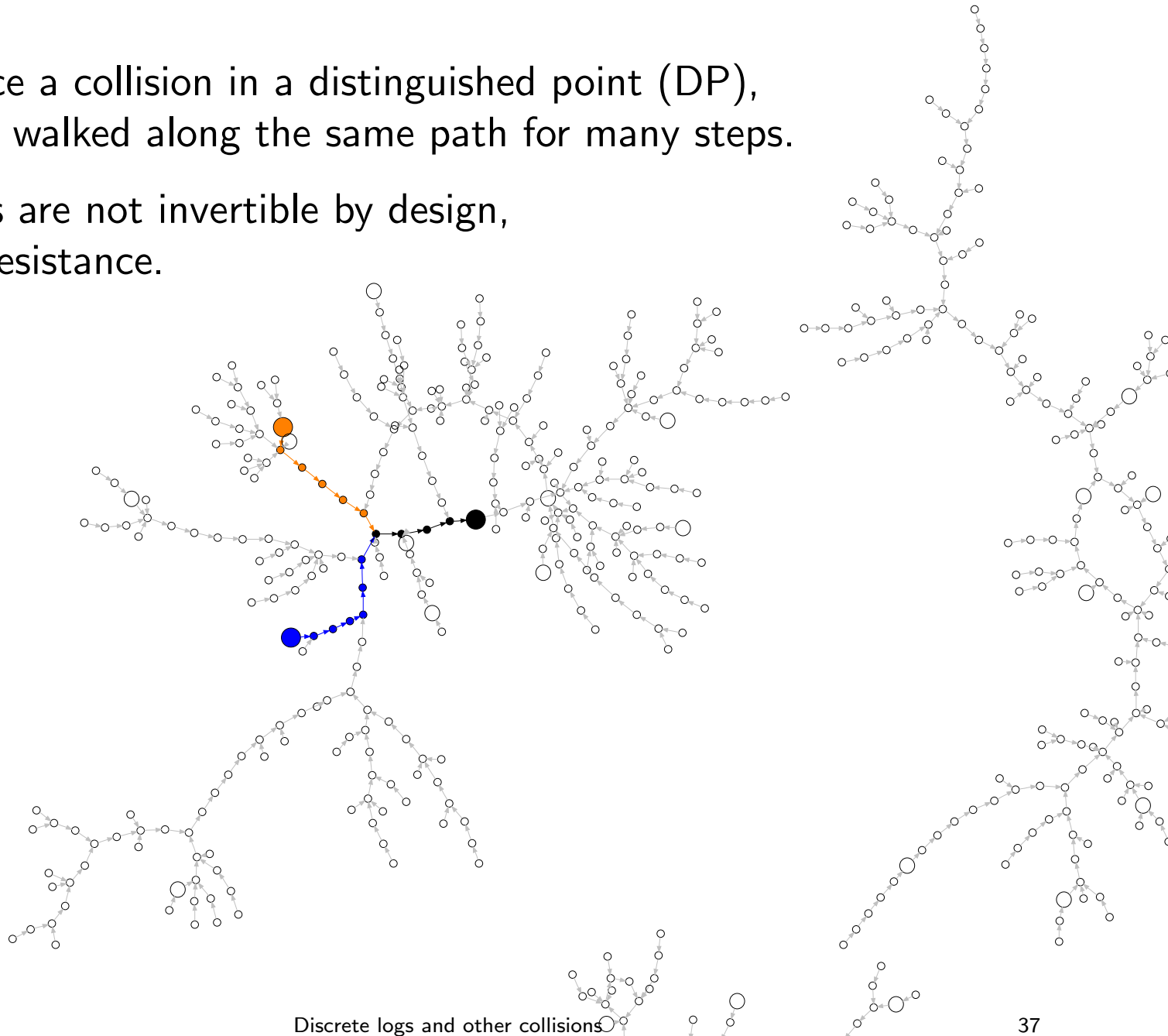

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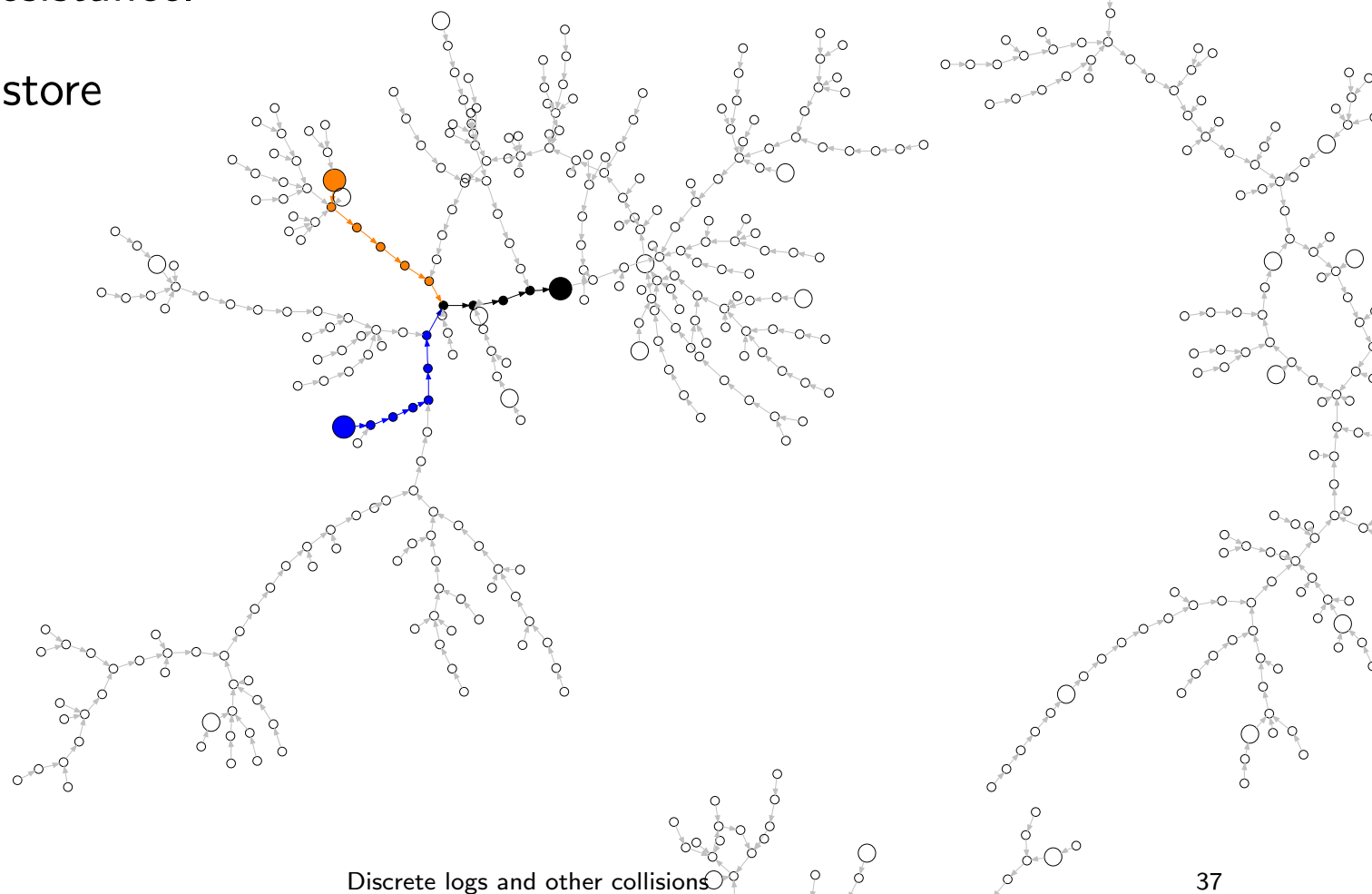

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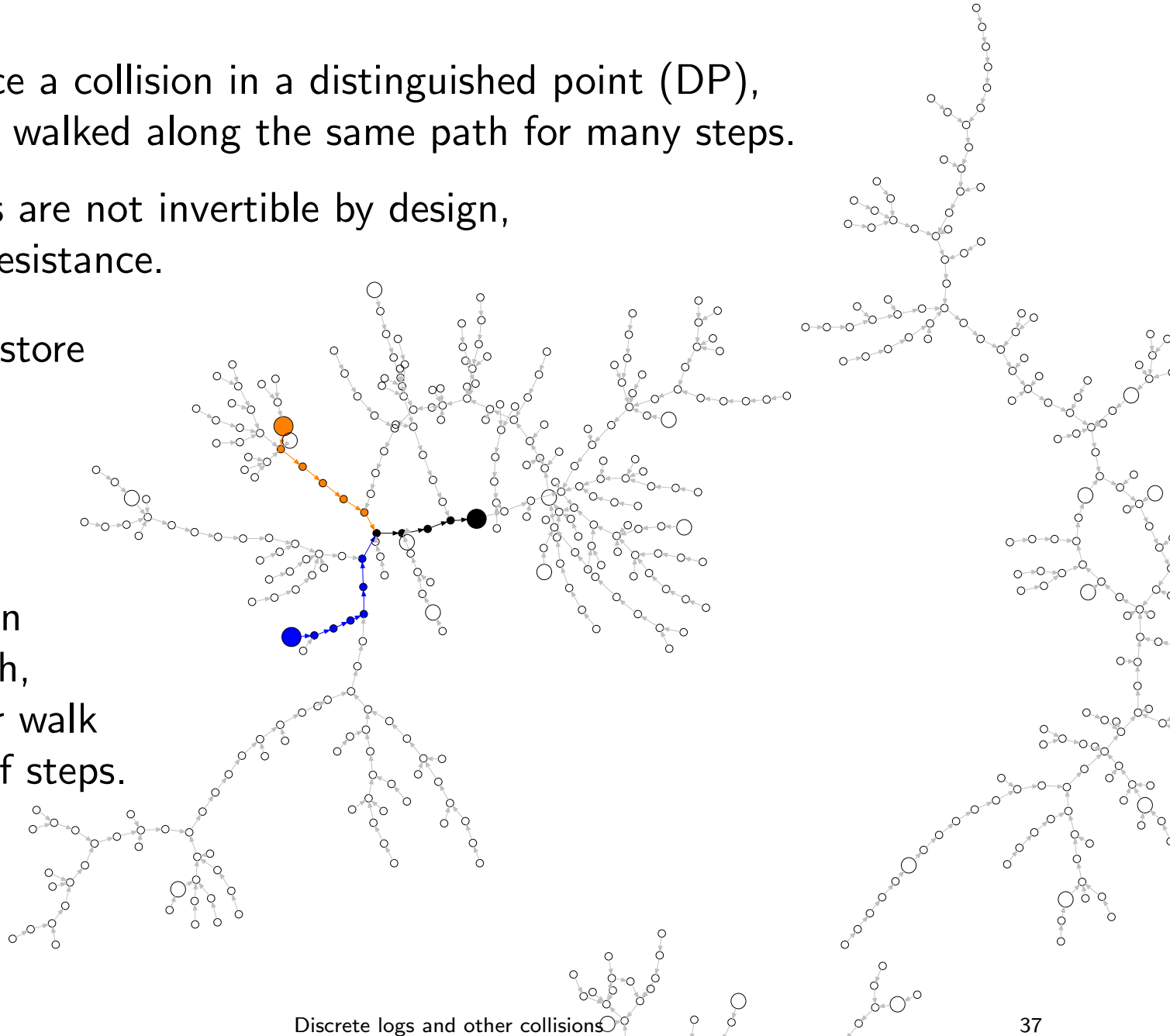
To find collision after DP match, advance longer walk by difference of steps.

Then advance both, checking each time.



Factorization overview and Pollard rho for factorization

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But: trial factorization is a useful step when factoring normal numbers.

Short summary of factorization methods

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- For RSA numbers: Number field sieve
  - Works by turning hard factorization of one number into many easier factorizations.
  - Uses sieving (think of Eratosthenes) to find small factors.
  - Uses the above to find medium size factors.
  - Also needs a stage of linear algebra at the end.
- The number field sieve has subexponential complexity, so we need to more than double the bit length to make the attack twice as hard.

Will use n for RSA numbers (hard to factor) and m for normal numbers. Typically, m is odd without very small prime divisors.

Pollard's rho method for factorization

Define $\rho_0 = 0$, $\rho_{k+1} = \rho_k^2 + 11$.

Every prime $\leq 2^{20}$ divides
$S = (\rho_1 - \rho_2)(\rho_2 - \rho_4)(\rho_3 - \rho_6)\cdots(\rho_{3575} - \rho_{7150})$.
Also many larger primes do.

If such p divides m, it divides $\gcd(S, m)$.
Computing S takes $\approx 2^{14}$ multiplications mod m, very little memory.

Compare to $\approx 2^{16}$ divisions for trial division up to 2^{20}.

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Problem: We don't see collision as we work modulo m, not p.
But p divides $\gcd(\rho_i - \rho_j, m)$.

S implicitly uses Floyd,
product reduces number of gcd steps:
$\rho_i \equiv \rho_j \bmod p \Rightarrow \rho_k \equiv \rho_{2k} \bmod p$
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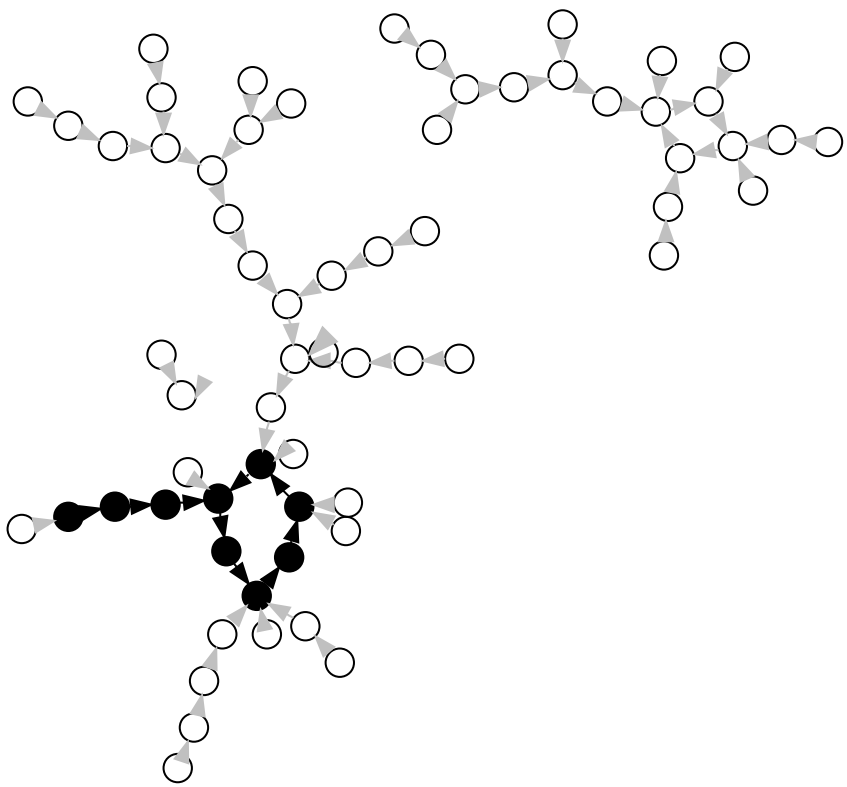

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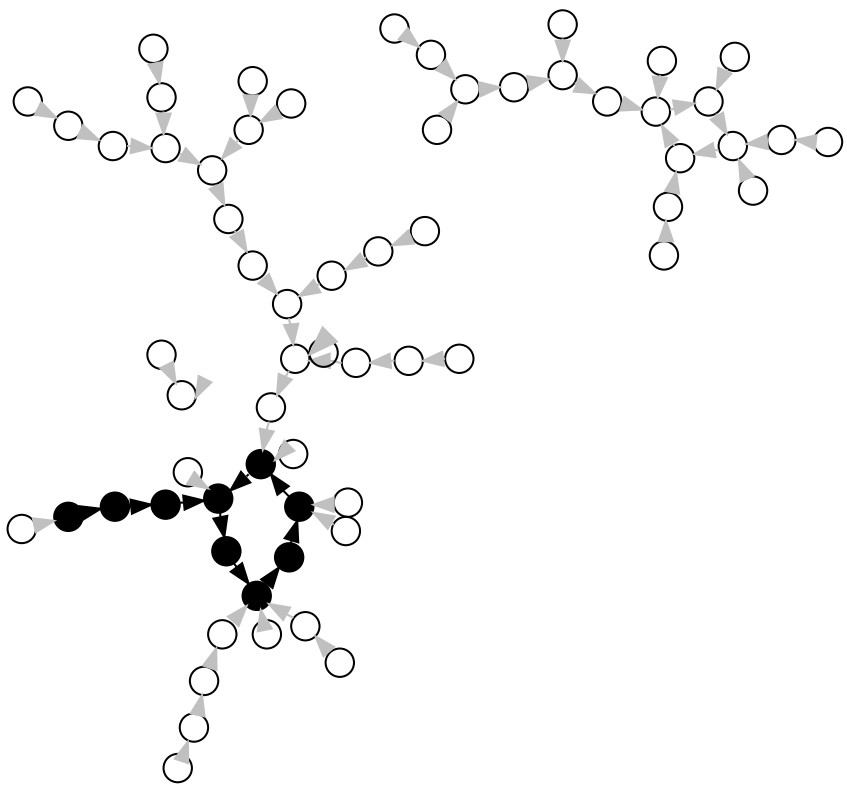

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for $k \in (j-i)\mathbf{Z} \cap [i, \infty] \cap [j, \infty]$.

Plausible conjecture: $y^{1/2+o(1)}$; so $y^{1/2+o(1)}$ mults mod m.